Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
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Lecture 58
Discrete Time Markov Chain Examples

Hello, welcome to another module in this massive open online course. We are looking at discrete time Markov chains and the analysis of discrete time Markov chains, essentially characterizing their properties using the one step transition probabilities, build the transition probability matrix and also looked at a couple of examples. Now, let us continue our discussion further. We were looking at for instance the industrial reliability or the machine reliability.

Example, where we have an operating machine and that can be two states that is operational or faulty and it can go, can transition from one state to the other. So, you look at the state at the beginning of each hour. So, it is either operational or faulty, by the next hour it can either continue to remain operational or faulty or can transition from operational to faulty or faulty to operational.

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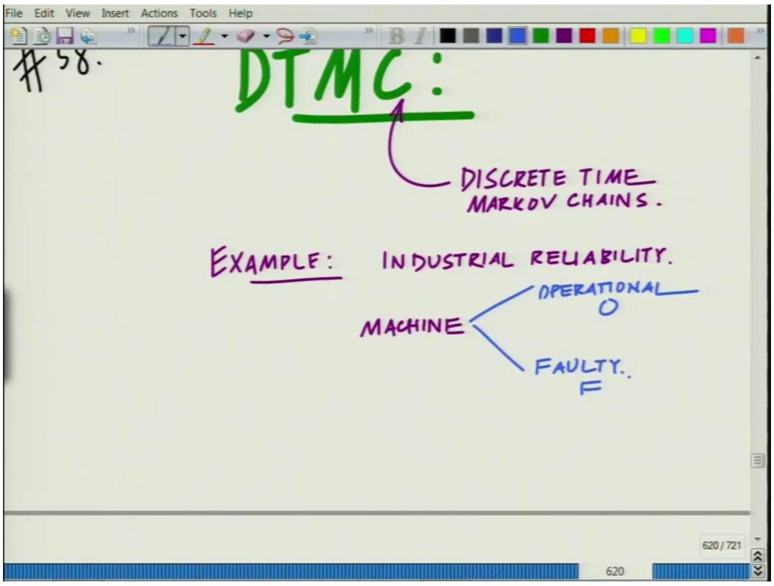

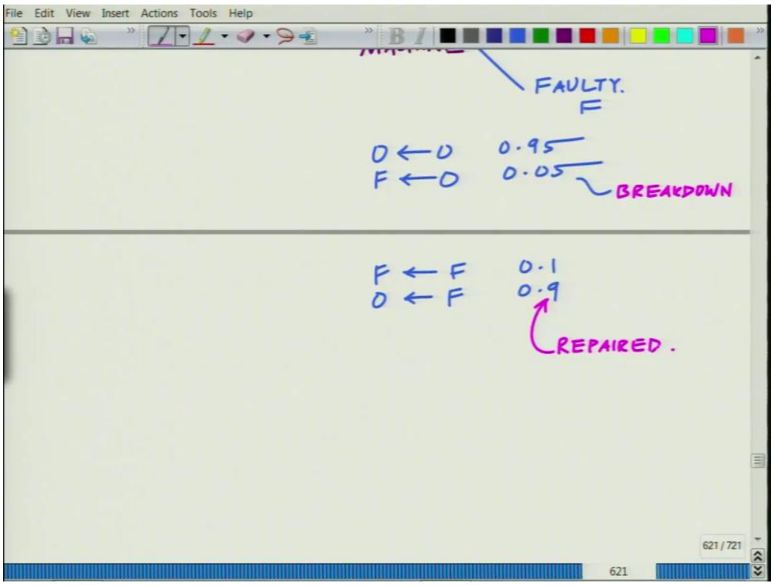

So, let us continue our discussion on that. So, basically, we are looking at DTMCs, that is your discrete time Markov chains, not to forget, that is stochastic processes with the Markov property. And we are looking at the industrial reliability property, the example of industrial reliability. So, we are looking at this example of industrial reliability, where the machine can be in one of two states.

So, the machine, a particular machine or device can be in one of two states, it can either be operational, which I will call as the state O, or it can either be faulty which I will call as a state F. And the various transition probabilities are, the transition from operational to operational, that is it continues to remain operational is 0.95, the probability that it goes from operational to faulty that is 0.05.

The probability that it goes from, if it is faulty, it continues to remain faulty is 0.1. And the probability that if it is a faulty, it transitions to operational is 0.9, that is. And these are what we call as the breakdown. This is the probability of for instance, breakdown. And this is the probability that faulty to operational, this is the probability that it can be repaired within the hour. That it breaks down and it is repaired by the next hour.

So, it goes from faulty to operational. Now, let us consider extending this example to two machines. So, the general question might be, so we have considered a simple example with one machine. Now, what happens? Let us say we have two machines operating independently, then what are, how can you model this system? What would be the states and how do you characterize the transitions?

