Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
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Lecture No. 07
Application of Matrices: Solution of System of Linear Equations, MIMO Wireless Technology

Hello, welcome to another module in this massive open online course. So, we have seen the concept of a matrix inverse and how to evaluate the matrix inverse. Let us now start looking at applications of matrices. Now predominantly one of the main applications of a matrix is to formulate a system of linear equations and to solve the system of linear equations.

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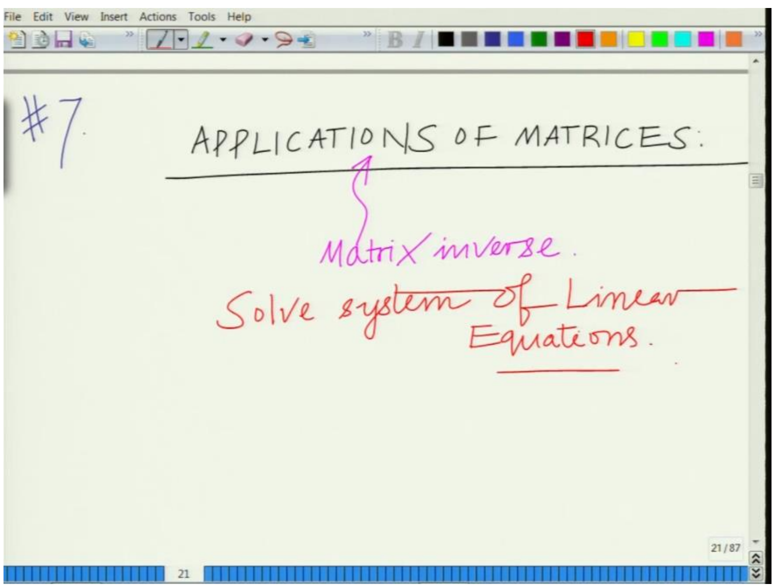

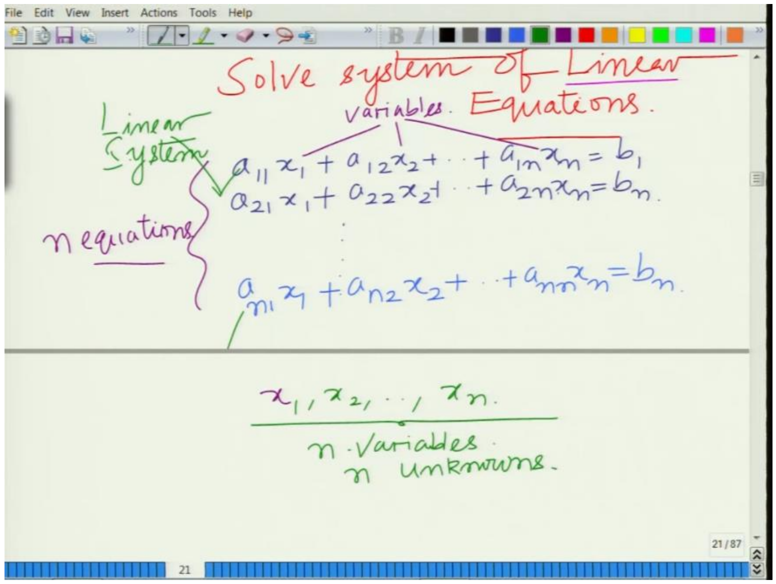

So, let us look at a broad class of applications of matrices where we use the matrix inverse. Specifically, where we want to use the concept of the matrix inverse that we have just learned. This is to solve a system of linear equations. I think that is important in the linear equations which is essentially you have the system of the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\dots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n.$$

So, you have the system of n equations and these $x_1, x_2, \dots, x_n$, these are the variables or basically the unknown. So, you have $x_1, x_2, \dots, x_n$, which are your n variables or n unknowns. So, you have a system of n equations and n unknowns and this is a linear system of equations because it is linear in $x_1, x_2, \dots, x_n$.

So, this essentially is your linear system of equations and why is it a linear system because it is linear in these variables $x_1, x_2, \dots, x_n$. It does not involve the powers x_1^2, x_1^3, it is simply depending on $x_1, x_2, \dots, x_n$. So, this is a linear system of equations because you have n equations because this is linear.

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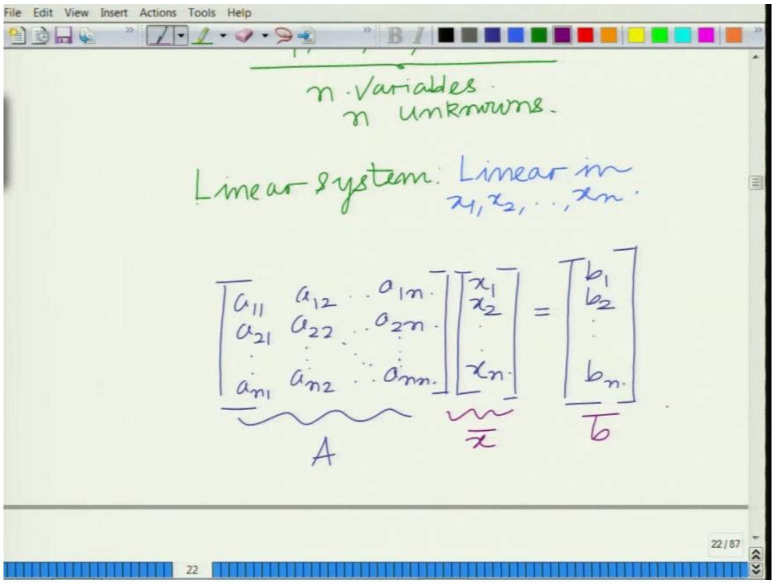

