Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning

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Lecture 43

IFFT/FFT Application in Orthogonal Frequency Division Multiplexing (OFDM) Wireless Technology

Hello, welcome to another module in this massive open online course. So, we are looking at IFFT and the FFT, IFFT operations, that is to convert signals and systems and their representation and the time and frequency domain go back, back and forth between the time and frequency domains. Let us now of course, that by itself has several implications for signals and system and; signals and system analysis. Let us now look at another interesting application of these transform domain representation that is the matrix representation of FFT and IFFT operations in the context of something that is very, very interesting and relevant and that is in modulation technique or a wave form known as OFDM.

Many of you might have heard about it, OFDM stands for Orthogonal Frequency Division Multiplexing and this forms the basis for the high data rate transmission that occurs in your mobile phones and even goes through via cellular network or even via Wi-Fi in your laptop. So, OFDM is the dominant modulation or one could call it as the physical layer, the signaling strategy that is employed in most modern day wireless communication systems. So, let us try to understand this better and see what is the relevance of the FFT and IFFT operations that we have just studied in this context.

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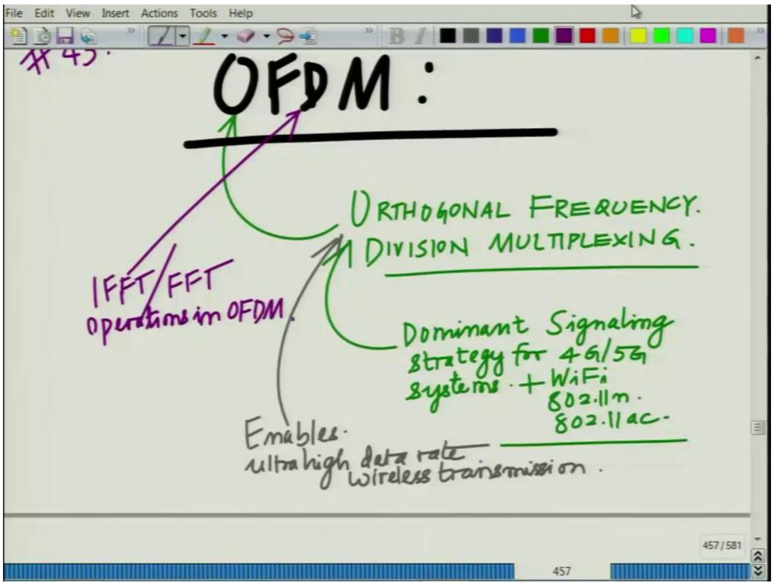

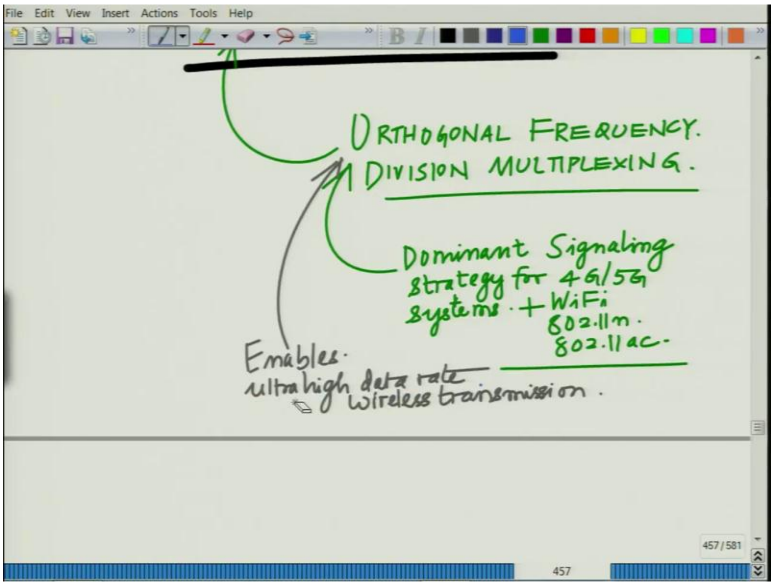

So, this is again a very interesting and are very relevant application of well IFFT and to begin with the IFFT and so, OFDM is Orthogonal Frequency Division Multiplexing which is the dominant, essentially the dominant for not only 4G but also 5G systems plus your Wi-Fi for instance your 802 dot 11 n, 802 dot ac, all these modern high data rate standards.

