Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning

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Lecture No. 09

Applications of Matrices: Graph theory, Social Networks, Dominance Directed Graph, Influential Node

Hello, welcome to another module in this massive open online course. So, let us continue our discussion on the applications of linear algebra, and that is yet another interesting area of application and that is basically in graph theory and how it can be applied in a very important and interesting area, that is, Social Networks.

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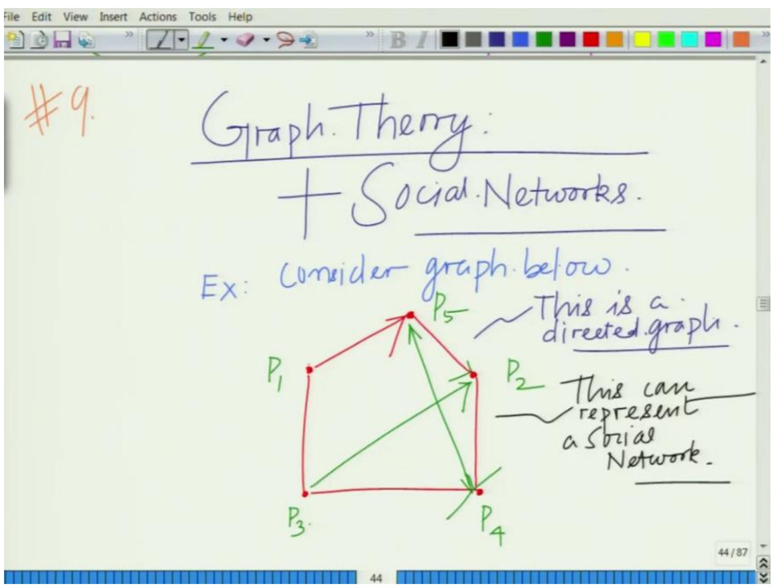

So, let us look at linear algebra application in Graph theory plus Social Networks. So, Graph theory can be used to analyze social networks. So, very interesting applications of linear algebra in graph theory which can be used for the analysis of social networks. Consider for example, consider the graph below I hope all of you know the meaning of a graph.

A graph basically contains nodes and edges, in particular, we are looking at a directed graph which contains directed edges. So, we are looking at this directed graph. So, we have these points P_1, P_2, P_3, P_4, P_5. So, these are the 5 points which can represent for instance people in a social network. So, you have this diagram this can show, for instance, the relations we are about to see this.

So, we have this directed graph and directed graph means basically you have the edge and edge is pointing from node A to node B and the edge is pointing either in the direction of node A or node B, and you can have in fact bi-directional edges also, but here we are considering only a single direction for each edge. For example, this is a directed graph, because each edge has a direction. And this can represent for instance a social network. What we can say? I am going to come to that.

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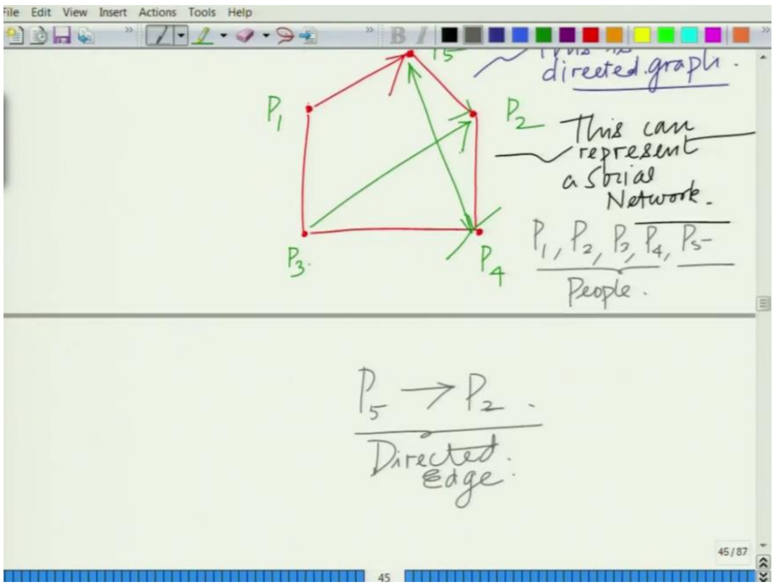

So, for instance, social network with P_1, P_2, P_3, P_4, P_5. These represent the people, and we have this directed edge for instance from P_5 to P_2.