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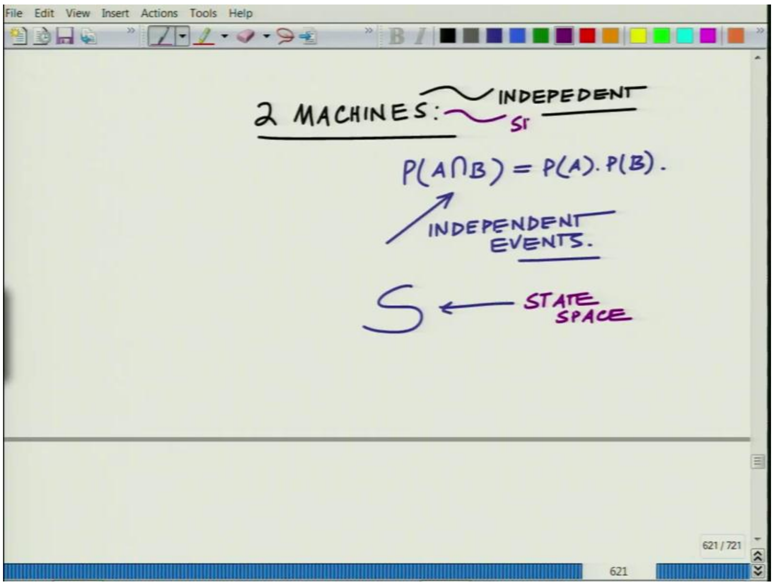

So, now consider two machines, consider the industrial setting, because typically in industry you have several such machines and naturally, we would like to generalize this to let us say two machines. And the key here is that these are independent. That is their operation is independent. That is, the problems occur independently, they break down independently and because the problems are independent, they can be that the probability that they are repaired or not being repaired is also independent.

And we know from probability theory that if two events are independent, then the probability of A intersection B equals probability of A into probability of B. This is for independent events. Now what does this mean? Now let us look at, now let us first characterize the state space for this problem, what is going to be the state space? So, these machines are similar machines and their operation is independent.

Now, since these machines are similar, we do not care which particular one of this is operational or faulty. Let us say there are two machines, if either one of them is operational then the purpose they serve is the same. So, we can characterize this by three states that is operational, operational both are operational, both are faulty or one of them is operational and one of them is faulty. It does not specifically matter which one is operational, which one is faulty because essentially the output of each machine is the same, operation characteristic you might say or the output is same.

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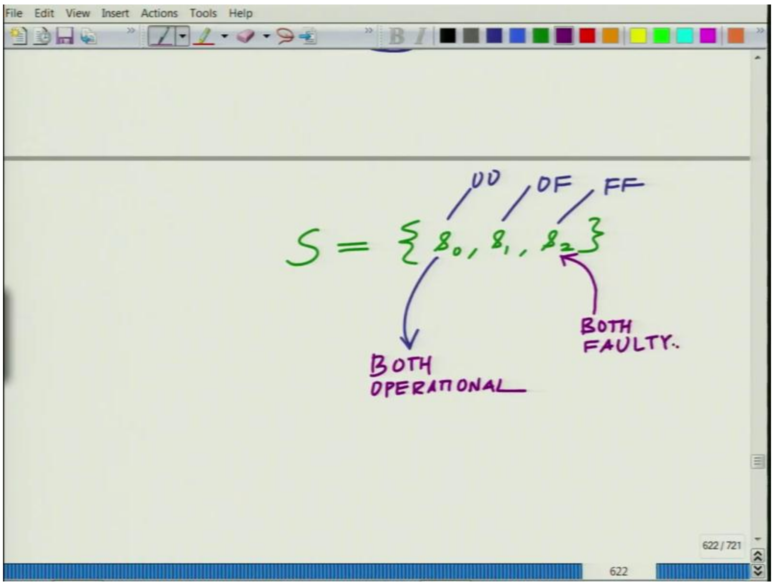

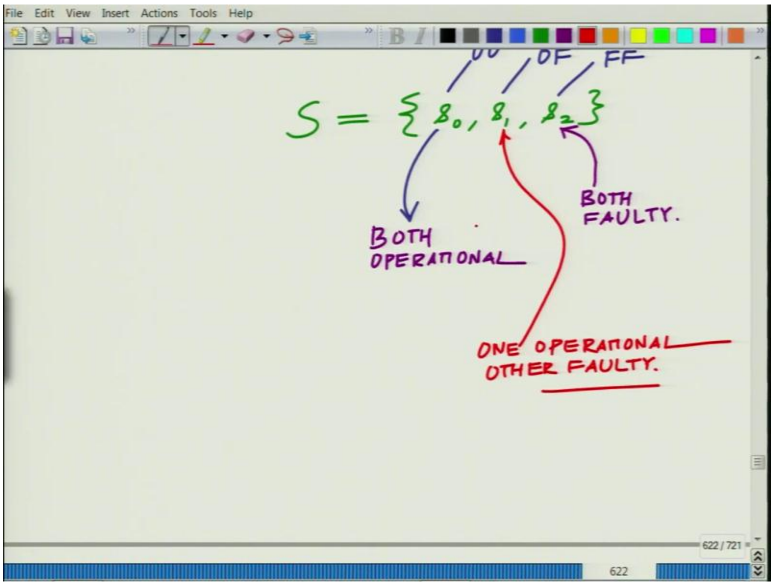

So, the state space can therefore be modeled as follows, the S can be modeled as, s0, s1, s2, where s0 is both are operational, one is operational, other is faulty and both are faulty. So, this is basically both operational. This is both faulty, this is both faulty and this is essentially one operational, other faulty. This is state where you have, this is a state where we have one operational other faulty.