So, enables OFDM enables high data rate wireless transmission. In fact, I might add ultra-high. This OFDM enables ultra-high data rate wireless transmission and naturally this is key modern cellular networks and as well as Wi-Fi and the IFFT and FFT operations play a very important role in OFDM and that is what we want to understand, what we want to explore via this module is what is the role of the IFFT and FFT operations in OFDM.

In fact, there is a reason I mentioned IFFT followed by FFT, look at it not FFT followed by IFFT. We traditionally think of fast Fourier transform followed by the inverse but here interestingly, it is going to the IFFT followed by FFT and for reasons that are going to be clear as we go through this module. So, what essentially is OFDM? So, OFDM as I already told you enables ultra-high data rate.

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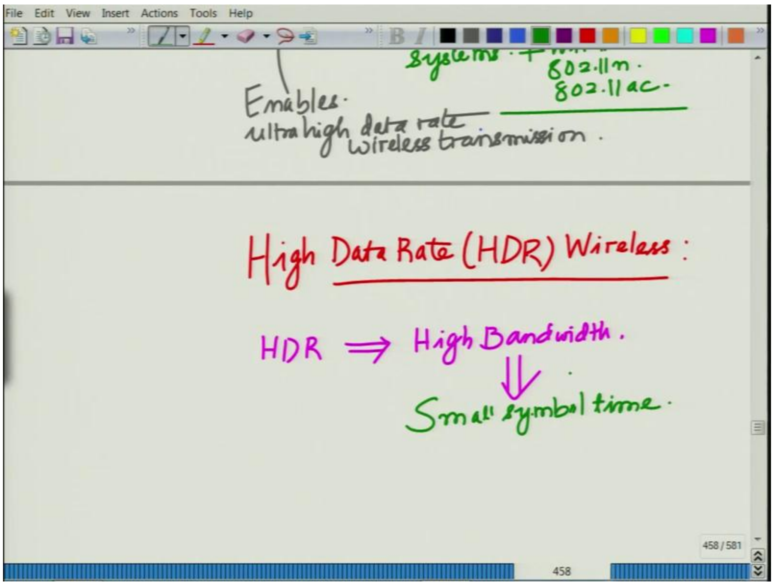

Now, what is the problem in the wireless system? So, if you look at a basic high data rate wireless system, what we call as an HDR high data rate, what is the HDR? What is the issue? The issue, the central issue of the fundamental problem here is, HDR, High Data Rate implies you have high bandwidth but the moment you have high bandwidth, this implies small, ultra-small symbol time. So it implies, so naturally if the data rate is rising, the symbol time is decreasing.

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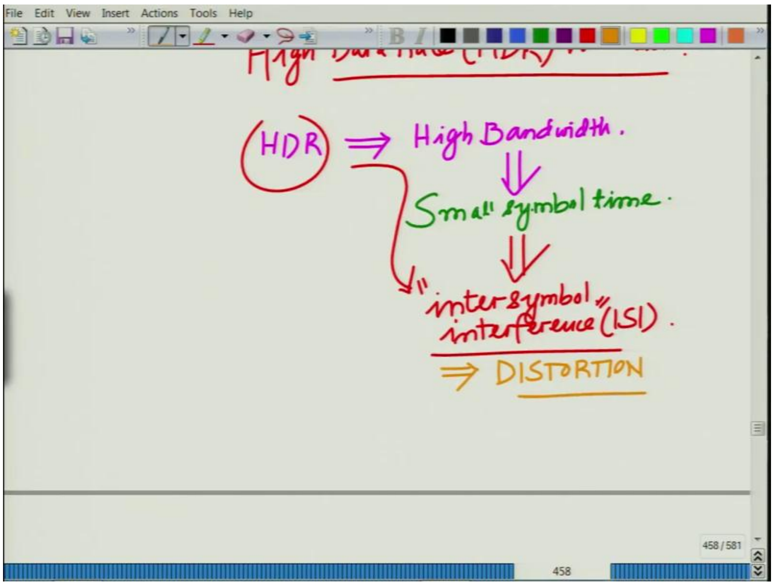