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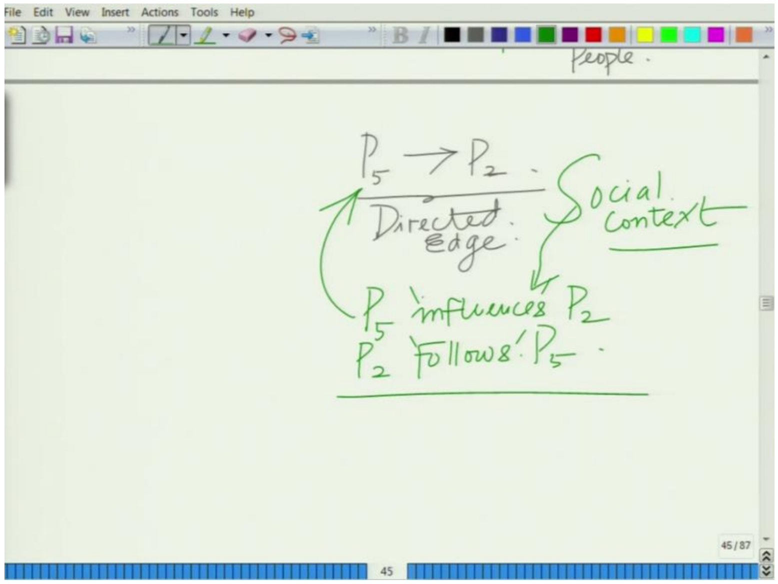

And now, this kind of directed edge is going to present, for instance, a relationship such as influence. So, P_5 influences P_2 or P_2 is following P_5. So, depending on the notation, so you can say P_5 influences, for instance, Twitter, you can think of this as Twitter. P_5 influences P_2 where P_2 follows P_5, P_2 follows P_5. So, this is how the social network works. Graphs are very important applications in modern social networks.

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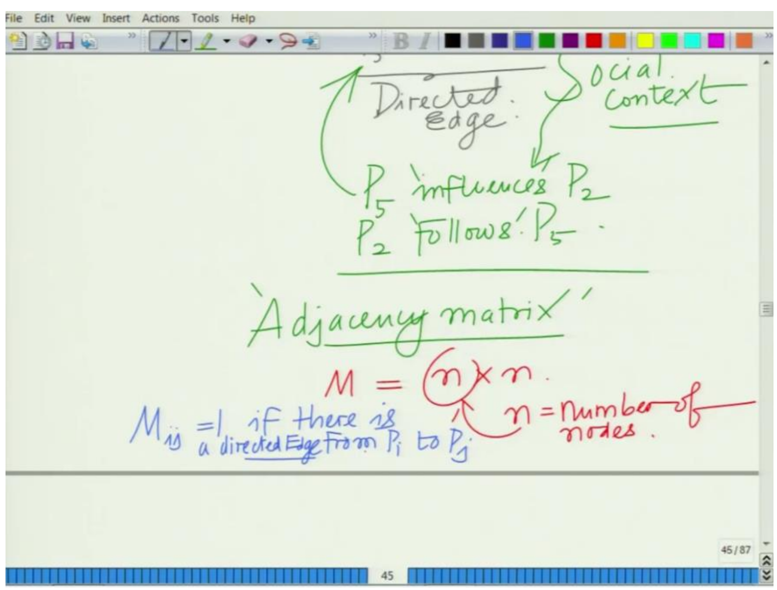

So, you say and now, to analyze this network one can build what is known as an Adjacency Matrix or a Vertex Matrix. This is the matrix **M** of size $n \times n$, where n is the number of nodes and M_{ij} is equal to 1 if there is a node from P_i to P_j, M_{ij} is equal to 1 if there is a node from P_i to P_j, if there is a directed edge. So, if there is a directed edge from P_i to P_j or P_i influences P_j or P_j follows P_i, then we say P_{ij} is equal to 1. For instance, in this case P_5 influences P_2, so, P_{52} will be equal to, that is, M_{52} will be equal to 1.