Naturally, you can write this as you can put all these things together as a matrix. You have our matrix

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}.$$

You can also equivalently write the above set of equations in a compact form as

$$\mathbf{A}\bar{\mathbf{x}} = \bar{\mathbf{b}}.$$

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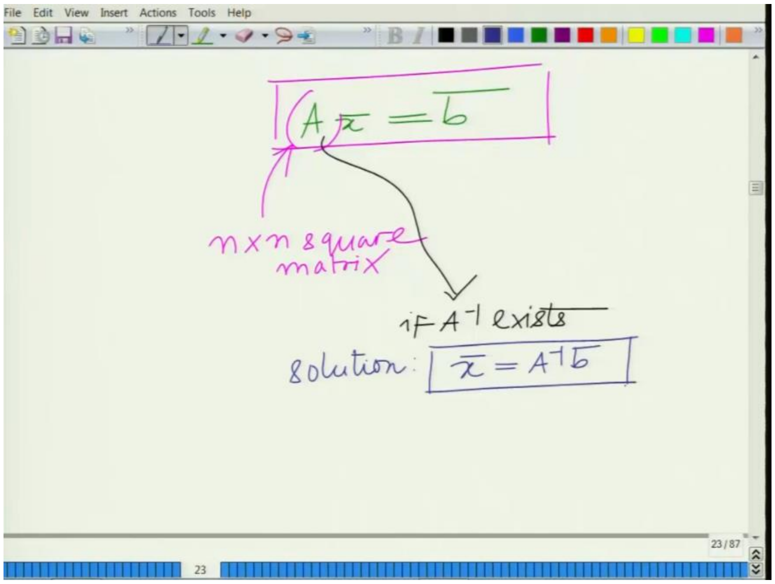

So, this is your compact representation of the linear system of equations, where $\mathbf{A}$ is your $n \times n$ square matrix, simplest formulation where you have a square matrix. Now what happens when the matrix is not square then the problem becomes more interesting.

So, in this simple example, in this the simplest form, we consider n equations and n unknowns and furthermore, if the matrix $\mathbf{A}$ is invertible, if $\mathbf{A}^{-1}$ exists, then the solution is solution is simply given as

$$\bar{\mathbf{x}} = \mathbf{A}^{-1}\bar{\mathbf{b}},$$

this is the solution to the above linear system of equations, of course, as we have said this is based on the assumption that the matrix $\mathbf{A}$ is invertible and $\mathbf{A}^{-1}$ exists.

Now, of course, what happens when $\mathbf{A}^{-1}$ does not exist or in fact $\mathbf{A}$ is not a square matrix that is you have more equations than unknowns or less equations than unknowns. Then it becomes very interesting and throughout this course we will encounter such scenarios. But this is to begin with, you can look at the simplest problem where you have n equations and n unknowns and the matrix $\mathbf{A}$ is invertible, then the solution is given by $\bar{\mathbf{x}} = \mathbf{A}^{-1}\bar{\mathbf{b}}$.

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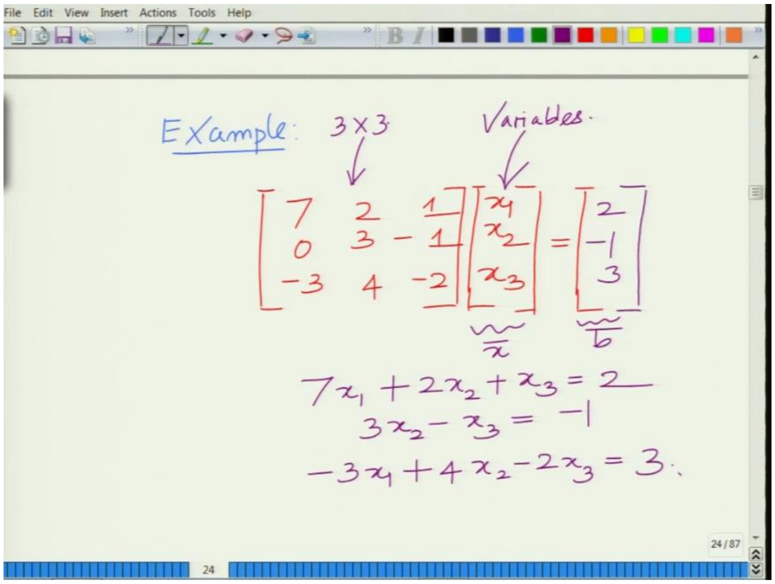

And let us take another example, a very simple example to understand this point. For instance, you have let us say the matrix

$$\mathbf{A} = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}, \bar{\mathbf{b}} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \text{and } \bar{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$7x_1 + 2x_2 + x_3 = 2$$

$$3x_2 - x_3 = -1$$

$$-3x_1 + 4x_2 - 2x_3 = 3$$

These are the equations and the solution is given as $\bar{\mathbf{x}} = \mathbf{A}^{-1}\bar{\mathbf{b}}$.