And therefore, so yeah both operational or both working, s2 is both faulty, one operational other faulty, as I said we do not specifically care about which one is operational, which one is faulty, because both machines serve the same purpose.

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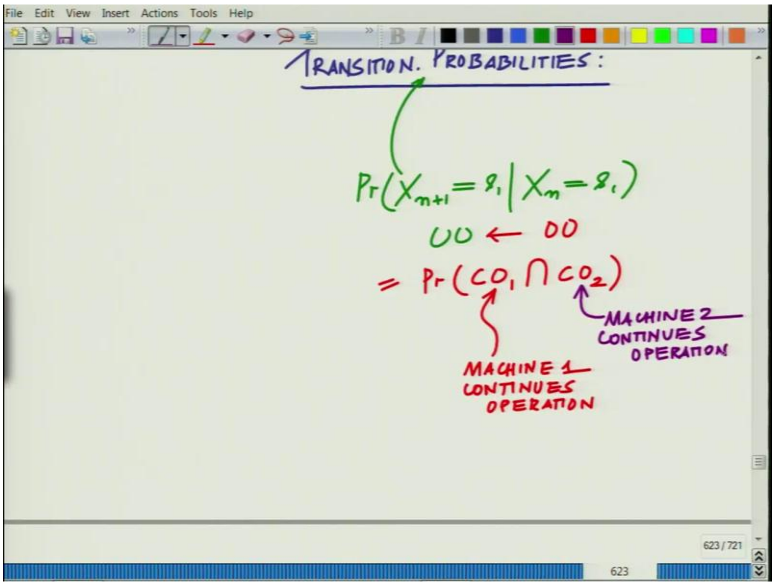

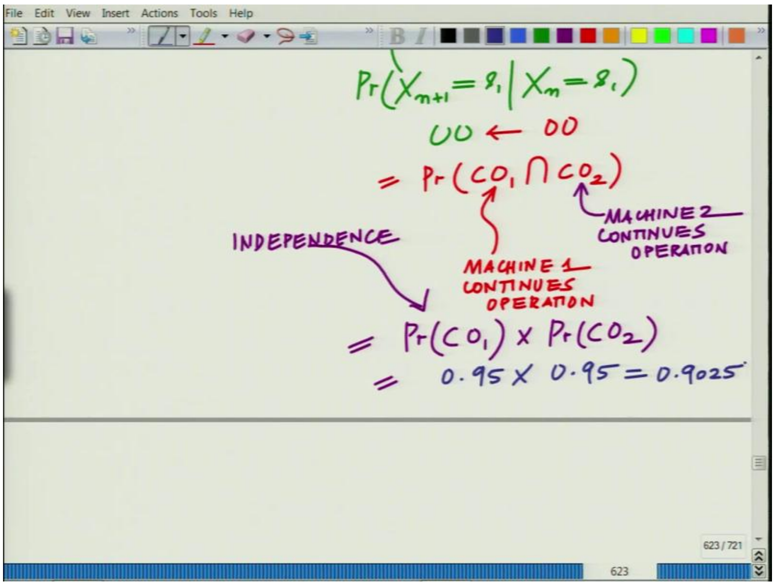

Now, let us find out what the transition probability is. Now let us remember, to characterize the DTMC we need the transition probabilities, the one step transition probabilities. We need the one step transition probabilities. What are these one step transition probabilities? Let us look at probability, start with probability Xn plus 1 equals s1 given Xn equal to s1 that is both are operational and both continue to remain operational, both are operational and both continue to remain operational.

What is the probability of this? Probability of this is remember operational, both continue to remain operational, that is you can say it is operational intersection operational that is both continue to remain, continue operational, intersection continue operational. So, if you think of continue (inter), operational as the event co. So, this will be probability of continue operational one, co1 that is machine 1 continues to be operational, machine 2 continuous to, so even think of machine 1 continues operation. And machine 2 continues operation.

And this would be, which as I said if these is, if these machines are independent now this becomes a product of these probabilities, that is the interesting thing. So, this becomes probability, continues to be operation 1 times the product continues to be operation 2. So, this is because the product is because of the independence. And this becomes your 0.95 remember that is a product probability, that machine one continues to operational 0.95 probability machine two continues to operational, the product is 0.9025.