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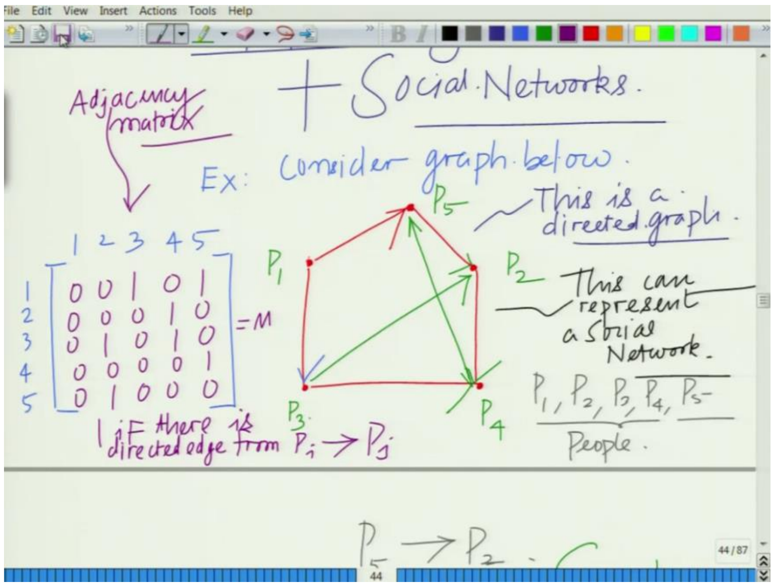

So, we can build this Adjacency Matrix. I will just go back to the figure for a moment to illustrate this adjacency matrix. So, we have the adjacency matrix for this. It can be built as follows, for instance, you have the directed graph, so, this corresponds to nodes 1, 2, 3, 4, 5 and1, 2, 3, 4, 5. So, you have the for instance from node 1, let me also mark this for node 1, you have directed edge from P_5 and P_3. So, these two will be 1, and the rest of the entries will be 0.

$$
\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}
$$

And this is basically what we are calling as the Adjacency matrix for this graph, which basically is 1 if there is a directed edge from the node P_i to P_j. So, this is basically the adjacency matrix for this graph.

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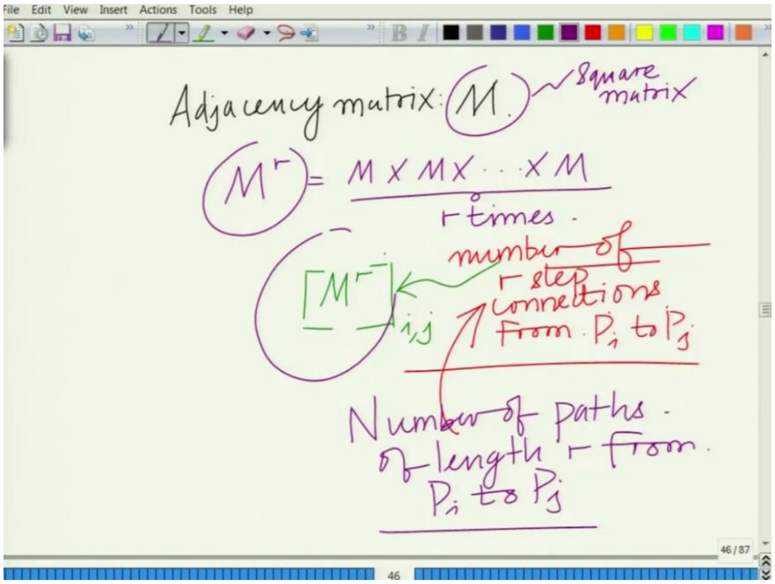

So, in this graph, now, we have the Adjacency matrix $\mathbf{M}$ and $\mathbf{M}^r$, that is, if you perform this product, $M \times M \times ... \times M$, r times, since it is a square matrix, you can multiply it with itself. Now, if you take this matrix $\mathbf{M}^r$, and look at the (i, j)th element of this. Now this is the number of r step connections from P_i to P_j in the graph.

