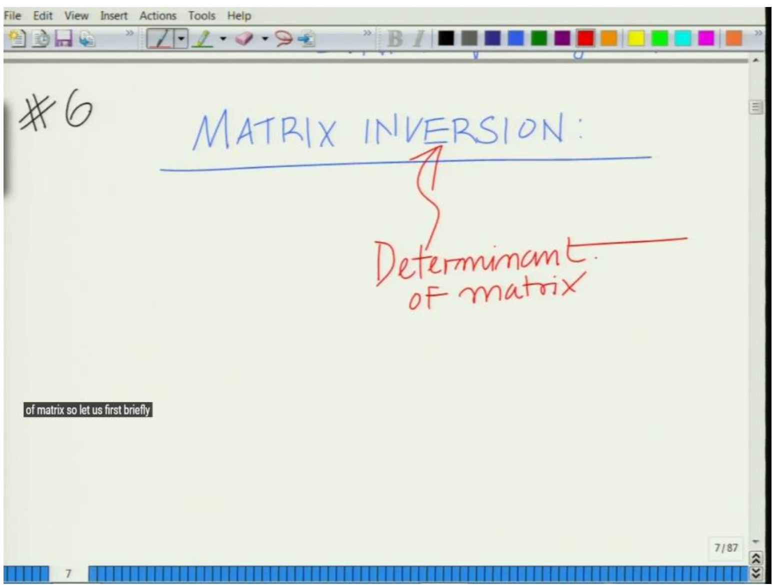
Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
Professor. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur
Lecture No. 06
Matrix: Determinant, Inverse Computation, Adjoint, Cofactor Concepts

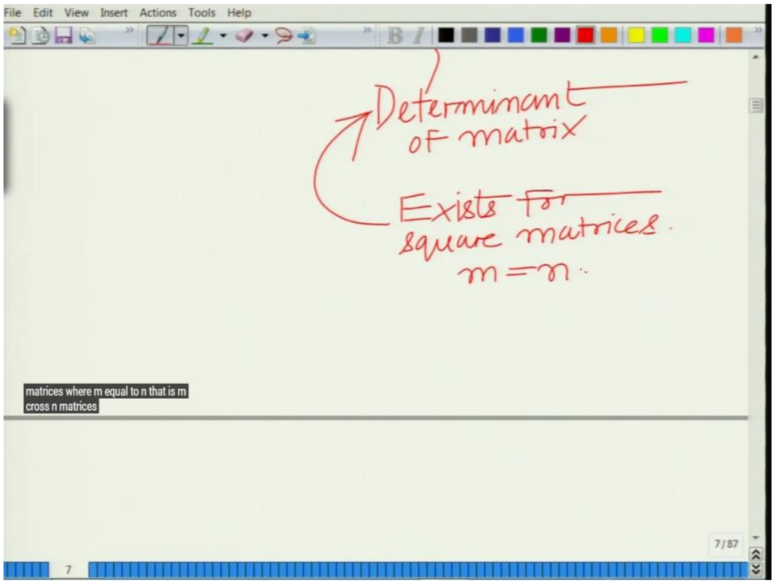
Hello, welcome to another module in this massive open online course. So, let us continue our discussion. And we are going to start discussing about the inverse of a matrix or matrix inversion, alright?

(Refer Slide Time: 00:31)



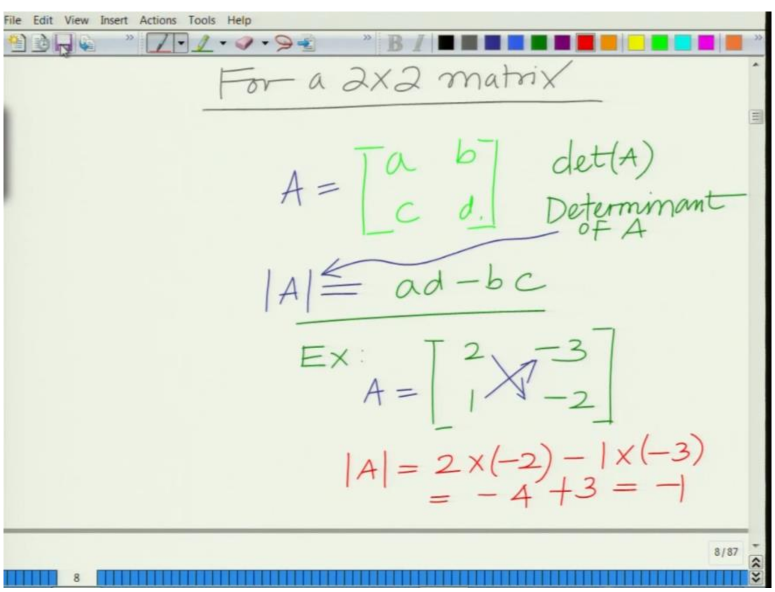
So, the next topic that we are going to cover is matrix inversion. And before we talk about the inverse of matrix, we need to talk about another concept, which is essentially the determinant of a matrix. So, let us first briefly talk about the determinant of a matrix.

