Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
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Lecture 44
OFDM System: Circulant Matrices and Properties

Hello, welcome to another module in this massive open online course. So, we are looking at OFDM and OFDM system and the specific application of linear algebra and the modelling and analysis of this system. So, let us continue our discussion.

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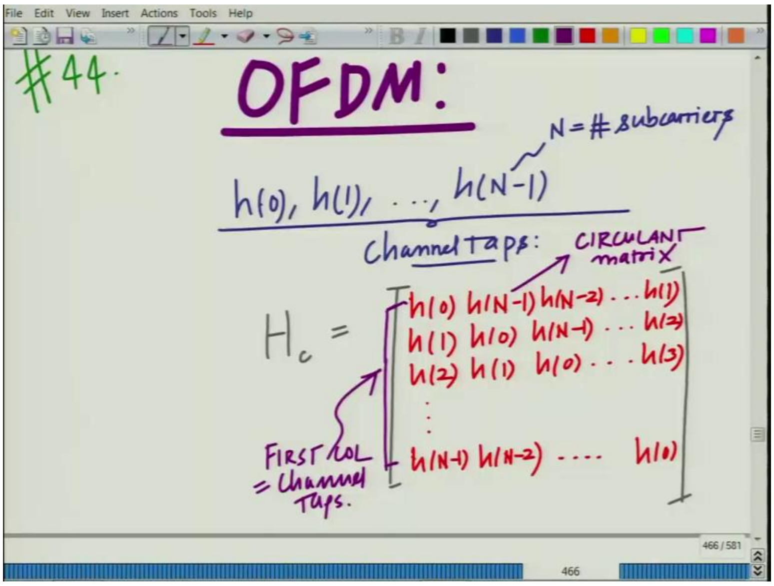

So, we are looking at OFDM and where we are left in the last module is the following thing: considering an ISI channel with the channel taps h 0, h 1, h N minus 1 where N equals number of sub carriers, where N equals the number of sub carriers. What we had said is that these are the channel taps and the channel matrix if you look at the OFDM system after the cyclic prefix and so on, the channel matrix can be has a very interesting structure.

So, this will be h 0, h N minus 1, h N minus 2 so on h 1 and then h 1 comes over here h 0 pushes to the right h N minus 1 also pushes to the right you have h 2 then h 2 comes rotates here, you have 0, you have h 3 so on and the last row will be h N minus 1, h N minus 2 so on until h 0.

And what we had said is very interestingly if you look at each row, each row is obtained by circularly shifting the previous row at the same time each column is also obtained by circularly shifting the previous column and therefore, is a very interesting matrix such a matrix is known as a circulant matrix and this is the characteristic feature of OFDM circulant.

This is the circulant matrix with if you look at the first column, if you specifically look at this is the first column, the first column basically comprises of the channel taps. So, how to construct this matrix? Simply take the right, the first column as the channel taps h 0, h 1 up to h N minus 1 and then each subsequent column you shift it down shifted by 1 and the 1 that pushes down you bring it back onto the top of the next column.

So, the second column will be h N minus 1, h 0, h 1 up to h N minus 2. Third column will be h N minus 2, h N minus 1, h 0 so on up to h N minus 3 and so on and let us now look at interesting properties of this matrix.

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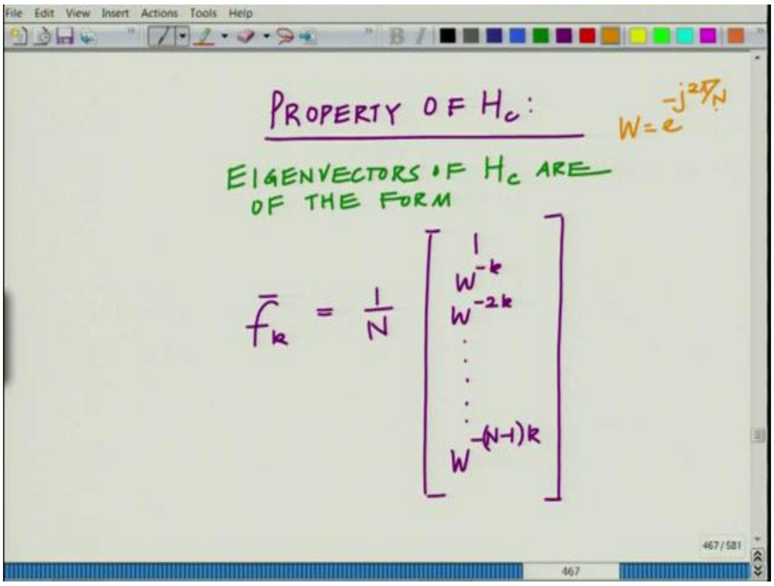