So, basically this says or you can say r step connections are, this also you can easily show that this is the number of paths. This is the number of paths of length are from P_i to P_j. So, this is something that is interesting that you can get. So, if you ask for instance, what is the number of paths of length 2, what is the number of paths of length 3 these can be obtained from the, from the respective powers of the adjacency matrix.

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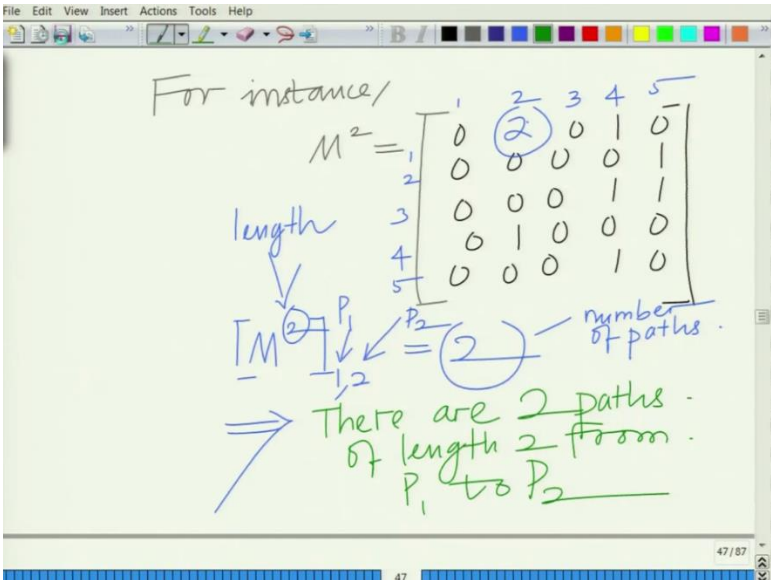

So, like for instance, let us take a simple example, for instance $\mathbf{M}^2$. Let us take a look at for instance, we have $\mathbf{M}^2$

$$\mathbf{M}^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

This is your $\mathbf{M}^2$ and we can see these are basically are nodes 1 2 3 4 5 and nodes 1 2 3 4 5. Now, look at this entry the $(1,2)th$ entry is 2. So, essentially if you look at this so, if you look at the $(1,2)th$ entry this is 2, so $[\mathbf{M}^2]_{1,2} = 2$.

This implies what does this imply? This implies that if you look at P_1 to P_2 there are two paths of length 2, this is basically your number of paths and this is basically the length number of r length connections. So, 2 paths of length 2 from P_1 to P_2 this is basically P_1, this is basically P_2 so, that is essentially the anatomy of this. And if you look at the graph, you can easily see that you have P_1 to P_5 to P_2, this is 1 path of length 2, and then you have P_1 to P_3 to P_2, that is another path of length 2. So, these are which can we easily see these are P_1 to P_5 to P_2, P_1 to P_3 to P_2.