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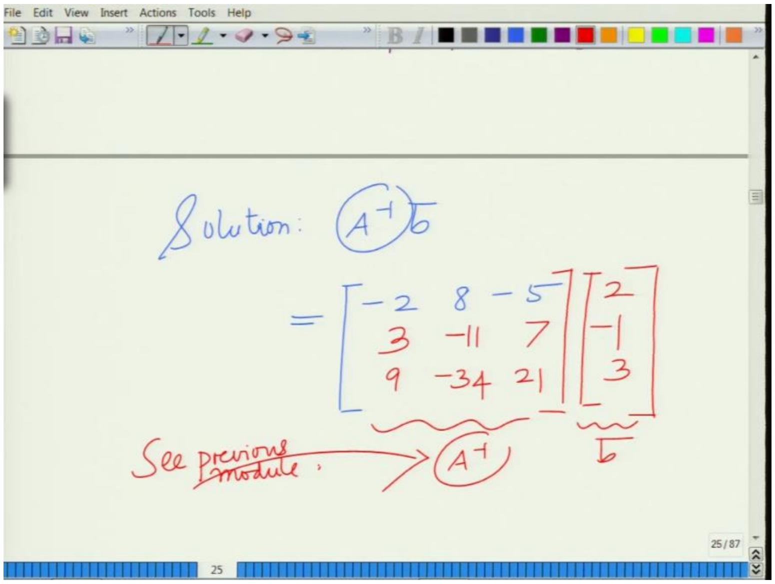

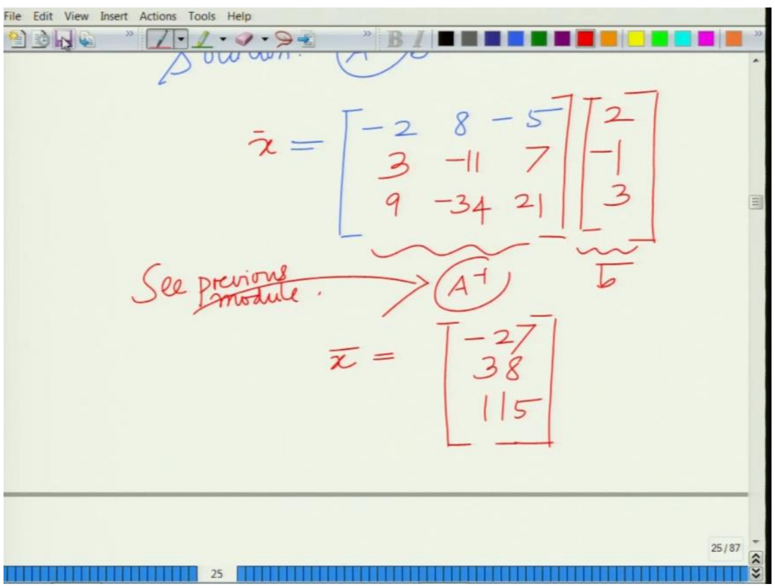

We have already calculated $\mathbf{A}^{-1}$ in the previous module, if you remember, we have already calculated $\mathbf{A}^{-1}$. And therefore, the solution is simply given as $\mathbf{A}^{-1}$ which is essentially your matrix

$$\mathbf{A}^{-1} = \begin{bmatrix} -2 & 8 & 5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix}.$$

You can look at the previous module to check how to evaluate $\mathbf{A}^{-1}$.Then you get

$$\bar{\mathbf{x}} = \begin{bmatrix} -2 & 8 & 5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -27 \\ 38 \\ 115 \end{bmatrix}$$

This is the solution of this problem of the system of this 3×3 system of linear equations. which is $\mathbf{A}\bar{\mathbf{x}} = \bar{\mathbf{b}}$, 3 equations, 3 unknowns and the matrix $\mathbf{A}$ is invertible. We have evaluated its inverse, the solution can be computed using $\mathbf{A}^{-1}\bar{\mathbf{b}}$ and that is what we have evaluated.

Now, let us move on to other interesting examples. Of course, this is an abstract example. Let us move on to other interesting practical real life applications of linear algebra, the theory of linear algebra and matrices. So, let us start looking at practical applications of this.

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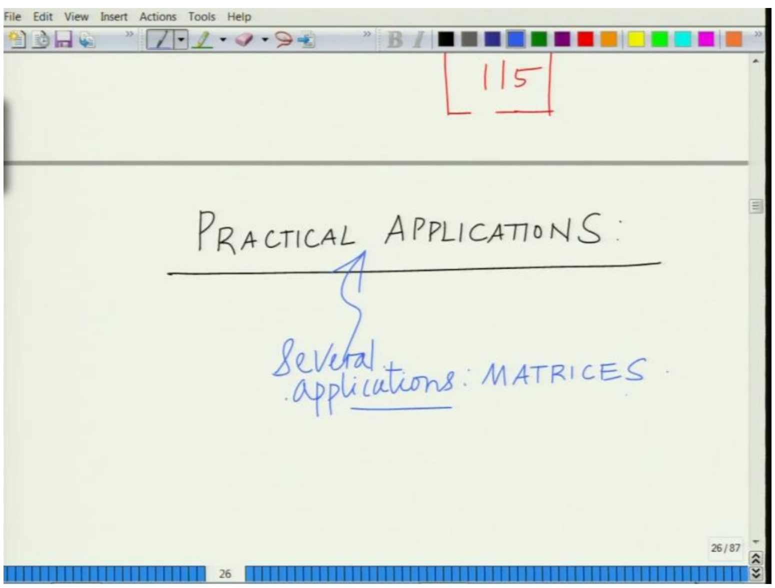

And remember, this course is applied linear algebra. So, we are not simply going to look at the abstract or the theoretical aspects of linear algebra, but we are going to integrate these concepts. I think that is the most important part of this course, that is not only look at the concepts. But alongside look at the application, so as to relate these concepts to the applications which will help you understand better and appreciate the concepts and more importantly, better understand how and where to apply these concepts that have been learned.

So, that is one of the important goals of this course that is to teach linear algebra, but not purely in a theoretical fashion, but from an application perspective. So, there are several interesting and in fact once one of the interesting things is probably, we have discussed at some point that linear algebra is probably one of the most important and one of the most extensively used theoretical tools.

Because linear modeling arises whenever you have linear systems and a large number of systems in science and engineering can be represented or modeled or analyzed using linear systems therefore the principles of linear systems, which is essentially based on linear algebra. So, linear algebra is used very very extensively, and its applications are numerous and we are only going to scratch the surface of that. I encourage all of you to look at further and explore many more such applications, I mean, even outside of this course.