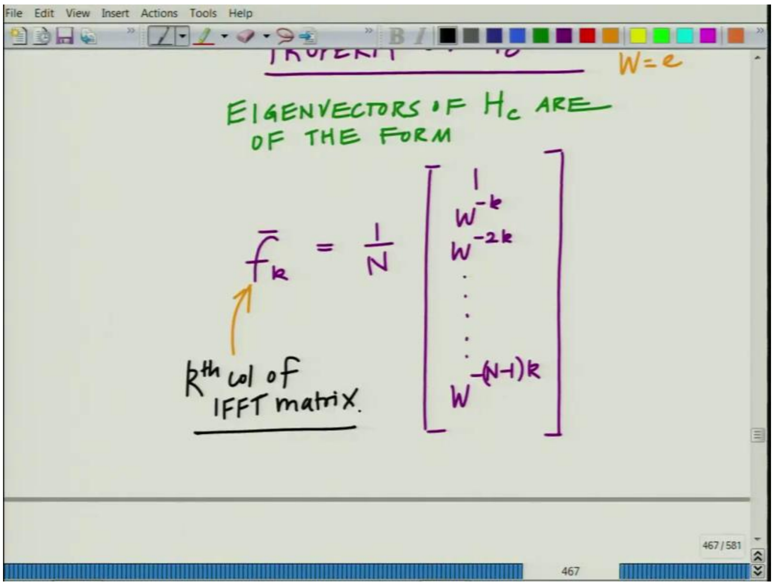

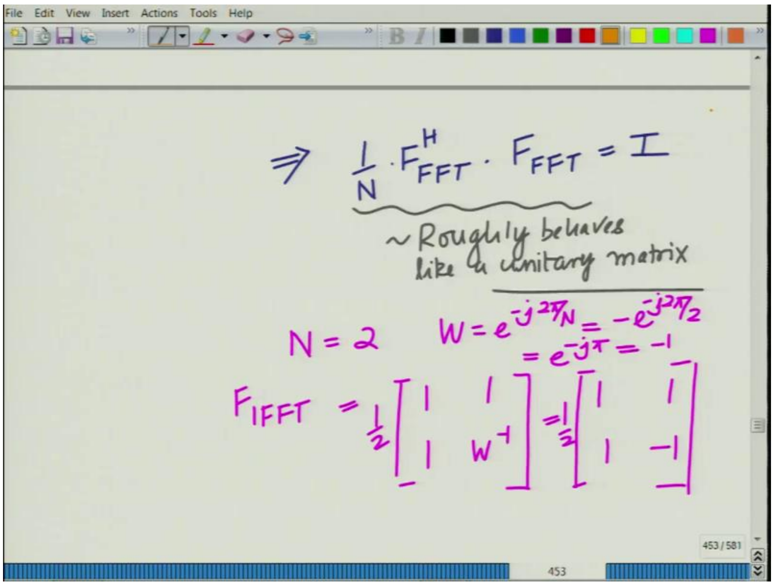

Now, what are the interesting properties? The most interesting properties now, there are several the most interesting property of this matrix is in fact the property of any circulant matrix, property of the matrix H c is that the Eigen vector, this is very interesting; the Eigen vectors H c are of the form that is if you look at the kth Eigen vector, I can write this as f bar k which is 1 over N times 1 W raised to minus k, W raised to minus 2 k, W raised to 2 minus N minus 1 k.