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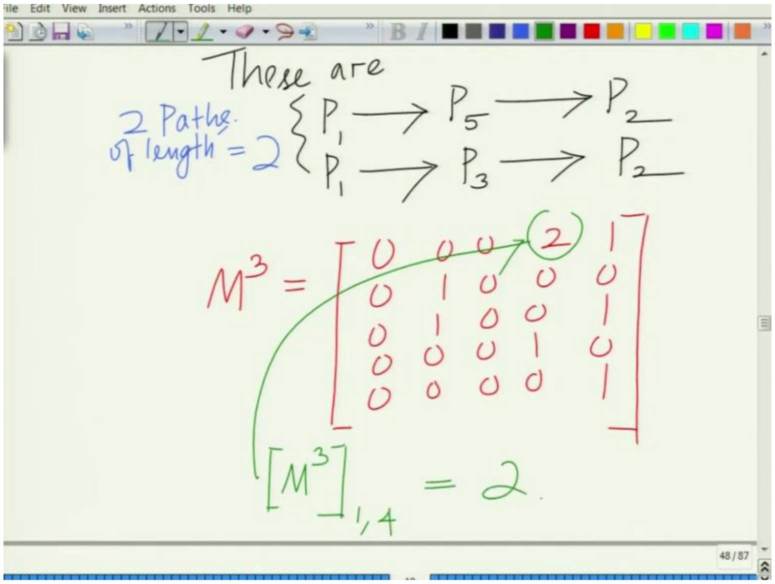
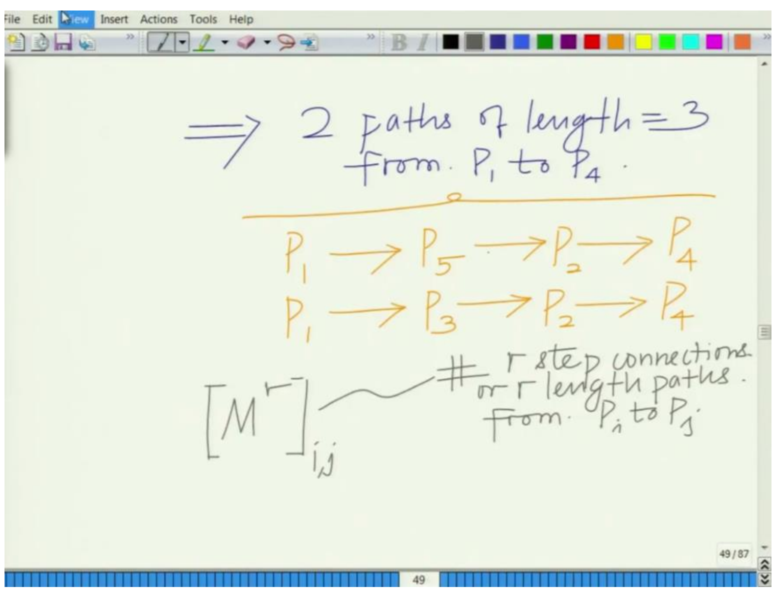


Now, $\mathbf{M}^3$ this is equal to $\mathbf{M}^3 = \begin{bmatrix} 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

So, this is $\mathbf{M}^3$ and if you look at the $(1,4)th$ element that is equal to 2. This essentially implies there are 2 paths of length 3. So, 2 paths of length 3 from P_1 to P_4. So, you have P_1 to P_5 to P_2 to P_4.



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And you have P_1 to P_3 to P_2 to P_4. So, there are 2 paths of length 3. So, these are essentially your 2 paths of length 3 from P_1 to P_4. And you can also find similarly, other r step connections from P_i to P_j by looking at the ijth entry of $[\mathbf{M}]_{ij}$ raised to the power of r. So, let me just repeat that once again, matrix $\mathbf{M}$ raised to the power of r length ijelement ij this gives number of r step connections or r length paths in the graph from P_i to P_j.

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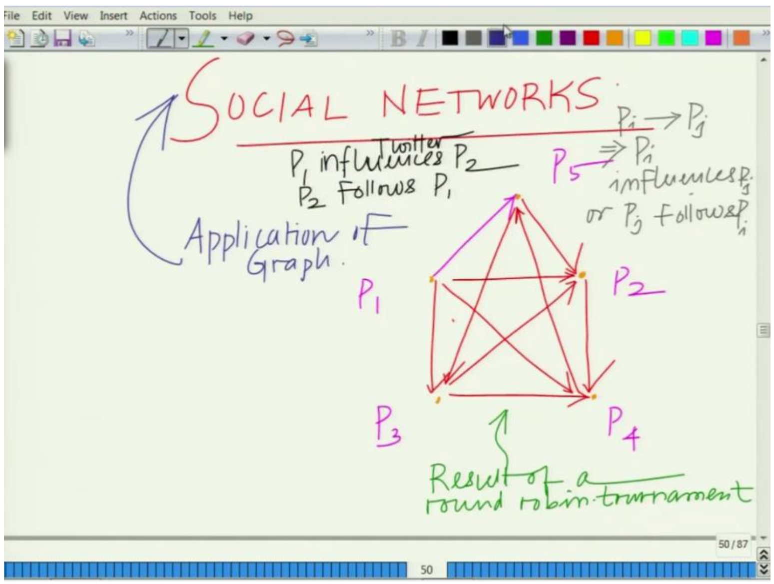