So, there are several important applications. We have already noted several applications of linear algebra and in particular the concept of matrices and the inversion of matrices, that we have studied so far. So, I am just going to simply say that concept of matrices.

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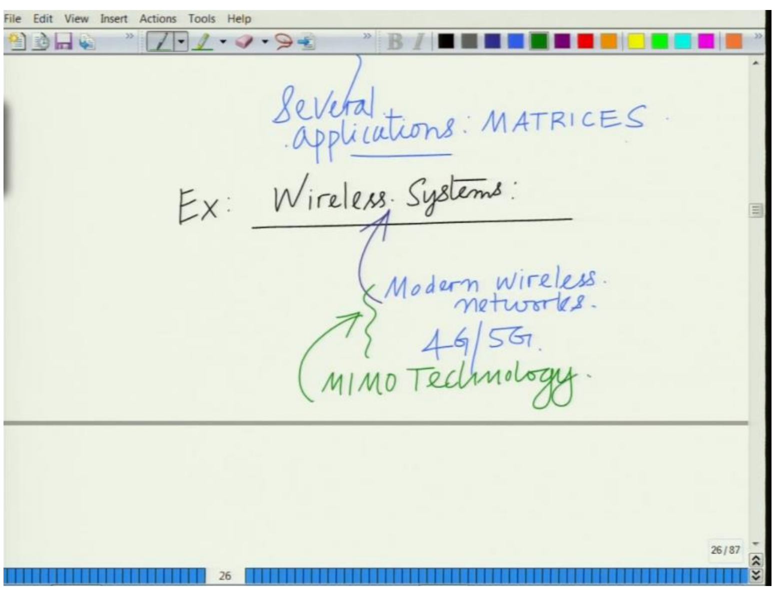

The first example that we are going to look at is basically in wireless communication or modern wireless system or wireless system design you can also say, where, for instance, you must all of you must be familiar that the most, the modern wireless systems are basically 4G. In modern wireless networks, you have the 4G and in the future you will have 5G and of course you will have the Wi-Fi and these are based on a very interesting technology. These modern wireless systems, these are based on what we call as the MIMO technology.

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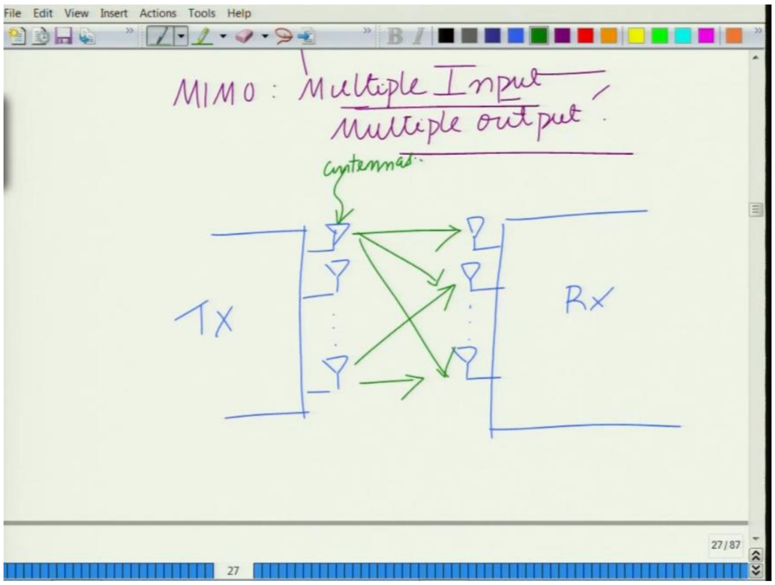

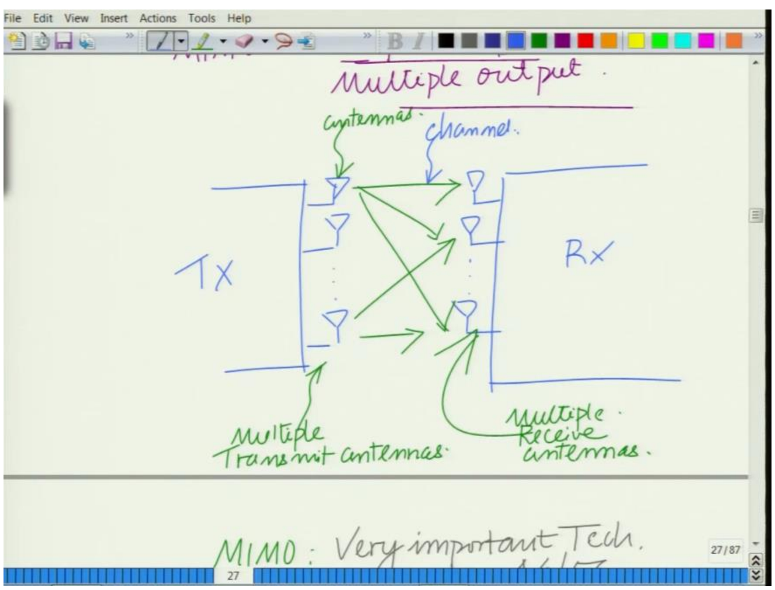

Now, what is MIMO, MIMO stands for multiple input multiple, and what does this mean? This is a very interesting technology, multiple input multiple output, what it means is that in a typical wireless system, you have the transmitter.

So, for instance let me draw this thing, you have the transmitter, you have the receiver and then you have the antennas installed on them, similar to what we have seen, each of these is an antenna and what MIMO means is that, you have multiple antennas at the transmitter from which you can transmit multiple transmit antennas and you also have the multiple receive antennas. So, you have the multiple transmit antennas from which you can transmit multiple symbols and you have multiple receive antennas from which you can receive multiple symbols.