This is the kth Eigen vector where you remember W equal to e raised to minus j 2 pi over N, that is the definition of W. Let me just quickly check how we define W? W equal to if you remember W equal to, we have W equal to e raise to minus j 2 pi over N indeed and now, if you look at this, this quantity this is nothing but what is this f bar k? If you look at this this f bar k is nothing but if you observe this closely this is the kth column of the IFFT matrix.

This is the kth column of the IFFT matrix and that is the interesting property of this. It is very interesting property and remember that is what we said so, it is naturally because it is circular that it is gives the impression that it is periodic. So, it naturally it has to have relation to sinusoids at complex exponentials and what is that relation? That relation is essentially if you look at this Eigen vector f bar k the Eigen vector of this H c is essentially given by the kth column of the IFFT matrix, this is the interesting property and let us prove this.

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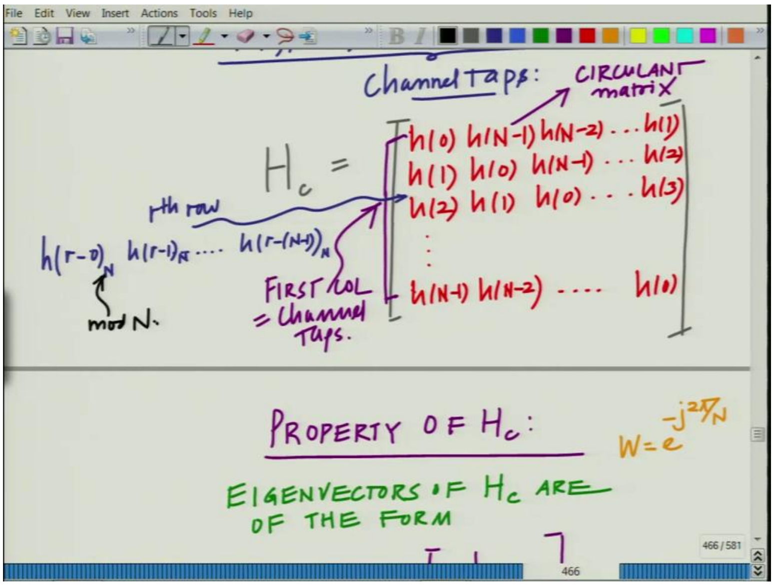