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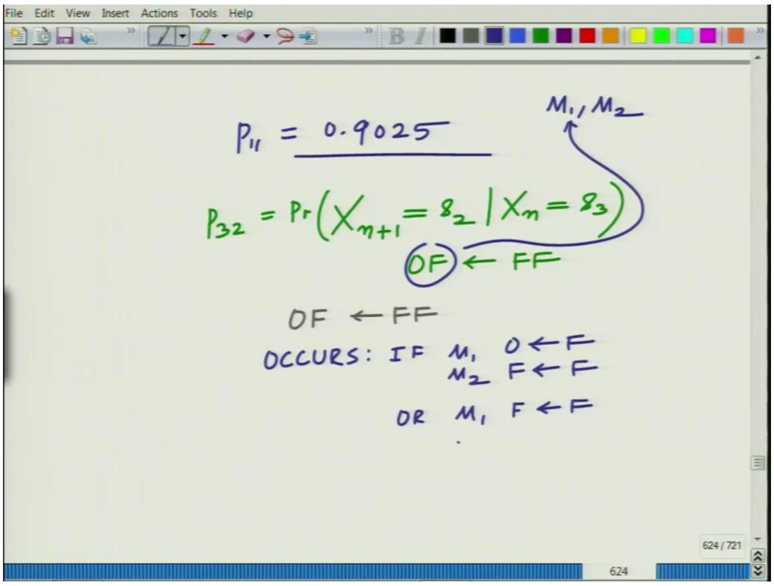

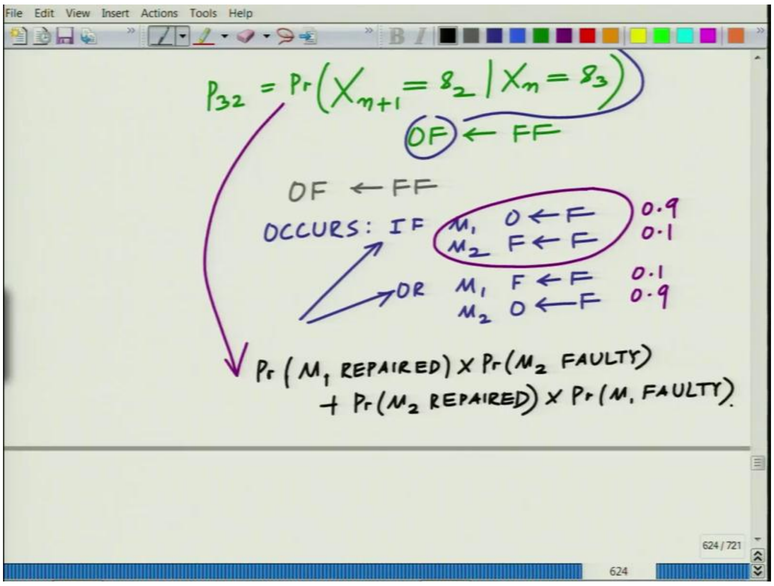

So, P 1 1 equals 0.9025, this is a product probability that Xn plus 1 equals s1 given Xn equals s1. Let us compute one more, we will not compute all of them, I will just illustrate the general procedure and you can extend it to the other transition probabilities. So, let us for instance look at another thing that is slightly different, let us look at P of 3 comma 2. That is the probability Xn plus 1 equals s2 given Xn equals s3.

That is basically it goes from both faulty to one is operational, it goes from both faulty, it goes from both faulty to one is operational, other is faulty. Now, remember in this you have two machines M1 comma M2. So, in this operational faulty, you can either have one of them is working, the other is faulty, either M1 is operational, M2 is faulty or M2 is operational, M1 is faulty, so both of them.

So, OF basically encompasses both of them. So, FF to OF can occur, so, this occurs. So, if you analyze this FF to F occurs, occurs if M1 goes from faulty to operational, that is M1 is repaired while M2 continues to remain faulty, that is M2 goes from faulty to faulty, M2 is not repairable or that is M1 cannot be repaired continues to remain faulty and M2 goes from faulty to operational. Now, these two events now, these two are mutually exclusive. So, their probabilities can be added and these two are independent.

Now, this probability is going from faulty to operational, that it can be repaired, this is 0.9 faulty remains faulty this is 0.1, faulty remains faulty this is 0.1, M2 goes from faulty to operational, this is 0.9. So, therefore, this net transition probability you can write it as follows, this is probability M1 repair times probability M2 continues to remain faulty plus probability, now M2 repaired cross probability M1 continues to remain, M1 continues to remain faulty.

And this product once again, this arises because both these are independent, not to forget that. The probability that M1 is repaired and continues, because the problems occur independently and the probability that they can be repaired, these two events are independent. M1 is repaired does not mean that M2 can be repaired, M2 being repaired does not have a bearing on M1 being repaired. These two are independent and therefore, we multiply the probabilities.