(Refer Slide Time: 1:07)



Although many of you must know it, but just to refresh your memory, and for some of those who might have forgotten how to compute the determinant, or what the concept of a determinant is, the determinant can be calculated as follows. So, for a 2×2 matrix, of course, it is very simple the determinant of course, the determinant, I think, all of you should also be familiar that the determinant exists for only square matrices or is defined only for square matrices, that is, for an $m \times n$ matrix, m should have to be equal to n.

(Refer Slide Time: 1:54)



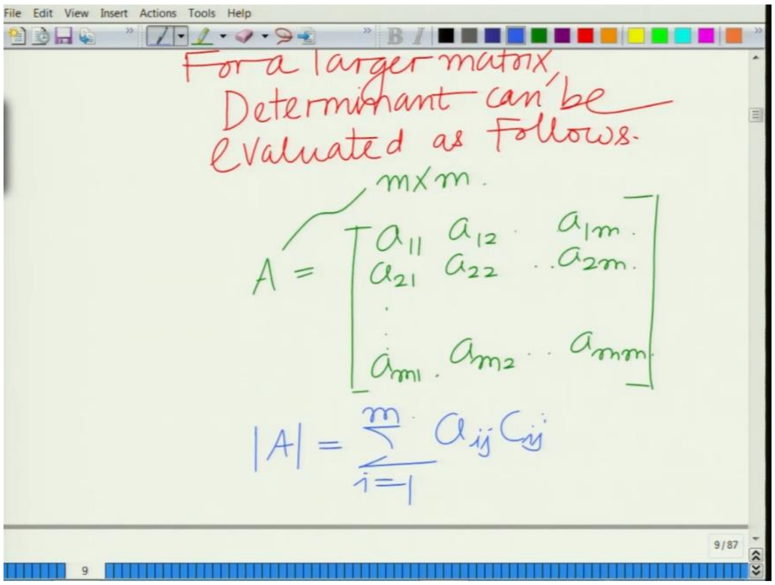
For a 2×2 matrix that is our canonical matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, which is a 2×2 matrix, the determinant of this matrix which is denoted by $|\mathbf{A}|$, this denotes the determinant of $\mathbf{A}$, that is mod symbol for real numbers. Sometimes it is also denoted by $\det(\mathbf{A})$, so on and so forth. So, this is simply your $|\mathbf{A}| = ad - bc$, this is the determinant of this 2×2 matrix.

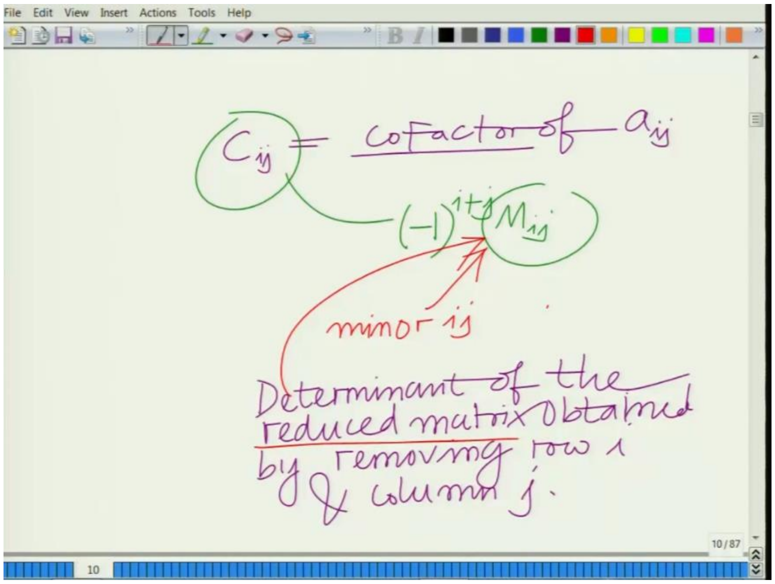
For example, let us take a simple example of a 2×2 matrix; $\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$. So, the determinant of this is

$$|\mathbf{A}| = 2 \times (-2) - 1 \times (-3) = -1.$$

So, the determinant of this simple 2×2 matrix is -1. Now, let us consider a general $n \times n$ matrix and see how the determinant is evaluated.