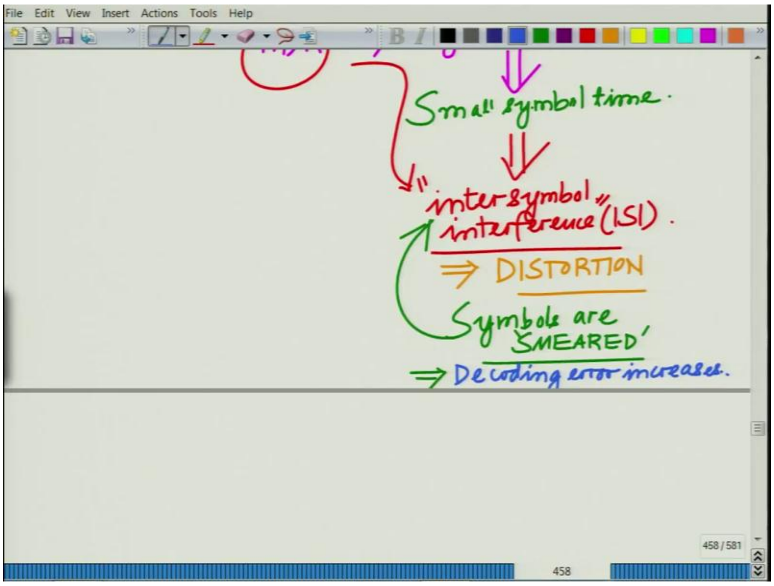

So this small symbol time now causes significant problems, small symbol time leads to inter-symbol interference which is also termed as ISI. So this is ultimately the problem that is your high data rate leads through this complicated, leads through this chain to inter-symbol interference and naturally inter-symbol interference implies distortion and so the long and short of the story is that is whenever you go towards a high data rate systems the symbol time becomes very small, the symbol time when, this symbol times becomes small, the channel sort of spreads these symbols, the channel sort of smears these symbols and these symbols start interfering with each other.

That is essentially inter-symbol interference which leads to destruction. Now once the symbols interfere with each other naturally it becomes very difficult to recover or reliably decode these symbols at the receiver. So this inter symbol interference roughly speaking if you want to get a physical feel for it the symbols, you can think of it this, as if the symbols are smeared, they interfere with each other implies decoding is very difficult or the decoding error increases.

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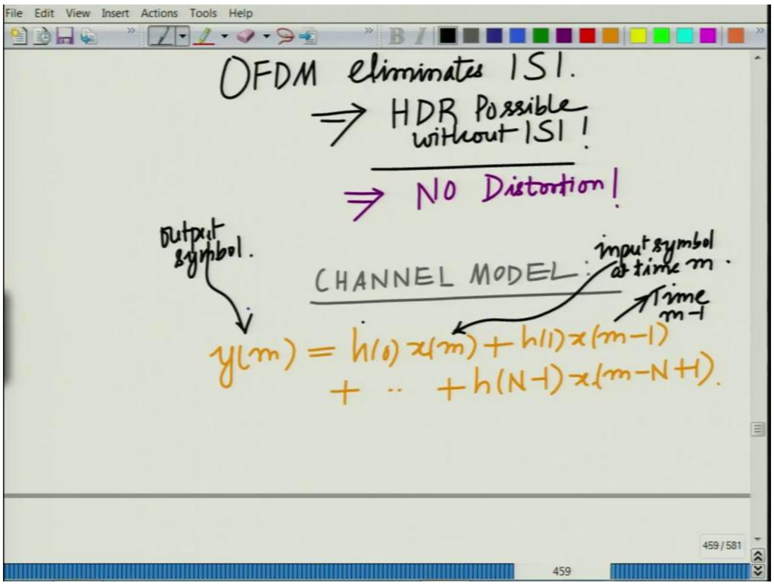