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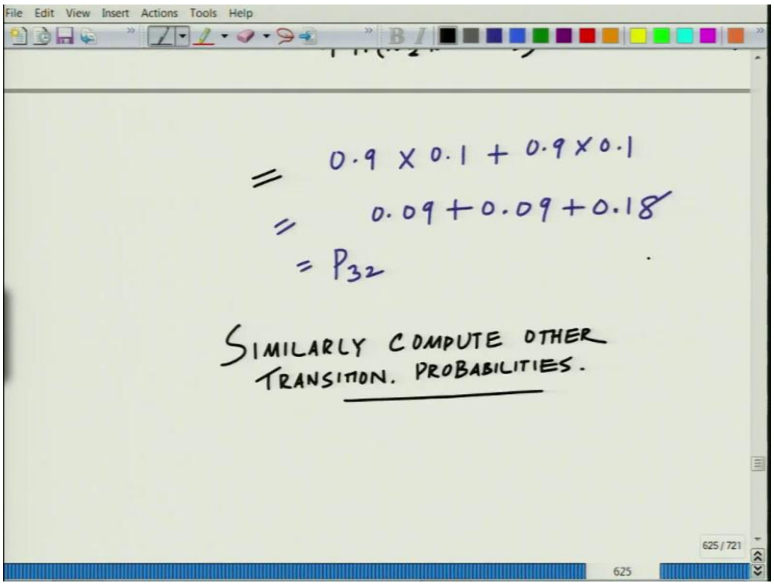

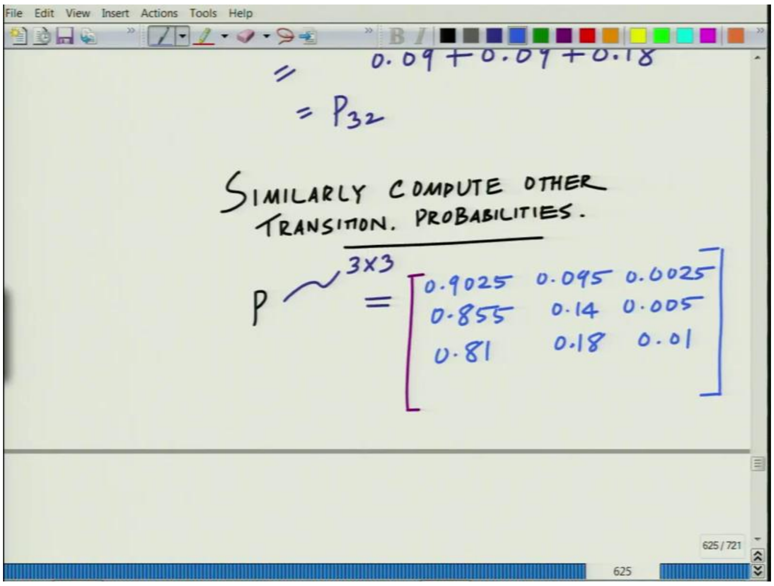

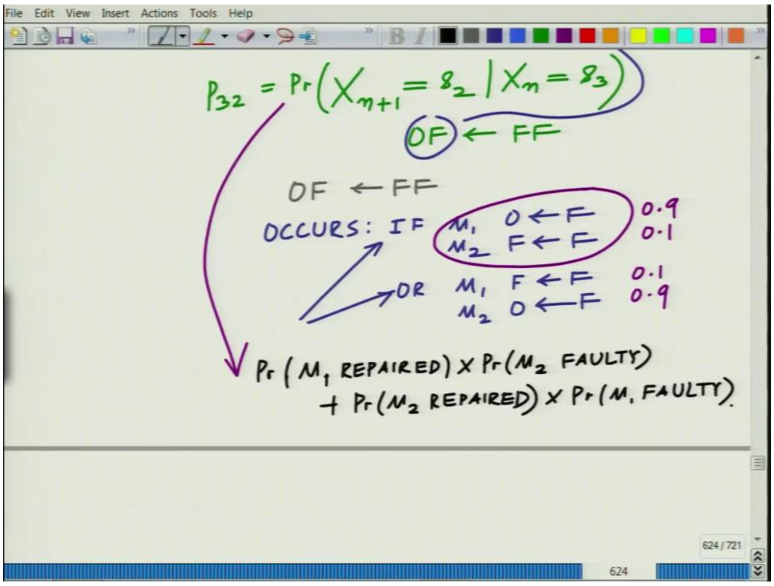

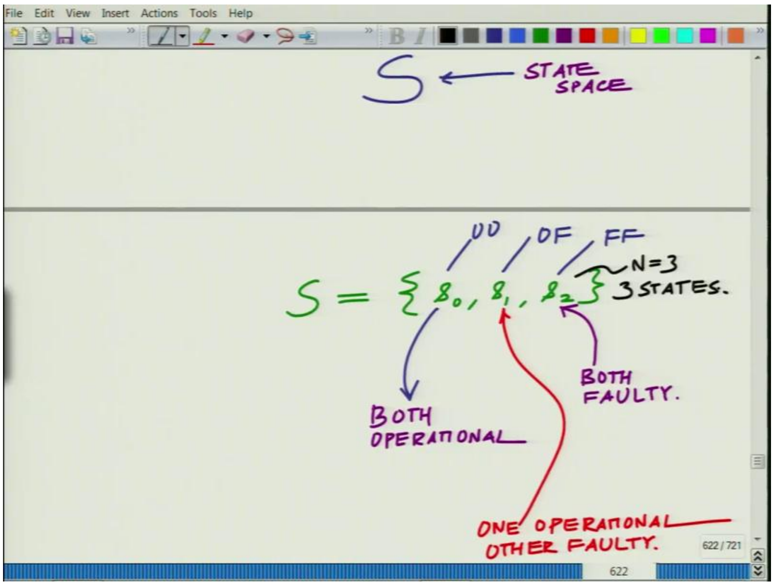

And so, therefore, this is equal to now probability M1 repaired this is 0.9, probability M2 continues to remain faulty is 0.1 plus probability M2 repaired 0.9 probability M1 continues to remain faulty is 0.1. So, this is 0.09 plus 0.09 that is 0.18. This remember is a probability, probability that is the probability Xn plus 1 equal to s2 given Xn equal to s3. And therefore, now you can similarly compute the other transition probability.