So, you have multiple inputs, there will be multiple transmitted symbols and you have the multiple outputs, that is, you will have multiple received symbols. That is the reason this is known as; such a system is known as a MIMO system. This is a very advanced wireless technology, which has been used, as I have told you in 4G, 5G wireless system and that makes it possible to transmit at very high data rates in such systems, alright? So, MIMO is an important technology.

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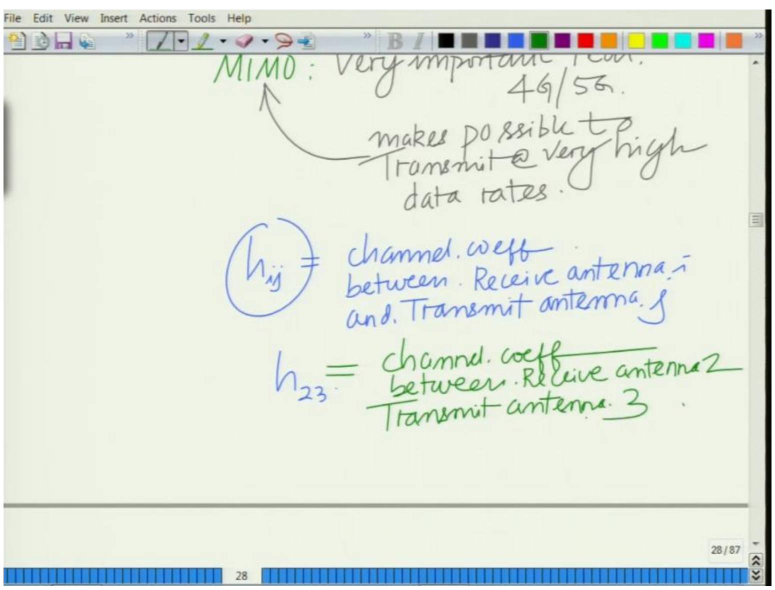

I would say very important technology for 4G/ 5G which makes possible to transmit at high data rates. Now, just to give you a brief background of this MIMO, so we have, for instance, h_{ij}. So, every wireless system is characterized by the channel coefficient h_{ij}, this is the channel coefficient between TX antenna j and the RX antenna i.

For instance, h_{23}, this is the channel coefficient between receive antenna 2 and transmit antenna 3. So, that is the concept of the channel coefficient h_{23}. And naturally, you can put all these channel coefficients together and you can make them into a matrix.

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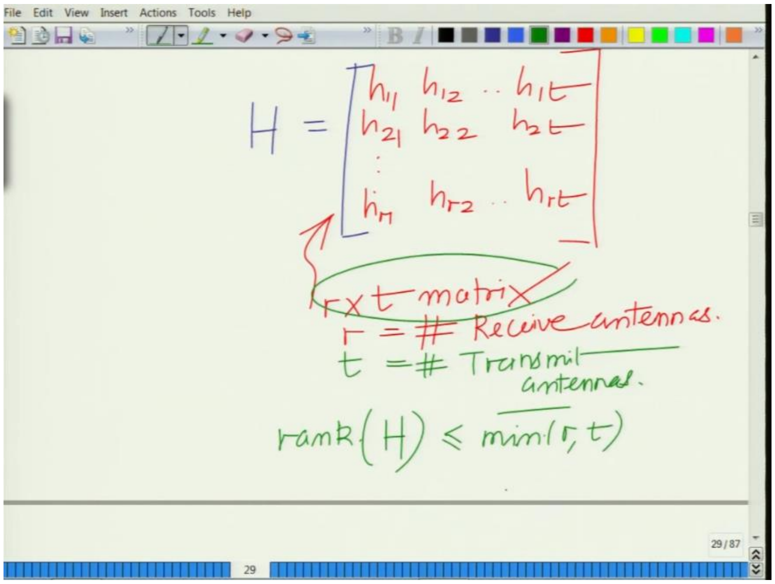

So, once you put all the channel coefficients together it becomes a matrix which is basically termed as the MIMO channel matrix. So, therefore, we will have the channel matrix **H** as

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1t} \\ h_{21} & h_{22} & \dots & h_{2t} \\ \vdots & \vdots & \vdots & \vdots \\ h_{r1} & h_{r2} & \dots & h_{rt} \end{bmatrix}.$$

This is the channel matrix **H.** As you can see, this is an $r \times t$ matrix where r equals number of receive antennas and t is equal to the number of transmit antennas, and this is very interesting.

Now, let us look at the rank of this MIMO channel, this has very interesting implication for communication. So, the rank of this, we know it is less than or equal min (r, t), i.e., number of receive and transmit antennas. And what can be shown is that in this MIMO system, the maximum number of symbols that can be transmitted at any time instant which basically has a bearing on the transmitted data rate that is the larger the number of symbols that can be transmitted the higher is the data rate that depends on the rank of this channel matrix.