(Refer Slide Time: 04:24)





For a larger matrix the determinant can be evaluated as follows, that is, for instance, let us take a simple example. Again, let us look at an $m \times m$, so a square matrix. So, you have

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}.$$

Then the determinant of A equals

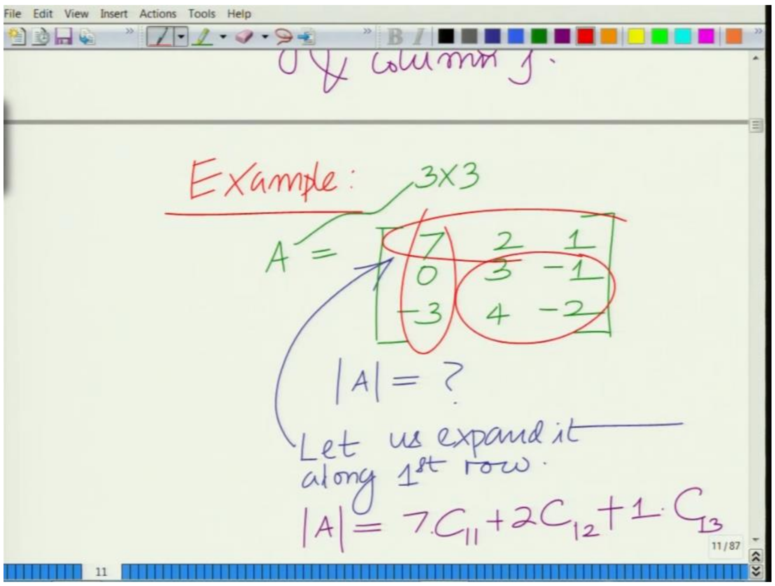
$$|\mathbf{A}| = \sum_{j=1}^{m} a_{ij} c_{ij},$$

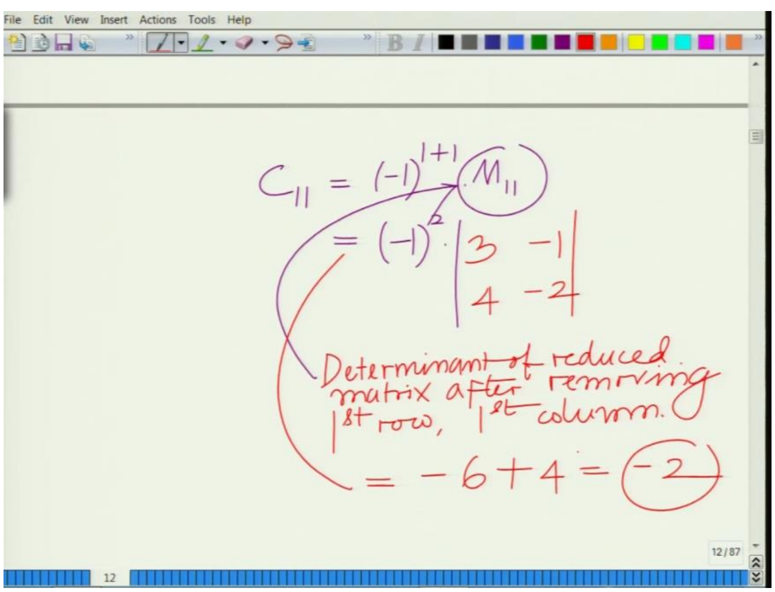
What is this concept c_{ij}, c_{ij} is the cofactor of the element a_{ij}. This is an interesting concept, I hope some of you might remember this is a cofactor of a_{ij}, what is the cofactor of a_{ij}? This is simply defined as $c_{ij} = (-1)^{i+j} M_{ij}$.

So, this M_{ij} is essentially the minor of **A,** this is the minor ij. I cannot say minor of a_{ij}, this is ijth minor, which is essentially nothing but the determinant of the reduced matrix obtained by removing row i and column j. So, this is the determinant of the reduced matrix that is essentially you remove the ith row and the jth column and look at the reduced matrix the determinant of that is essentially the minor.

So, the definition of a determinant, it works recursively, and hence the determinant of a 3×3 matrix depends on the 2×2 determinant the determinant of a 4×4 matrix depends on the determinants of the 3×3 essentially the minors, right? so on and so forth. So, the definition of the determinant works recursively, so, let us look at a simple example to understand that.

(Refer Slide Time: 08:53)





So, let us look at a simple 3×3 matrix that is and that will make it clear. For instance, let us say you have this 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}.$$

So, this is your 3×3 matrix. Now, we want to ask the question, what is the determinant of $\mathbf{A}$? Well, let us expand this along the first row. Let us expand it.