So, I will simply add here, similarly compute other, other transition, similarly compute other transition probabilities. Now, recall for this problem we have three states. So, if you look at this problem with two machines, we have n equal to 3. So, we have, we have three states, this implies the matrix P is of size 3 cross 3 transition probability matrix P. This is of size 3 cross 3 and this matrix will be given as and you can verify this, this will be given as 0.9025 0.095 0.0025 0.855 0.14 0.005 0.81 0.18 0.01.

This is essentially the transition probability matrix, which is a 3 cross 3 matrix for this problem, with the industrial reliability problem with two operating machines which are operating, independently two similar machines which are operating independently. So, essentially now, you can see, get an idea of how to construct more and more complex problems and complex examples starting with a simple example.

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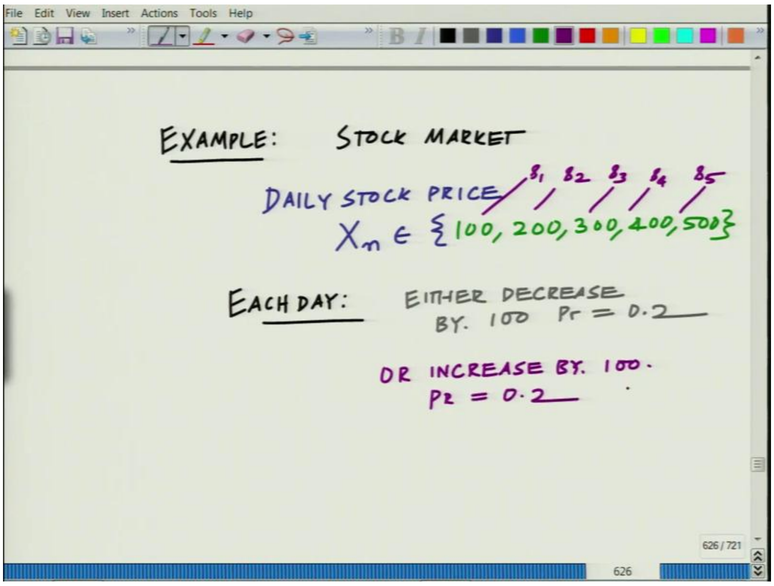

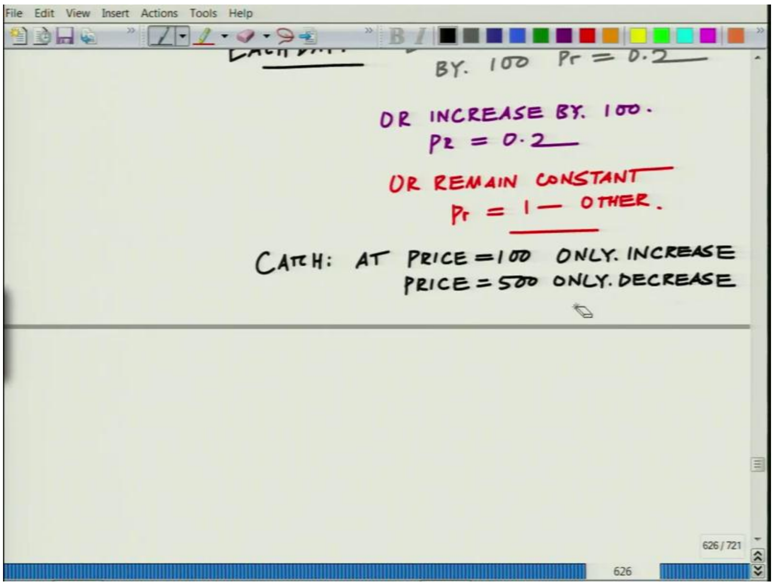

Let us look at another example, the DTMC have a large number of applications, such as inventory I have already told you, inventory management so on and so forth. Let us look at a simple example. Remember, we also talked that stock price reduction. So, let us look at the stock market. So, let us say the stock price, daily stock price, we characterize this as the state. This can be either, let us consider a simple example although you can build more complex examples starting from this 100, 200, 300, 400, 500.

And let us call these, these states as s1, s2, s3, s4, s5 and now each day, each day either it can, either of things can happen, each we look at we monitor the stock price every day either at the beginning or at the closing. Each day either decrease by 100 with probability equal to 0.2

or the stock price can increase by 100 probability equal to 0.2 or it can remain constant, or remain constant.