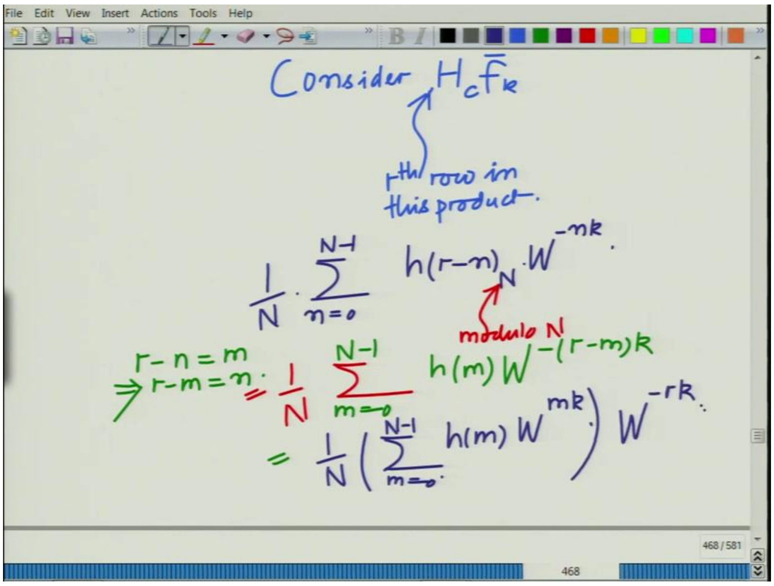

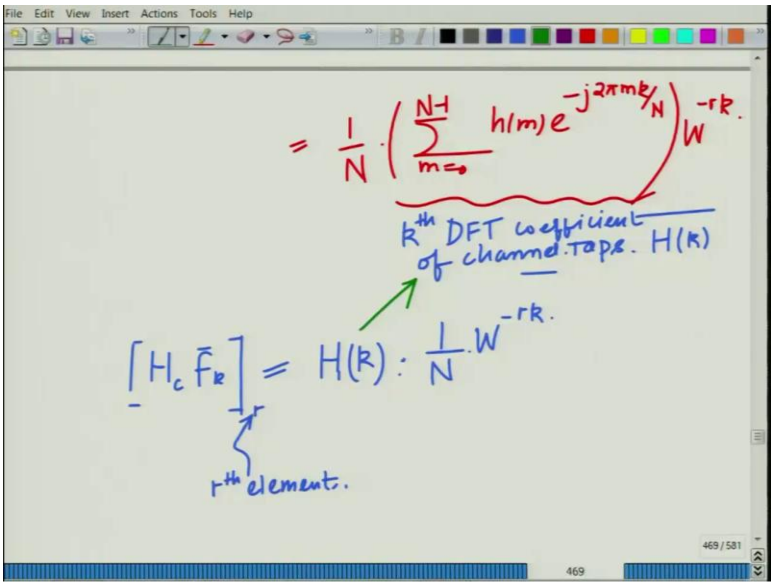

So, let us try to see how this is, this is as follows can be seen as follows: if you look at this now, first let us realize that the rth column, rth row if you look at the rth row that will be h r minus 0, h r minus 1 so on h I can, the only thing I need to do here is I can add modulo N. So, this will be h r minus 0 modulo N, h r minus 1 modulo N so on and h r minus N minus 1 modulo N.

So, this small N here this indicates modulo N and therefore, if you look at the rth row and if you look at the inner product with this that is let us consider, consider the product H c times f bar k and let us look at what happens to the rth row in this product. That is output is a vector and we are asking the question what is our element of that vector and that will be 1 over N summation N equal to 0 to N minus 1 h r minus N modulo N, W raise to minus n k.

This is once again to remind you this indicates that this is modulo which is equal to 1 over N summation N equal to 0 to N minus 1 or now substitute N equal to 0 to, now make r minus n equal to m, substitute r minus n equal to m So, m also goes from 0 to N minus 1 this will become h m there is no need for modulo N because m anywhere goes to 0 to N minus 1.

So, modulo m modulo N will simply be m because m is going from 0 to N minus 1. So, therefore, this will be h m W raised to the power minus N minus n k. So, this implies r minus n equal to m. This implies r minus m equal to n. So, this will be r minus of this is equal to 1 over N summation m equal to 0 to N minus 1 h m W raised to the power of minus or in this case plus m k times this other quantity will be common W raised to the power of minus r k and now, if you look at this quantity, this is interesting.

This is summation, this is nothing but if substitute for W this will be 1 over N summation m equal to 0 to N minus 1 h m e raised to the power of minus j 2 pi m k over N times W raised to the power of minus r k and if you look at this quantity this is nothing but the FFT or the DFT coefficient of the channel taps. This is nothing but the in fact this is the kth DFT coefficient.

In fact, there is a kth DFT coefficient of the channel tap. So, I can write this so, this is essentially your H of k. So, this is basically your H of k times 1 over N W raised to the power of so, if you look at H c times f bar k and if you ask the question, what is the rth element that is what is our element of this? That is simply the H k that is the kth DFT coefficient H k times 1 over N W raised to minus r k. This is the interesting aspect.

So, H k where H k, remember what is H k? H k is the kth DFT quotient that is summation m equal to 0 to N minus 1 h m e raise to minus j, e raised to minus j 2 pi m k divided by N. That is if you take the channel taps into the kth DFT coefficient that is what it is. Now, write all this, so therefore, if you write H c times f bar k.