So, what are we going to have? We are going to have

$$|\mathbf{A}| = 7C_{11} + 2C_{12} + 1C_{13}.$$

Now, what is C_{11}? that is,

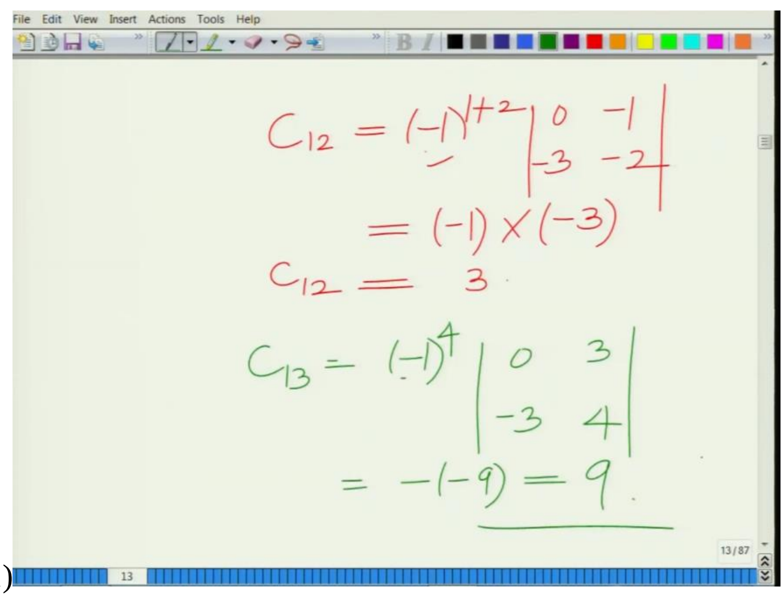
$$C_{11} = (-1)^{1+1}M_{11} = (-1)^2 \begin{vmatrix} 3 & -1 \\ 4 & -2 \end{vmatrix}$$

Now, remember M_{11} is the determinant of the matrix obtained by removing the first row and first column. So, we already know how to calculate the 2×2 determinants. So, this is equal to

$$C_{11} = 3 \times (-2) - (-1) \times 4 = -2.$$

This is the cofactor of the $(1,1)$ element, which is essentially -2, that is your C_{11}, this is equal to -2. Similarly, we can evaluate the other cofactors, that is, C_{12} and C_{13}.

(Refer Slide Time: 12:41)



So, now what is C_{12}? you can easily see C_{12} is

$$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 \begin{vmatrix} 0 & -1 \\ -3 & -2 \end{vmatrix}$$

$$= (-1) \times \big(0 \times (-2) - (-1) \times (-3)\big) = 3.$$

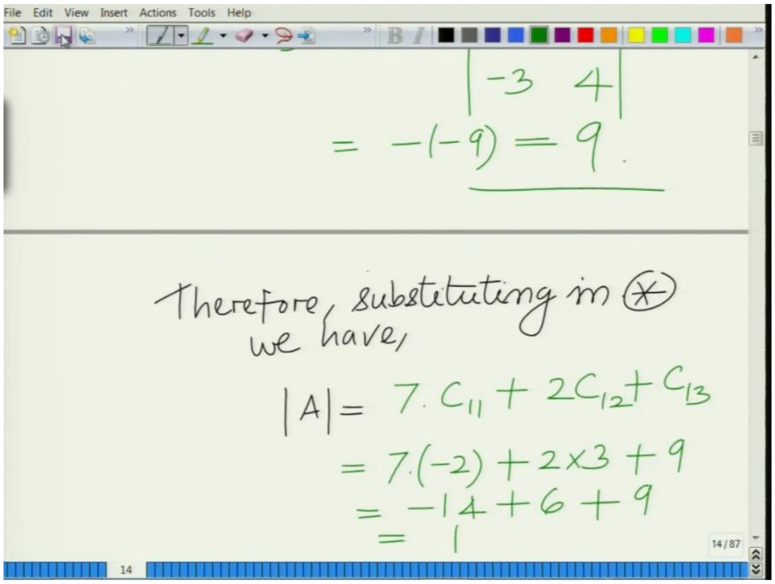
So, C_{12} is equal to 3. And now what about C_{13}? C_{13} is

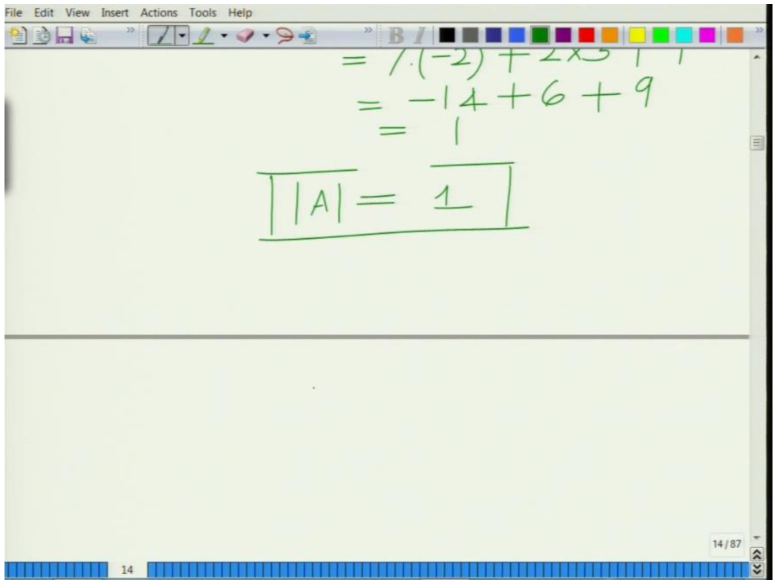
$$C_{13} = (-1)^{1+3} M_{13} = (-1)^4 \begin{vmatrix} 0 & 3 \\ -3 & 4 \end{vmatrix}$$

$$= (1) \times \big(0 \times 4 - 3 \times (-3)\big) = 9.$$

So, that is essentially what it is. So, you have C_{11} equals -2, C_{12} equals 3, and C_{13} equals 9.