Probability equal to 1 minus, I will simply say other probabilities because there is a catch here. What is the catch? The catch is whenever it hits the maximum 500, it can only decrease, it cannot increase, or whenever prices hits 100, it cannot decrease it can only increase. So, that is the catch. So, what is the catch? At price equal to 100 only increase, price equal to 500 we have only, we have only decrease, price equal to 500 it only decreases.

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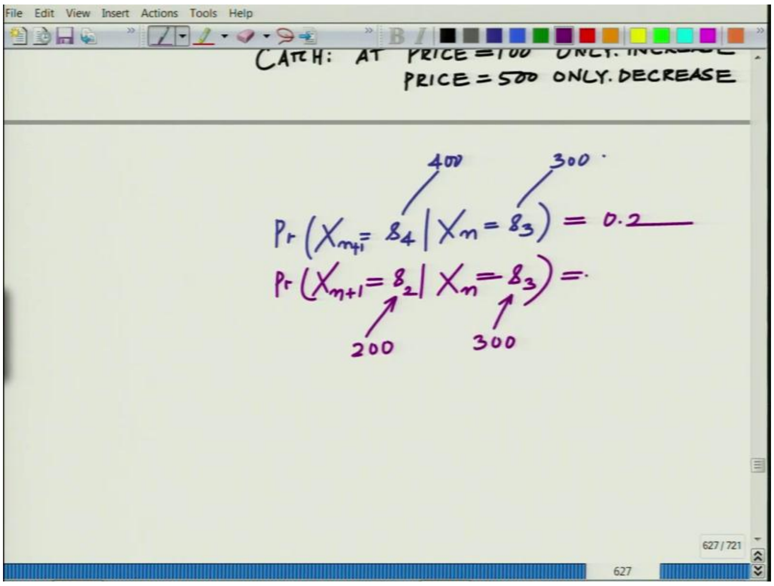

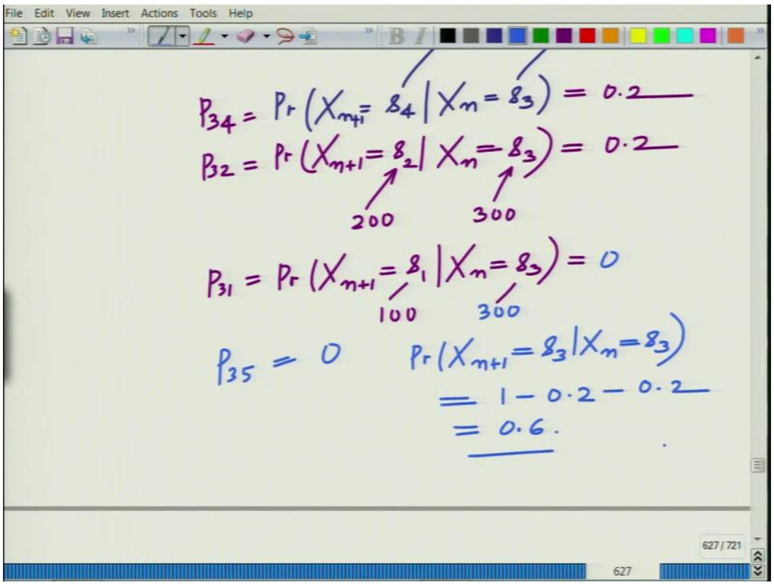

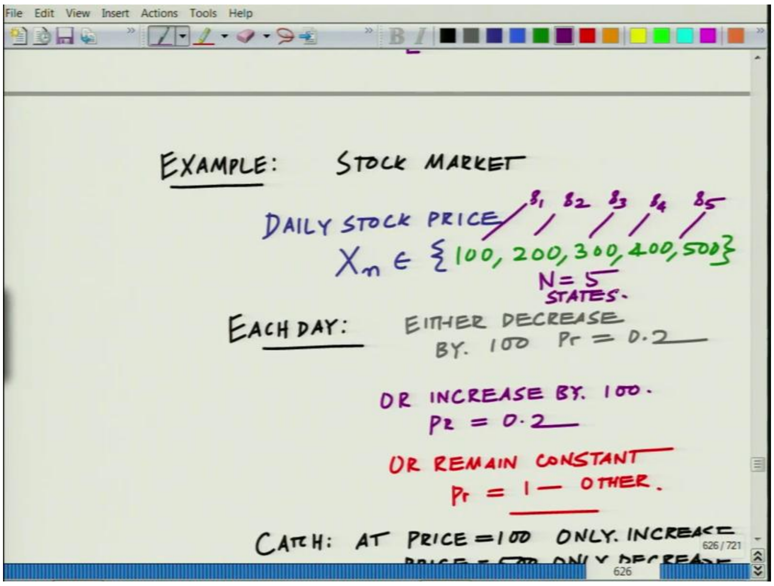

Now, what do we do? Therefore, now let us take a simple example, let us ask, what is the probability Xn equal to let us say 400 or Xn equal to s4, Xn equal to s4 given, or we can say Xn plus 1 equal to s4 given Xn equal to s3, remember s4 means price is 400, s3 means price is 300 probability price increases from 300 to 400 that happens with probability of 0.2. Similarly, what is the probability Xn plus 1 equal to s3 given or given Xn plus 1 equal to s3 given Xn let us continue on the s4, Xn equal to s3 or let us say s2, that is remember s2 is 200, s3 is 300 probability that decreases by 100, this is equal to 0.2.