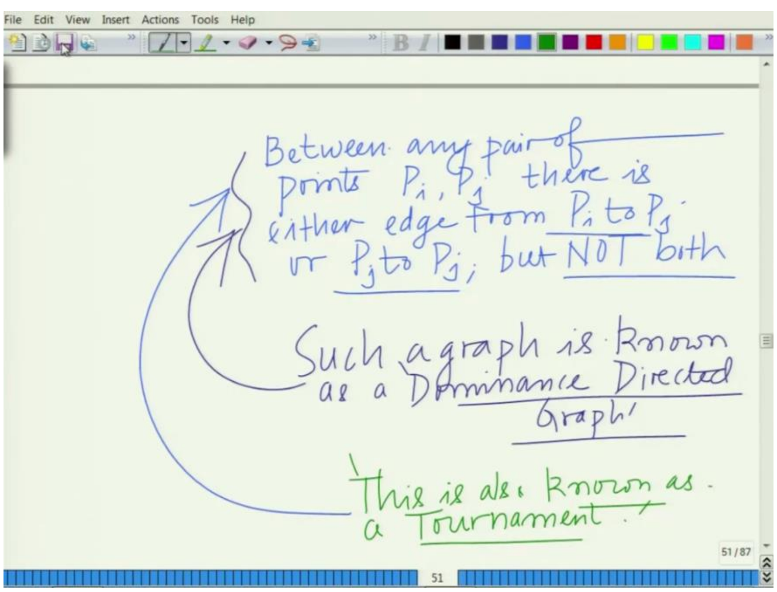

Now, let us look at another interesting application in the context of social networks, that is, what we call as the special kind of a graph, which I am going to show. Let us look at an application of graphs and therefore, linear algebra essentially and social networks. Now, let us consider a similar kind of graph but slightly modified. So, let us again look at our 5 points.

So, we have P_1, P_2, P_3, P_4, and the only difference here is that this is a fully connected graph. We are going to make a fully connected graph, that is, we are going to have some more edges in comparison to the previous one. So, now you can see this is very similar but slightly modified graph. Now, this is what is known as a Dominance Directed Graph. I will come to this, so in this graph if you look at any pair of points.

So, between any pair of points in the above graph, P_i, P_j, there is either a directed edge from P_i to P_j or P_j to P_i, but not both, that is, it is fully connected and there is a directed edge from either P_i to P_j or P_j to P_i, but not both, this is known as a dominance directed graph. So, this is the dominance directed graph and have another name for this, it is also known as a Tournament.

And remember, this is like a round robin. So, we can say that P_5, there is an arrow for P_5 to P_2, this implies that P_5 dominates P_2 through that. So, then there is the notion of dominance. So, you can also think of this as a round robin tournament, the result of a round robin tournament, because you have an edge between each set of players i and j.

And if there is an edge from P_i to P_j, then P_i dominates P_j and so on. And now, let us write, so, we have already seen this in the context of social networks, if there is an edge from P_i to P_j, this implies that P_i influences P_j or P_j follows P_i. For instance, here you can say P_2 influences P_4 or P_4 follows P_2 and for instance P_1 influences P_2 or P_2 follows P_1.

This can be an example, such as Twitter, where you have the different users who are connected. And now question is to find in this dominance directed graph, given this Twitter network or given this dominance directed graph we want to find the most influential node in this dominance directed graph.

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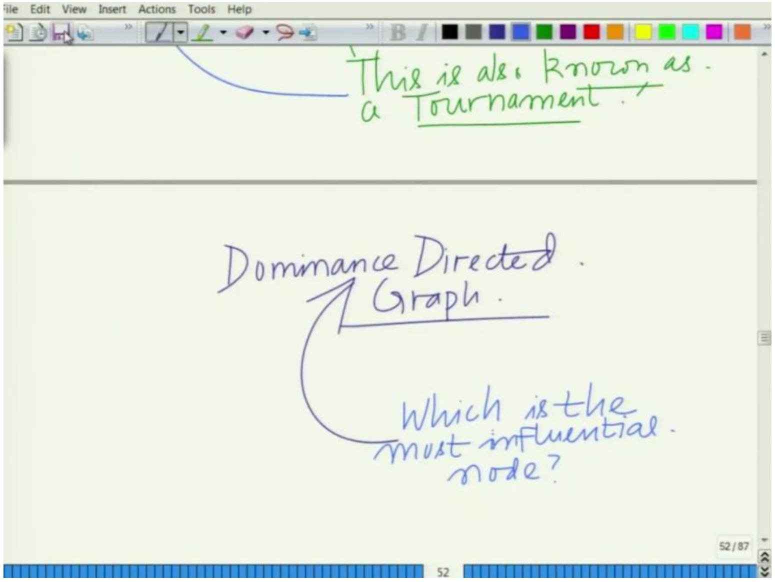

So, given this Dominance Directed Graph the question we want to ask is which is the most influential node or which is the most influential person? Which is the most influential node in this dominance directed graph?