(Refer Slide Time: 14:34)



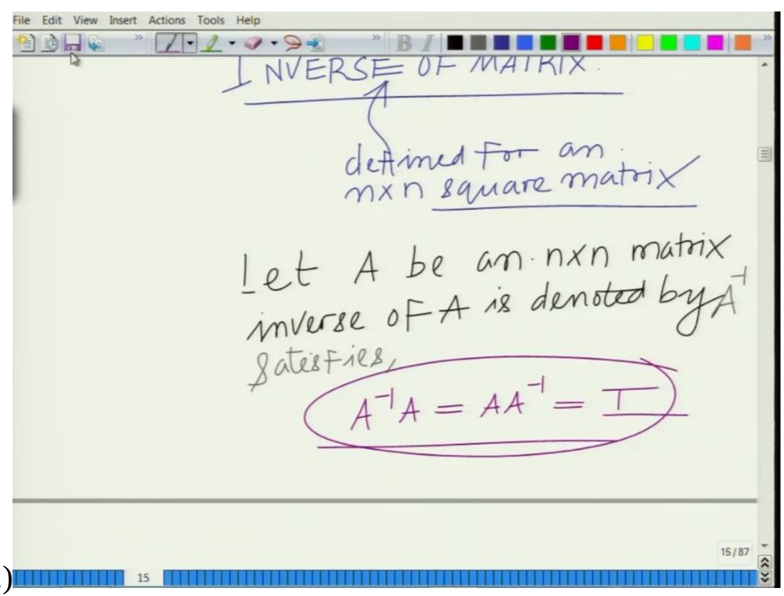


Therefore, now we can evaluate, now substitute in this, right? remember we let us call this equation as star. So, substituting in this equation

$$|\mathbf{A}| = 7C_{11} + 2C_{12} + 1C_{13} = 7 \times (-2) + 2 \times 3 + 1 \times 9 = 1$$

So, essentially, the $|\mathbf{A}|$ this is essentially equal to 1. So, for this simple example, we are able to evaluate the determinant using this concept of cofactors and minors and so on. Now, we are ready to talk about the inverse of a matrix remember this is what we started with the inversion of a matrix.

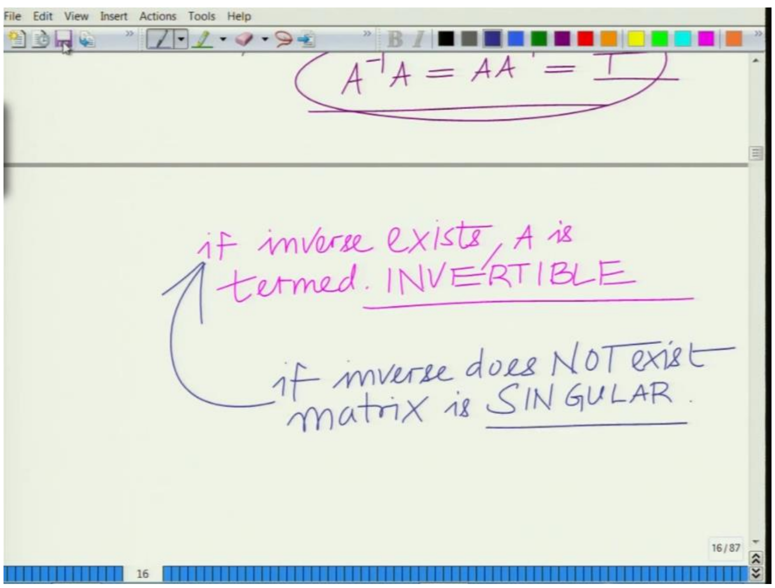
(Refer Slide Time: 16:12)