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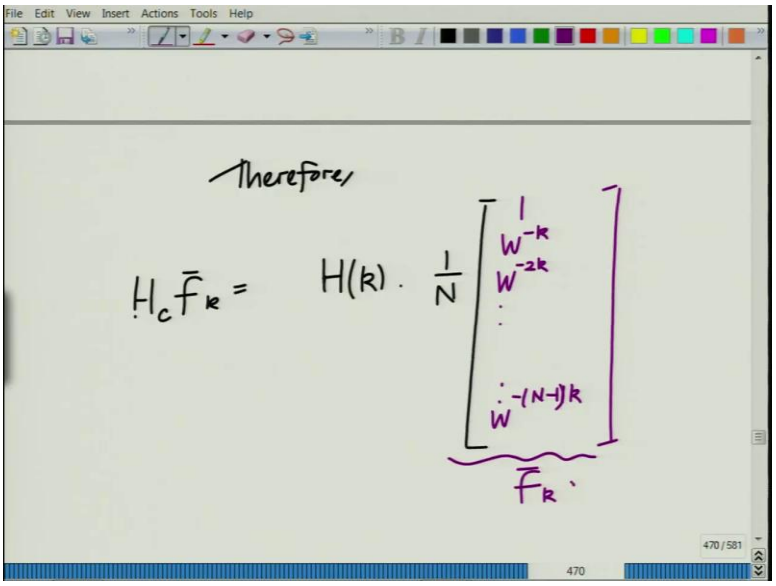

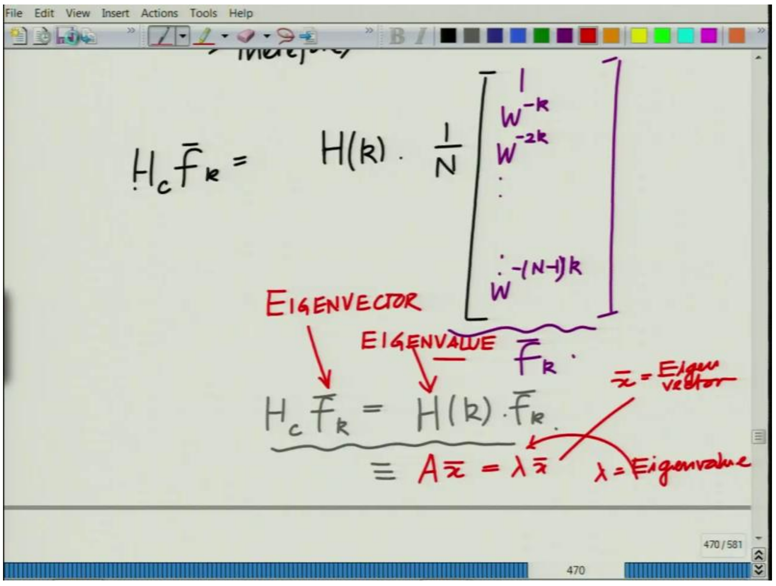

So therefore, write all the elements H c times f bar k equals H c. This will be H k or this will be H k times 1 over N. What are the elements? 1 first element will be W, each element is W raised to minus r k. So, W raised to minus k, W raise to minus 2 k so on, W raised to minus N minus 1 k and in fact, if you look at this now, interestingly if you look at this, this is nothing but essentially the vector f bar of k. So, that is the interesting thing. So, essentially you have the relation H c of f bar k equals 1 over N are equals. In fact, H k times f bar k.

So, this is the form A x bar. If you look at this, this is similar to A x bar equal to lambda x bar, where x bar equals the you know very well now, x bar equals Eigen vector lambda, equals the Eigen value. So, this implies that in this f bar k, this is the Eigen vector of H c and this DFT coefficient is the Eigen value. So, that is the interesting aspect of this.

So, the H k is that is a DFT coefficient, the Eigen vectors are the columns of the IFFT matrix and Eigen values are given by the FFT or the DFT of the channel taps. So, this is the interesting property of the circulant matrix.