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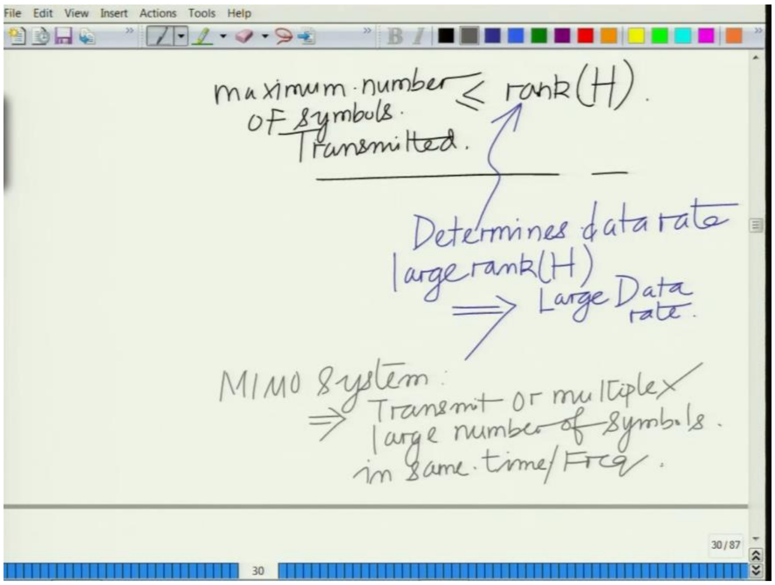

So, the maximum number of symbols that can be transmitted is equal to $rank(\mathbf{H})$, which means if your $rank(\mathbf{H})$ is large, you can transmit a large number of symbols at any time. So, the data rate is going to be very high and that is how it is possible to transmit at very high data rates in 4G and 5G system. So, rank plays a very important role. Rank of the channel matrix plays a very important role in determining the maximum data rate that is possible in a wireless communication system.

So, that is a very important, so, you can see linear algebra is everywhere, in particular, modern wireless communication to determine the data rate. For instance, what kind of data rate is it possible to transmit? For instance, by the base station to your cell phone. So, rank determines the data rate. So, larger the $rank(\mathbf{H})$ implies larger data rate.

And therefore, once you have the MIMO system you can transmit at a much larger number of symbols, that is, basically in the same time and frequency you can transmit a much larger number of symbols, that is, you can multiplex a larger number of symbols this property is termed as spatial multiplexing. So, MIMO system essentially implies transmit or multiplex large number of symbols in same time and frequency.

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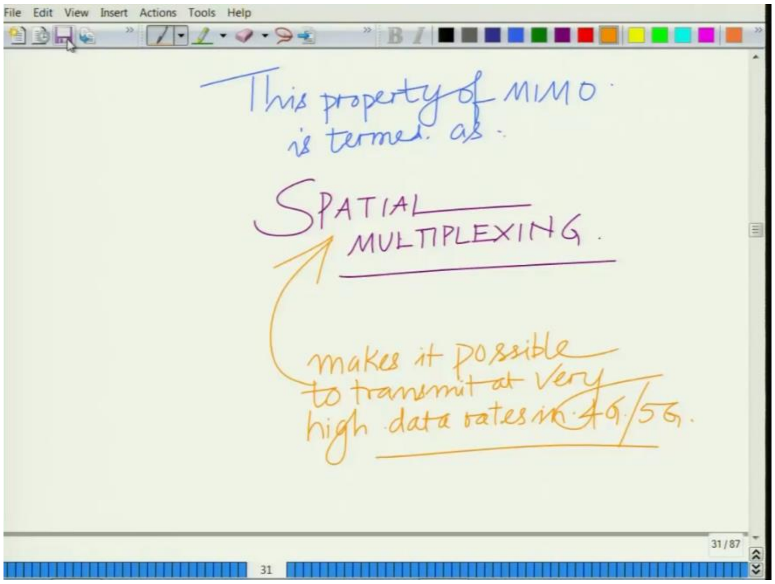

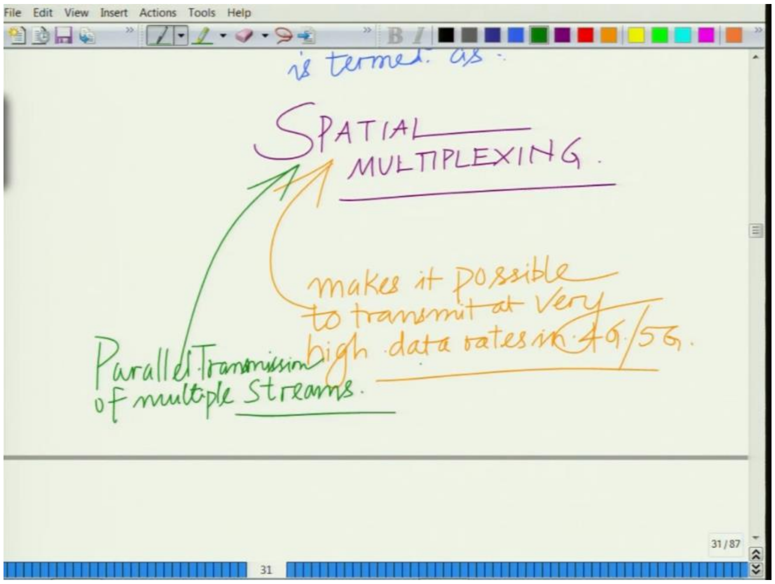

This property of MIMO is termed as spatial multiplexing. So, this is a very important term, and this is a very important property which makes it possible to transmit at very high data rates in 4G and 5G systems. So, via this property of spatial multiplexing in MIMO, it becomes possible to achieve very high data rates in 4G/5G wireless systems.

So, essentially what you are doing is? You are multiplexing multiple streams parallely through space. So, this is basically also termed as the parallel transmission of multiple data streams in same time and frequency. This is the parallel transmission of multiple streams. So, this is basically spatial multiplexing.