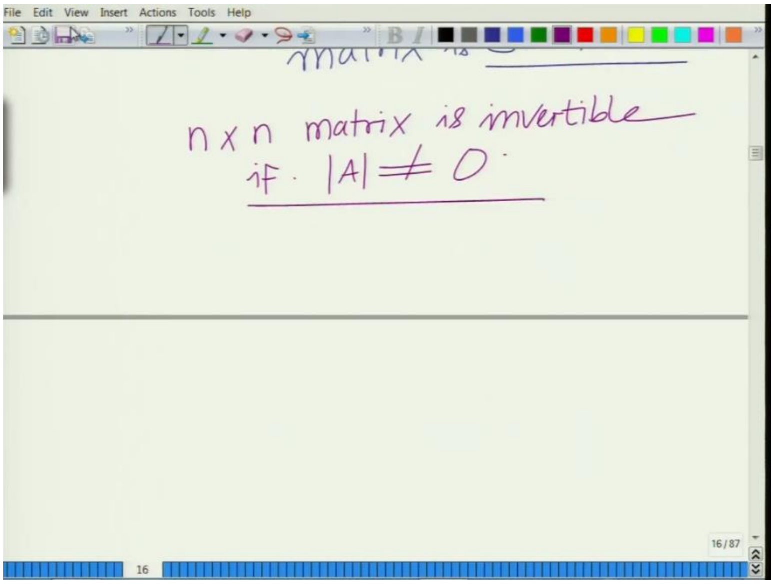


Now, how do we calculate? Now, we are ready to talk about this notion of the inverse of a matrix. So, once again this is defined for a square matrix this is defined for an $n \times n$ square matrix. So, let $\mathbf{A}$ be an $n \times n$ matrix. So, now, the inverse of $\mathbf{A}$ is denoted by $\mathbf{A}^{-1}$ or let me write it over here inverse of $\mathbf{A}$ denoted by $\mathbf{A}^{-1}$ satisfies, that is, if it exists it must satisfy the property $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Now, the other important thing to realize here is that $\mathbf{A}^{-1}$ is not guaranteed to exist for any square matrix. So, we are going to see that so, if it exists it must set satisfies the property $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, that is if you multiply it on the left of $\mathbf{A}$ or right of $\mathbf{A}$ or you multiply $\mathbf{A}$ with $\mathbf{A}^{-1}$ on the left or if you multiply $\mathbf{A}$ with $\mathbf{A}^{-1}$ on the right it gives you identity This is the property of the matrix inverse.

(Refer Slide Time: 18:25)

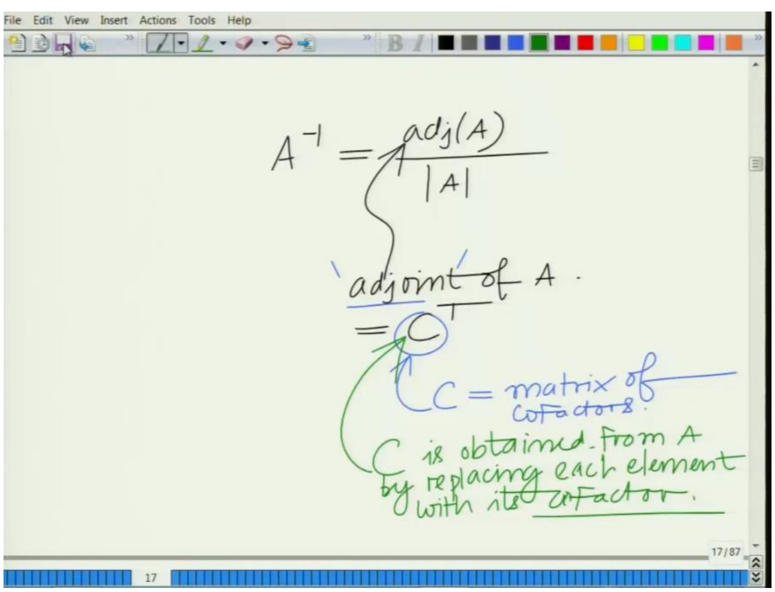




Now, it is not guaranteed that inverse exists for all matrices **A.** If inverse exists **A** is termed invertible. Now, if inverse does not exist then matrix is singular and this is an important.

Now, what is the condition for inverse to exist? It is very simple an $n \times n$ matrix is invertible if the determinant is not 0, that is, if the determinant of the matrix is not 0, then the matrix is invertible, if the determinant is 0, then the matrix is singular as simple as that if the determinant is 0, then the matrix is singular.

(Refer Slide Time: 20:36)



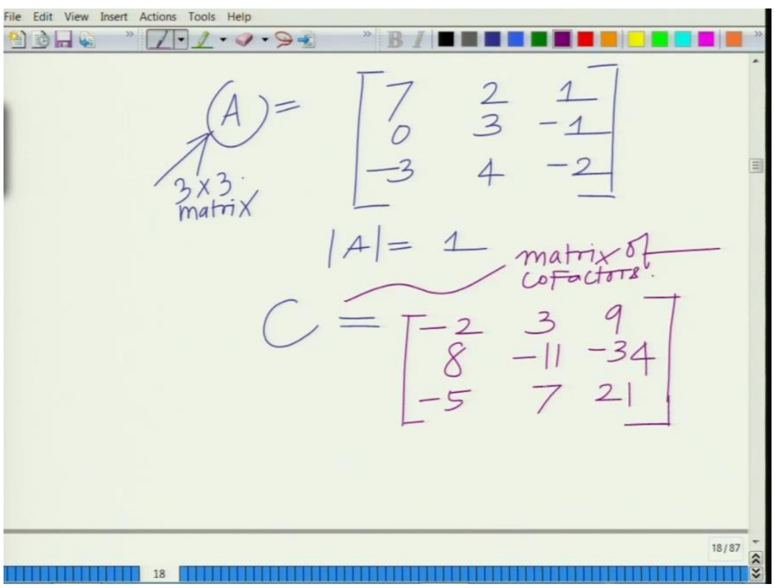
Now, let us look at the formula, how to evaluate the inverse if the inverse exists. $\mathbf{A}^{-1}$ is given as the