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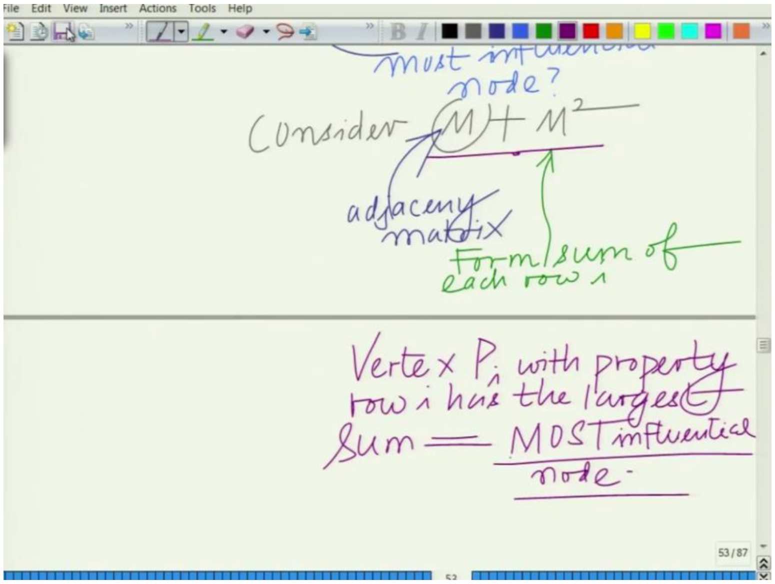

And to determine that, what we do is, it can be shown that you have to consider $\mathbf{M}+\mathbf{M}^2$where $\mathbf{M}$ is the adjacency matrix, we have the notion of the adjacency matrix, where the adjacency matrix $\mathbf{M}$ and $\mathbf{M}+\mathbf{M}^2$ remember $\mathbf{M}$ gives the number of 1 length connections, $\mathbf{M}^2$ gives the number of 2 length, length 2 connections. And the vertex P_i with the property that the row i has the largest sum is the most influential node. So, it can be shown that once you form $\mathbf{M}+\mathbf{M}^2$ you form the sum of each row now, the row i that has the maximum sum the corresponding vertex P_i is the most influential node.

So, basically you have this twitter network or you have this social network, in which or you have this tournament, in which you have these different players and each player has played with the other player and then one of the players has won, there are no draws. And now, after the results of this tournament, you want to find out which of these, which person is the dominant or influential player in this tournament. And similarly, once you have a Twitter network, where P_i influences P_j or P_j is basically following P_i and you have directed edges between each node i and j that is either $P_i$influences P_j or P_j follows P_i.

And now, you want to look at this network and you want to ask the question, which is the most influential person in the network? And the answer to that is you look at $\mathbf{M} + \mathbf{M}^2$ form the sum along each row, the ith row, the row that has the maximum sublets. So, the ith row has the maximum sublets the corresponding player P_i or the corresponding person P_i in the social network is the most influential person.