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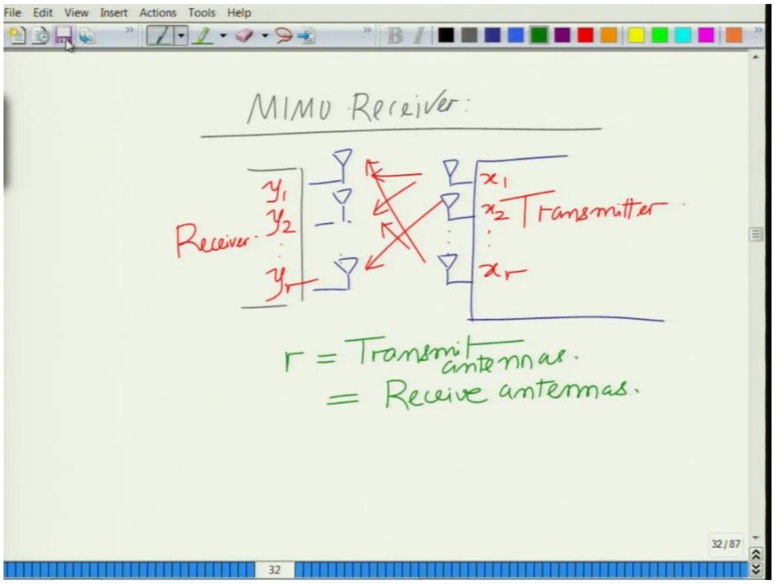

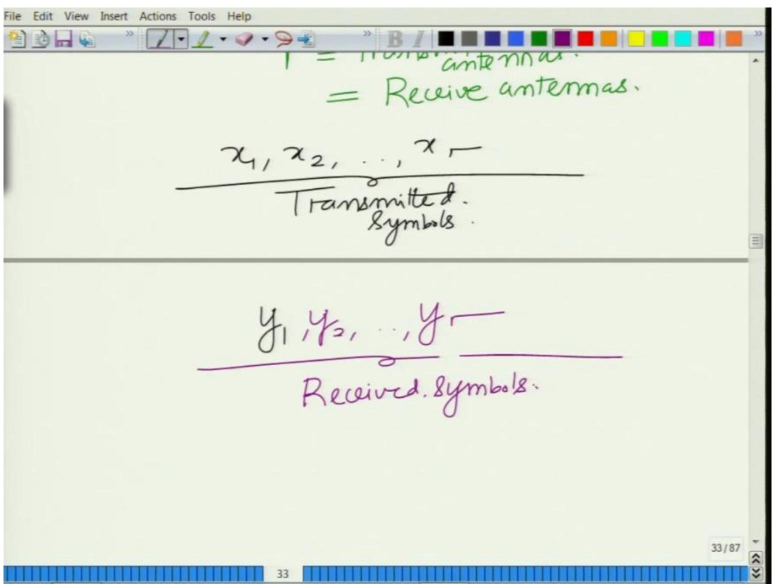

And now, let us look at how to design the receiver for a MIMO system? Very simple, let us look at a MIMO receiver. So, now, remember we have this multiple antenna system. Let us consider a special system with the number of receive antennas equal to number of transmit antennas equal to r and let us call these output symbols as $y_1, y_2, \ldots, y_r$. So, this is your receiver this is your transmitter and these are your transmit these are your transmitted symbols.

So, we are considering a special system in which r is equal to number of transmit antennas which in turn are equal to number of receive antennas. So, we are considering a special system we will see what happens if they are not equal. So, we are considering a system with number of transmit antennas r number of receive antennas r, so, it becomes a square channel matrix. So, these are basically the transmitters symbols $x_1, x_2, \dots, x_r$, these are basically your transmitter symbols and $y_1, y_2, \dots, y_r$ these are your received symbols.

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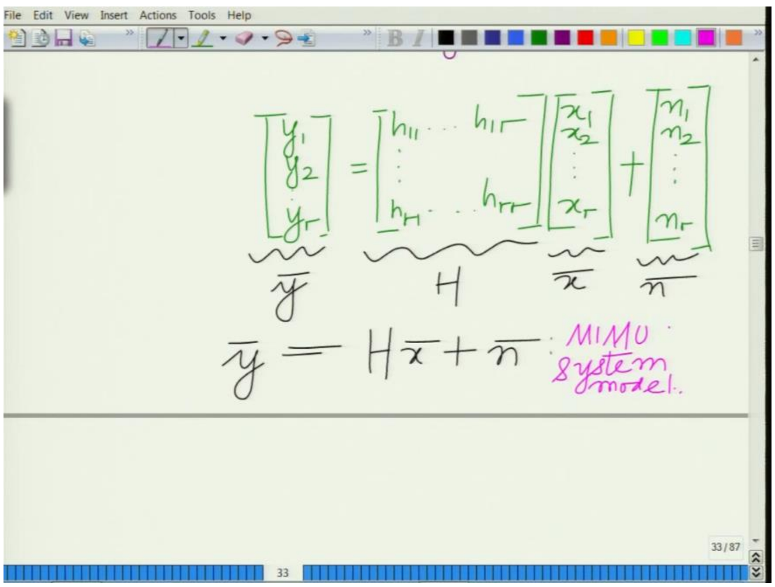

And these are related by the channel as follows, that is, the model for the MIMO system is given as

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1t} \\ h_{21} & h_{22} & \dots & h_{2t} \\ \vdots & \vdots & \vdots & \vdots \\ h_{r1} & h_{r2} & \dots & h_{rt} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_r \end{bmatrix}$$

$$\bar{\mathbf{y}} = \mathbf{H}\bar{\mathbf{x}} + \bar{\mathbf{n}}.$$

So, this is essentially what we call term as the MIMO system model. This is what we are going to use several times in this course.