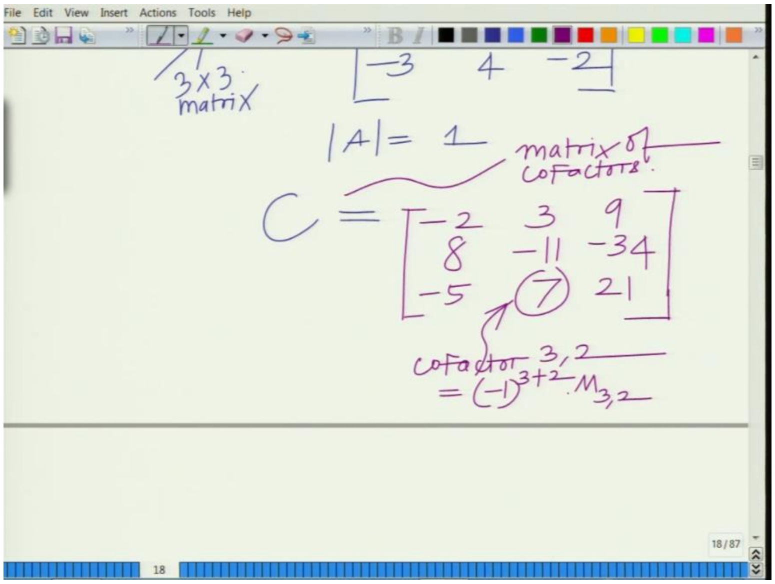
$$\mathbf{A}^{-1} = \frac{\mathrm{adj}(\mathbf{A})}{|\mathbf{A}|},$$

the adjoint of $\mathbf{A}$ denoted by $\mathrm{adj}(\mathbf{A})$ is equal to $\mathbf{C}^T$ where $\mathbf{C}$ is the matrix of cofactors, that is, you take the cofactor matrix replace it by its transpose to get the adjoint of matrix $\mathbf{A}$.

So, take each element of $\mathbf{A}$ and replace each element of $\mathbf{A}$ by its cofactor. So, $\mathbf{C}$ is obtained from $\mathbf{A}$. So, you take each element a_{ij} replace it by its cofactor c_{ij} and then we take the transpose of that, it gives the adjoint. So, adjoint is the transpose of the matrix of cofactors. Finally, the inverse is $\mathrm{adj}(\mathbf{A})$ divided by the determinant $|\mathbf{A}|$ and remember you can do this division because the determinant is nonzero. Therefore, the inverse exists only if the determinant is nonzero.

(Refer Slide Time: 22:42)





Let us take a simple example again simple example what we already seen. Let us go back to our example matrix

$$\mathbf{A} = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}.$$

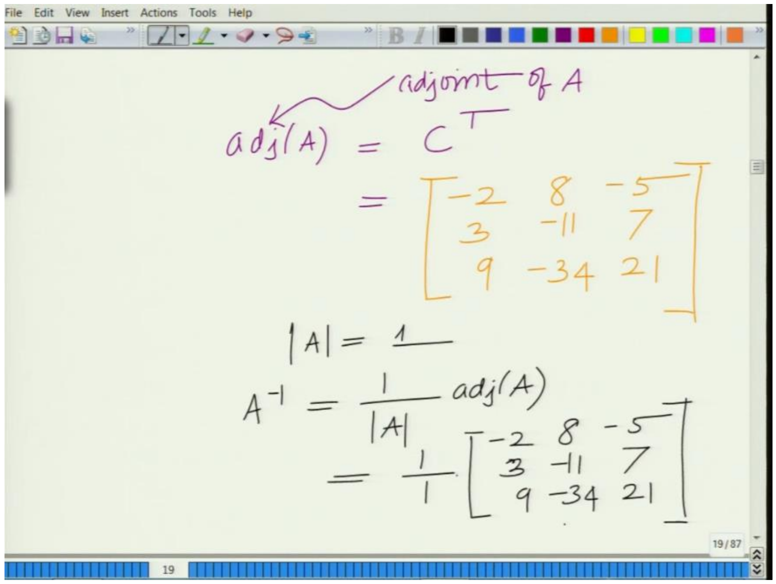
Take a look at this, if you take a look at this, we already seen this is our $3 \times\ 3$ matrix and its determinant is equal to 1. Now, let us look at the adjoint of this matrix **A**, that is, the transpose of the matrix of cofactors.