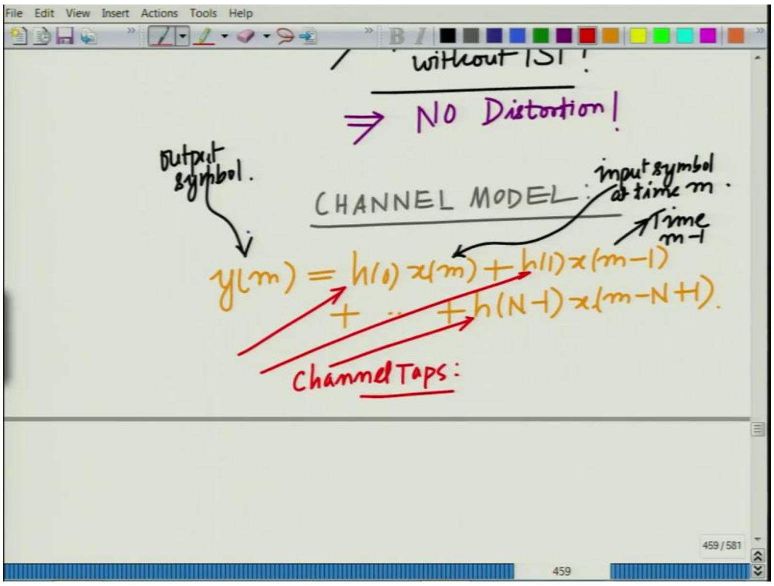

Now how to avoid this thing and OFDM does precisely that. OFDM is a strategy to overcome this, so enable so to put it in a nutshell OFDM is a strategy that enables high data rate transmission without inter-symbol interference and therefore there is no distortion and therefore it makes reliable transmission with high data rate possible. So what OFDM does is in a nutshell OFDM enables, avoids OFDM eliminates I think that is, implies HDR high data rate transmission possible without ISI and therefore this implies, no, this implies no distortion.

That is the advantage of OFDM and how does OFDM do that to understand that let us examine the model of the wireless channel. Let us assume the output of the channel, so what is the channel model? Let us now examine this and this can be understood in the following simple steps. Let us start with the channel model. The channel, this is the output symbol output symbol n. Let us call this as output symbol or output symbol m. This can be represented as h0 x m plus h1 x m minus 1 plus so on hN minus 1 x m minus N plus 1.

Now, what is y m? y m is the output symbol. Now, this x m is input symbol at time, similarly, x m minus 1 is input symbol at time m minus 1. So, this is at time you have the time m minus 1 and so on and these quantities h0, h1 so on hN minus 1 these are the channel taps. So, if you look at this h 0, h 1, hN minus 1.

Point here is that you see y m that is a symbol at time m depends not only on x m but also depends on past symbols x m minus 1, x m minus 2 so on. Therefore, these symbols are interfering with each other. There is interference in x m from the past symbols. Ideally you would like y m to depend only on x m so that you could decode x m but now look at this because of the high data rate that we are talking about y m depends on x m as well as there is interference from x m minus 1, x m minus 2 so on that is the past symbols of the inter symbol interference.

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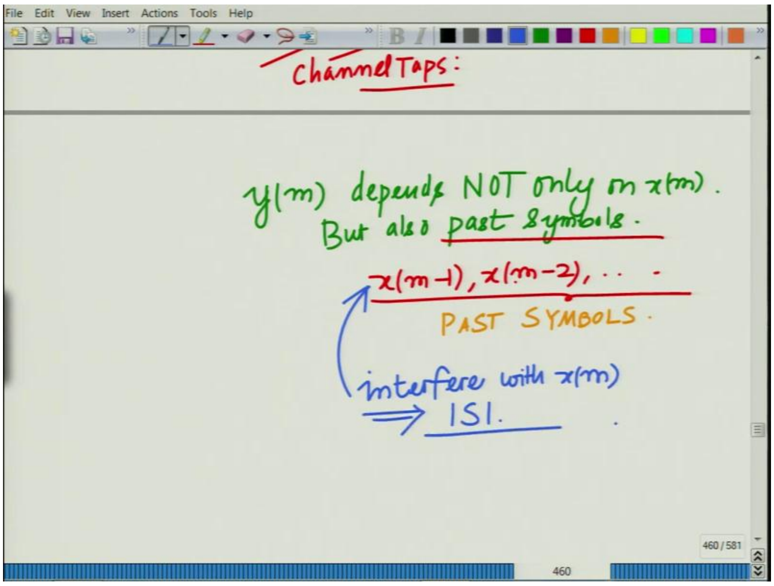