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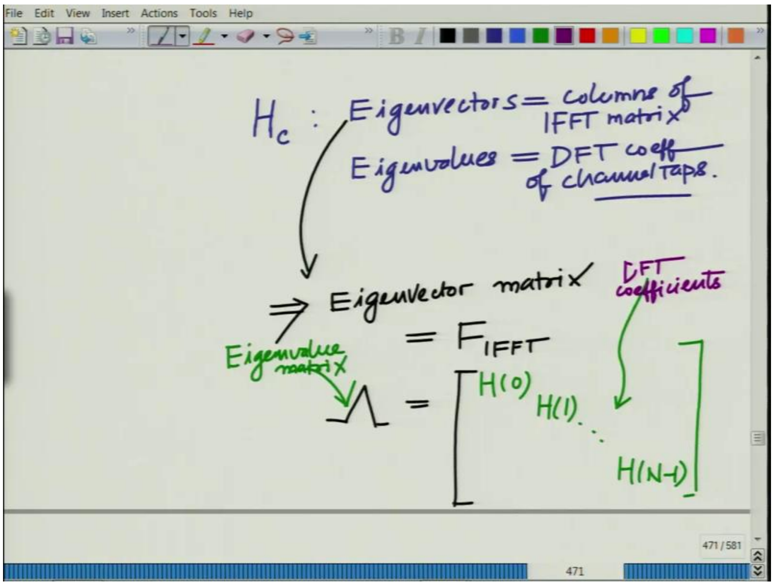

So, H c has the following very interesting property. H c Eigen vectors equals columns of IFFT matrix and Eigen values equals the DFT coefficient. Eigen values are the DFT coefficients of the channel taps and this also implies that the Eigen value matrix, implies the Eigen value matrix if you look Eigen value matrix, this is equal to simply the column of the matrix which contains the IFFT columns of the IFFT matrix.

So, this is simply the IFFT matrix. So, this is the Eigen, I am sorry this is the Eigen vector matrix and Eigen value matrix lambda, the Eigen value matrix is contains the DFT coefficients. So, this is the Eigen value matrix, this is the Eigen value matrix. So, this is the Eigen value matrix basically contains the DFT coefficients and we know what is Eigen value decomposition?

The Eigen value decomposition of a square matrix is nothing but the Eigen value matrix, I am sorry the Eigen vector matrix times the diagonal matrix of Eigen values times the inverse of the Eigen vector matrix. Eigen vector matrix is the IFFT matrix inverse of the Eigen vector matrix is nothing but the inverse of the IFFT matrix which is nothing but the FFT matrix and that essentially completes this analysis.

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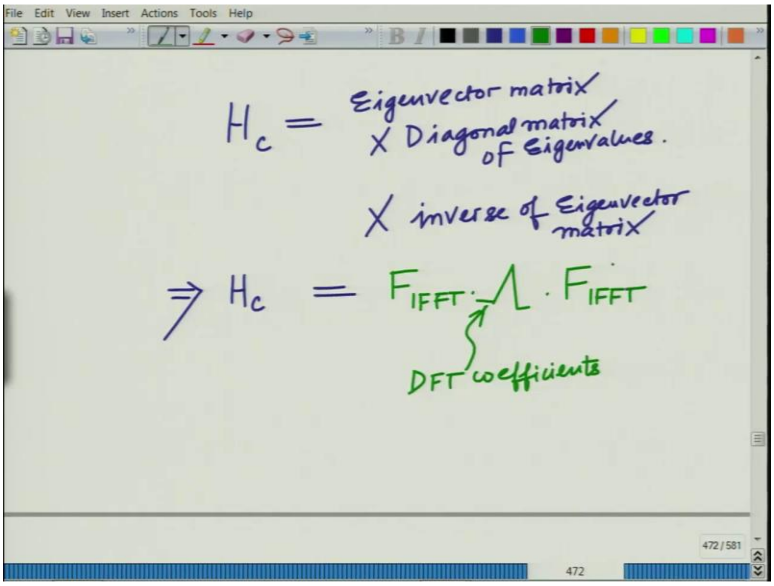