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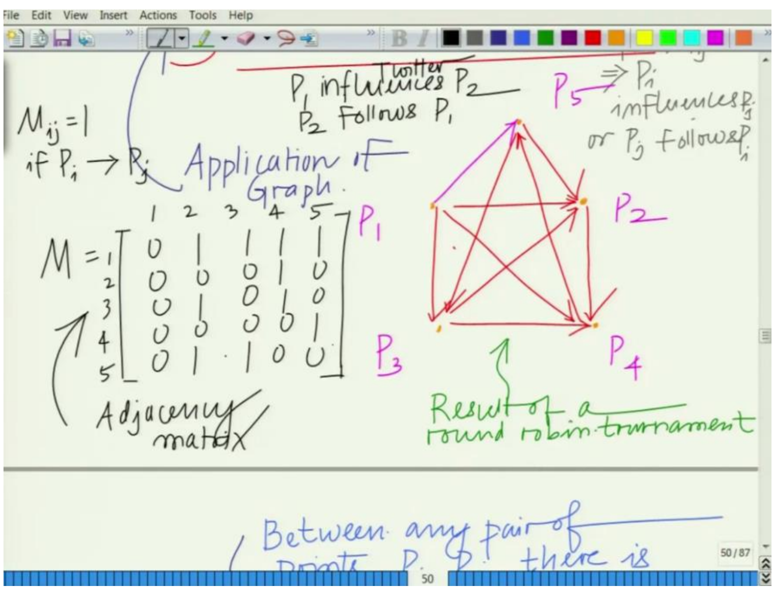

So, to find that, to determine that, let us write the adjacency matrix for this and this is not very difficult to write. What is the adjacency matrix for this? You will have $\mathbf{M}$ equals if you look at the adjacency matrix, node 1 influences 2, 3, 4, 5. So, it will have 1s in all these positions remember this is 1 2 3. Recall, what is our adjacency matrix, $[\boldsymbol{M}]_{ij}$ is equal to 1, if P_i influences P_j. That is there is a directed edge from P_i to P_j.

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

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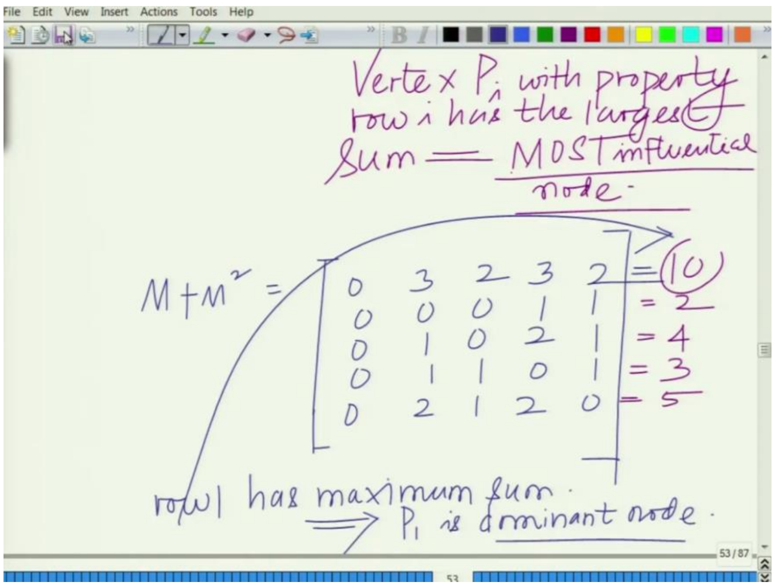

$\mathbf{M} + \mathbf{M}^2$ for this graph is

$$\mathbf{M} + \mathbf{M}^2 = \begin{bmatrix} 0 & 3 & 2 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 & 0 \end{bmatrix} \begin{matrix} = 10 \\ = 2 \\ = 4 \\ = 3 \\ = 5 \end{matrix}$$

And if you form the sum of each row. The first-row sum is equal to 10, correct? Second row sum is equal to 2, third row sum is equal to 4, fourth row sum equals 3, fifth row sum equals 5 and this is basically you can see clearly, row 1 has the maximum sum implies P_1 is the dominant node.

So, that is an interesting application of essentially graphs and linear algebra based on matrices constructed on graphs. We can say Graph theory as well as Social networks. We have looked at a dominance directed graph, you looked at the end, we have looked at the adjacency matrix of that and from the adjacency matrix $\mathbf{M}$, you compute $\mathbf{M} + \mathbf{M}^2$ and from that you can find which is the most influential node in this network, or which is the most influential person in this corresponding social network.

So, that is a very interesting practical application and Graph theory has a lot of applications in social networks and naturally linear algebra that is matrices based on graphs. And in fact, Matrix theory and Linear algebra of course, naturally therefore, implies that it has a lot of applications in analyzing social networks like this example. So, let us stop this module here. We will continue in the next module. Thank you very much.