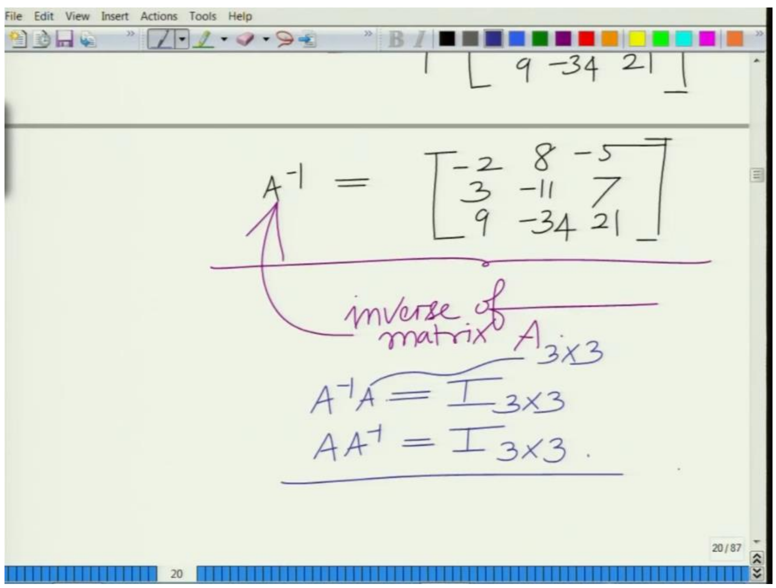
Now, what is **C,** this is the matrix of cofactors. This is given as

$$\mathbf{C} = \begin{bmatrix} -2 & 3 & 9 \\ 8 & -11 & -34 \\ -5 & 7 & 21 \end{bmatrix}.$$

So, these are the cofactors. So, we are replacing each element of **A** by its cofactor for instance 7. This is the cofactor $(3,2)$, which remember cofactor $c_{ij} = (-1)^{i+j} M_{ij}$. That is you remove the third row and second column and calculate the determinant of the residual matrix, hopefully that is okay.

(Refer Slide Time: 24:51)





Now you take the transpose of this. So, $\text{adj}(\mathbf{A}) = \mathbf{C}^T$, which is essentially now if you look at it, simply take the transpose of $\mathbf{C}$ where rows become column and columns become rows. So, this will be

$$\text{adj}(\mathbf{A}) = \begin{bmatrix} -2 & 8 & 5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix}$$

And now we have already seen that the determinant of $\mathbf{A}$ is one. So, in this case, $\text{adj}(\mathbf{A})$ itself becomes the inverse or in other words, you have

$$\mathbf{A}^{-1} = \frac{\text{adj}(\mathbf{A})}{|\mathbf{A}|} = \text{adj}(\mathbf{A}).$$

And, therefore, the inverse is also essentially the same as the adjoint matrix.

So, this is basically your $\mathbf{A}^{-1}$. And you can check this by doing for instance $\mathbf{A}^{-1}\mathbf{A}$ and this should be equal to identity and in this case 3×3 identity matrix because we have $\mathbf{A}$ of size 3×3 and $\mathbf{A}\mathbf{A}^{-1}$ should also equal to the 3×3 identity matrix, this is an important concept. So, you can verify this property is satisfied here. and later we are going to see a very interesting concept, another very interesting concept, which is known as the pseudo inverse.

So, please do not confuse the inverse and the pseudo inverse, the pseudo inverse is different from the inverse. Inverse exists only for a square matrix, if the matrix is invertible. Pseudo inverse is a different concept which is also very useful. But please do not confuse it with the inverse. Now relation between pseudo inverse and inverse is if the inverse exists, then the pseudo inverse is naturally also the inverse, I mean, the inverse is also the pseudo inverse. But the pseudo inverse also exists when the inverse does not exist. Anyway, that is a concept which we will look later in detail.

But the property of the inverse is that essentially, when you multiply it by $\mathbf{A}^{-1}$ on the left, that is if you multiply $\mathbf{A}^{-1}$ into $\mathbf{A}$ that is also identity. And if you multiply $\mathbf{A}$ times $\mathbf{A}^{-1}$, that also equals identity. So, I think some of these things are very basic. Some of you might already be familiar with these some of you might not be so familiar with these or some of you might be familiar but might have forgotten this thing. So, the concept of a matrix in inverse is very important in linear algebra, because this is something that we are going to use very frequently going forward especially in applications, for instance, this arises in the solution of linear equations and so on. And the solution of linear equations is something that is very fundamental. So, this is something that is very important and I think one has to have a good understanding of the matrix inverse, what are its properties and how to evaluate it. So, this basically completes our discussion on the matrix inverse. We are going to look at the applications of the matrices and the matrix inverse in the subsequent modules. Thank you very much.