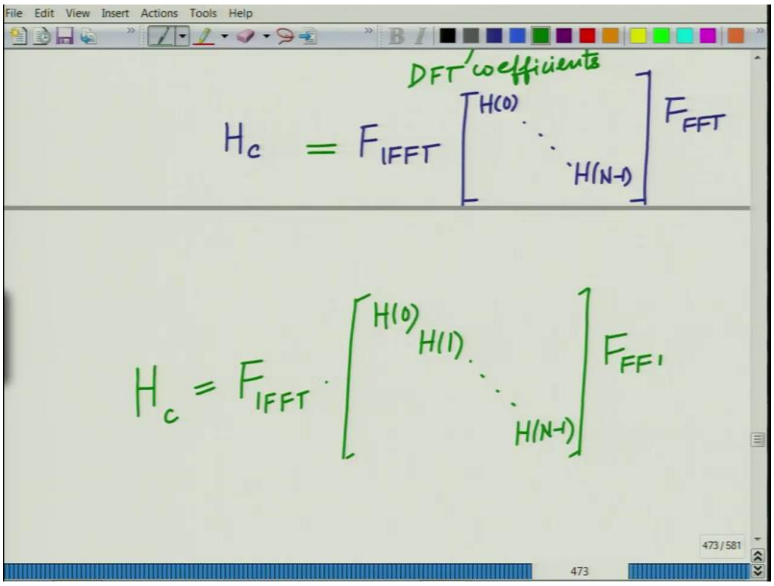

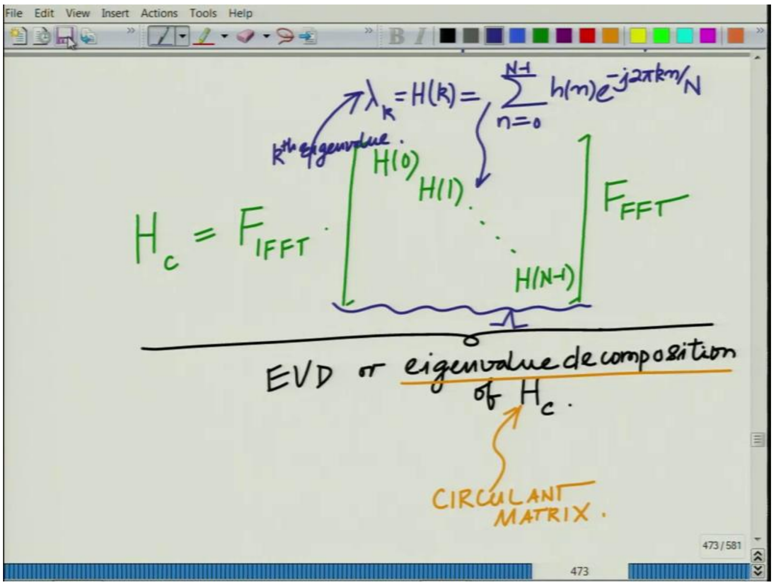

So, you have now very interestingly if you look at it, H c any matrix let me write the general property. This is can be written as the product of Eigen vector matrix times diagonal matrix of Eigen values, Eigen values times inverse of Eigen vector matrix which in this case implies H c equal to Eigen vector matrix is the IFFT matrix, F IFFT times the diagonal matrix of Eigen values which contains the DFT coefficients times F IFFT inverse.

But F IFFT inverse equals FFT. So, therefore, this is writing it explicitly. F IFFT times your H 0 up to H N minus 1 times F FFT. Let us see and this essentially, so if you look at this this is the Eigen value decomposition. Let me write this thing once again H equal to F IFFT times H 0, H 1 up to H N minus 1 times FFT and this is essentially the what we call as the EVD.

This is the Eigen value decomposition of H c which is now. So, for the circulant matrix in fact for any circle in matrix Eigen value decomposition is given by the IFFT matrix times the diagonal matrix of Eigen values which are nothing but the DFT or FFT coefficients of the channel taps followed by the FFT matrix and in fact, let us also write the formula for the Eigen value you have lambda k equals H of k which is basically summation m equal to 0 or you can write it as n equal to 0 to N minus 1 h of n e raise to minus j 2 pi k n by N.

So, this is your kth Eigen value and this is your matrix lambda that is the diagonal matrix of Eigen values. So, this is essentially the interesting aspect or this is the most interesting property of the circulant matrix and now, what we are going to do is we are going to use this matrix in the OFDM system model and we are going to simplify the OFDM system model and observe something very interesting and that is what we are going to do in the next and the final module. So, let us stop here and we will continue this discussion in the next module. Thank you very much.