Now, one can ask what is the probability Xn plus 1 equal to s1 given Xn equal to s3. Now, these are basically your probabilities 3 comma 4, I believe probabilities 3 comma 2, this is probability 3 comma 1. This is equal to 0, because remember this is essentially your 100 and this is essentially your 300, it cannot fall by 200. So, this is equal to 0, P 35 similarly, equals 0 probability increase from 300 to 500 that is 0.

And therefore, now, if you look at this probability Xn plus 1 equal to s3 given Xn equal to s3, remember all the probabilities have to add up to 1. So, this is 1 minus 0.2 minus 0.2 equals 0.6. So, so on and so forth you can compute all the transition probabilities, all the transition probabilities and therefore, the transition. Now, now, look at this again there are five states. So, n equal to 5 states in this DTMC.

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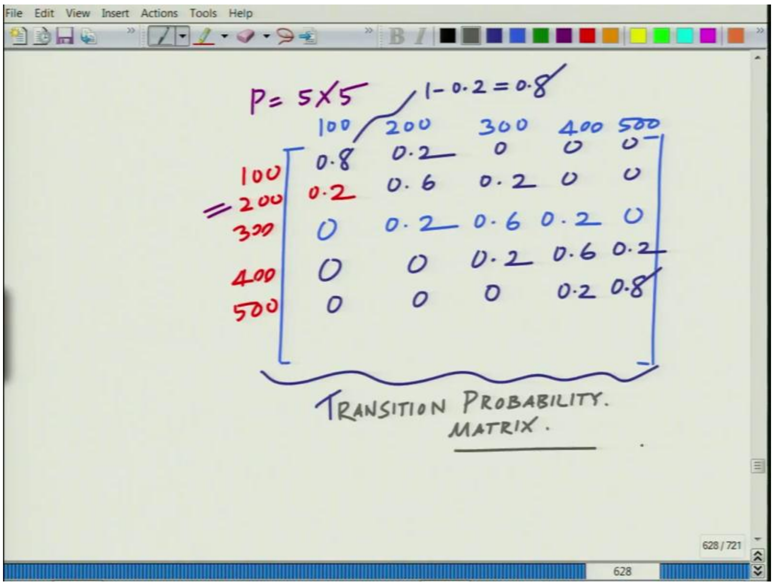

Therefore, we have the transition probability matrix P is 5 cross 5 matrix and therefore, this can be written as follows. We have already computed the third row, row number 3. So, that is going to be 0 0.2 0.6 0.2 0. And remember, I will just going to write the prices over here 100 200 300 400 500. And these are also 100 200 300, what we just written 400 500. Now, the probability it goes from again, let us look at the 200, probabilities goes from 200 to 100 it decreases by 100.

So, this is again 0.2 probability remains at 200 you will see this is 0.6, this is 0.2 0 0. Similarly, from 400 to 100 it decreases to 100 that is 0, it cannot decrease to 100, it cannot decrease 200. It can decrease to 300 with probability 2, it can 0.2, it can increase to 500 with probability 0.2, remaining is probability which stays at 400 that is 0.6. Now, what about the, these are the interesting cases, what about the 500?

Now at 100 remember it can, it can either go to 200 but it cannot fall to 0, remember that is not one of our states. So, the remaining is all the other probabilities are 0 and the accumulated rate comes into 100 that is, this becomes 0.8 remember 1 minus 0.2 because the sum of each row has to be 1. So, probability remains at 100 is 0.8 and the probability that, it goes to 200 is 0.2. Similarly, at 500 it cannot go to 600.

So, it can only go to 400, this probability is 0.2 rest all are 0 and this is 0.8. So, this is 0.2. So, this is 0.8. So, at this I believe is our 5 cross 5 transition probability matrix for the stock price problem and this is a simple problem and this idea is, this can be used to build more complex problems. So, this is the one step, this is the transition probability matrix. This is the transition probability matrix for this problem.

So, we have looked at several interesting examples, variation of original examples and also interesting example pertaining to stock price. So, as you can clearly see DTMC is a very powerful framework which has several applications be it either reliability or industrial what you call as the health monitoring of machines or be it stock price, stock market modeling I already told you about inventory modeling, prediction, inventory management insurance.

There are so many areas, again one of the very powerful frameworks, it can also be related, more complex models can also be built for stock markets, financial mathematics and futures and derivatives and that is basically how is stock price evolving. What is the price of future, what is the price of a derivative and all these kinds of complex questions can be readily answered using this framework. So, clearly a very powerful framework, I urge you to once again go through the lectures to understand and appreciate all the concepts completely. Thank you very much.