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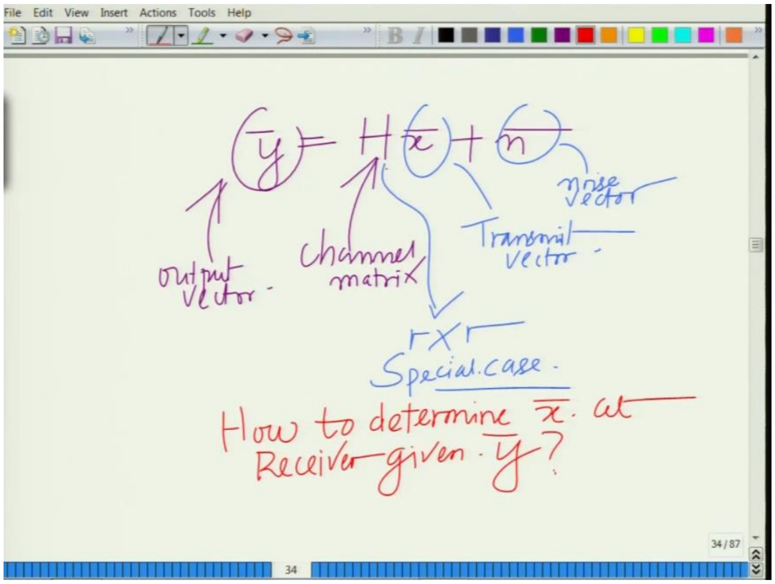

And now, the question is how to and remember we are considering a special case where this is an $r \times r$ matrix. So, remember this is only a special case, this is not a general. Now, the question is how to determine the transmit vector $\bar{\mathbf{x}}$. How to determine $\bar{\mathbf{x}}$ at receiver given $\bar{\mathbf{y}}$?

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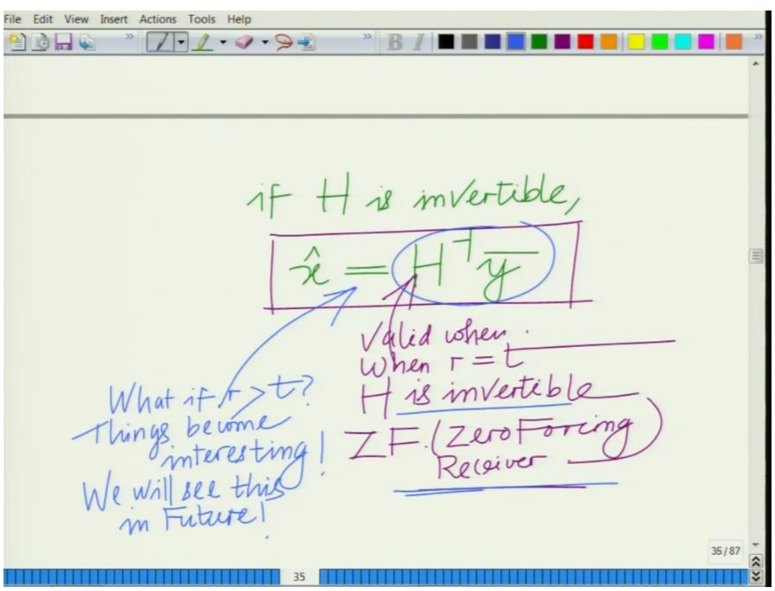

This is essentially for this case where $r = t = r$, if now you can see, if $\mathbf{H}$ is invertible then $\hat{\mathbf{x}}$, which is basically what we term as the estimate of the transmit vector at the receiver, that is the recovered transmit vector at the receiver based on the output symbols or the observed symbols $\bar{\mathbf{y}}$ that is given as simply $\hat{\mathbf{x}} = \mathbf{H}^{-1}\bar{\mathbf{y}}$. It is like solving a system of linear equations, but mind you, this is valid only if only when r equal to t and $\mathbf{H}$ is invertible.

This is, what is termed as the ZF or this is an interesting name. This is termed as a zero-forcing receiver. Or essentially it is equalizing the one you can also think of it as an equalizer, right? it is essentially, although it is not typically called as sometimes, also it is called as an equalizer. Typically, it is known as zero forcing receiver that is basically essentially, inverting the effect of the channel, but however, remember this is only valid when r equal to t and $\mathbf{H}$ is invertible.

Now, what happens when r is not equal to t, in particular, when the number of receive antennas r is greater than the number of transmit antennas t, i.e., $r \geq t$. That discussion is going to be very interesting, which we are going to see in the course, I mean, in this MOOC course. So, that discussion is going to be interesting.

So, what happens when r is not equal to t? So, let me also note that what if r greater than t then I would say things become interesting. We will see this in the future modules. So, this is the zero-forcing receiver and note that this particular application, that is, $\mathbf{H}^{-1}\bar{\mathbf{y}}$ this is valid when r equal to t and $\mathbf{H}$ is invertible.

So, essentially this is one of the, I mean this is these are very interesting applications of the linear algebra. So, essentially the idea is to convey to you the sense that why we are learning these different concepts, these are not general these are not just abstract theoretical constructs, but these are really really powerful concepts and techniques that have immense applicability in practice.

For instance, we have just talked about one such application in the context of MIMO technology. Multiple Input Multiple Output technology that employs multiple antennas for instance, at the base station and at your cell phone and naturally when you have MIMO so the channel becomes a matrix kind of channel. So, and therefore, linear algebra and the principles of matrices concepts related to matrices such as rank, the rows of the matrix, columns of the matrix and all these become very important.

As we have already seen, the rank has a very important role to play because that determines the maximum number of symbols that can be transmitted and larger rank implies larger number of symbols, which implies very high data rates. The matrix inverse that basically for an $r \times r$ system, can be used as a receiver to recover your transmitted symbol.

So, all these concepts have very, very powerful applications and we are going to see many such applications of these concepts in linear algebra as we go through this course. So, with this motivation, let me stop this module. We will keep discovering such applications in the subsequent modules. Thank you very much.