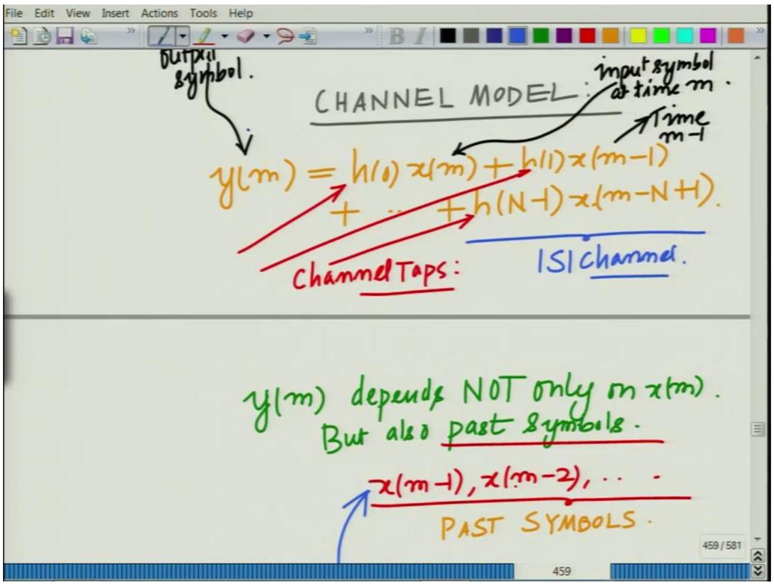

So, y m it can be seen easily depends not only on x m but also the past symbols x m minus 1. So, these are your past symbols which means there is inter symbol, this cause (inter) this interfere with x m, this implies, this is precisely ISI and this is essentially a model for your ISI channel. So, you have a model this is precisely what we are calling as the ISI channel. So, this is the channel which has inter-symbol interference, your y m depends on x m and there is interference from x m minus 1 and so on. So, how does OFDM (acquire), overcome this? What is the procedure that OFDM employs?

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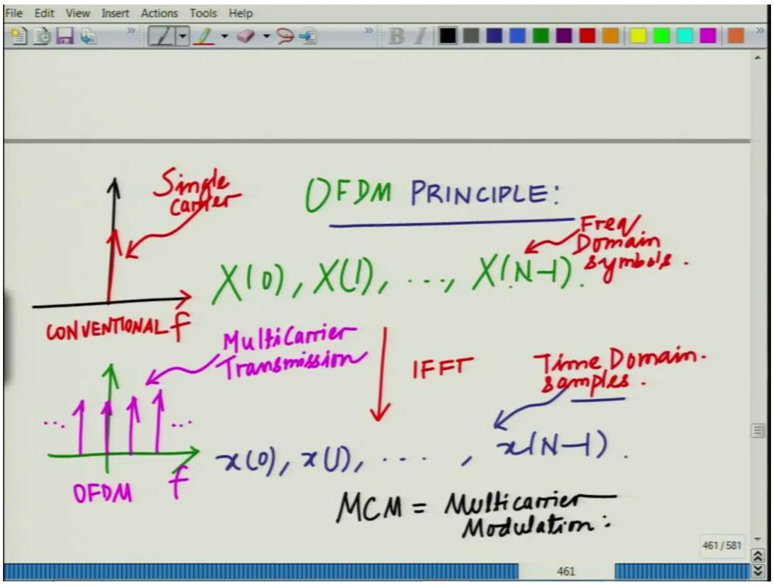

The procedure is as follows; it seems a little tricky to begin with. The point is we start in OFDM we start with x 0, x 1, x n minus 1. So, what is the OFDM principle? And we proceed with the IFFT of these symbols. So, at the transmitter, so these are you can think of this as the frequency domain symbols. And these are you get the time domain samples, that is we do not transmit the symbols directly but perform the IFFT unlike conventional communication transmit the time domain sample. So, these are the time domain samples.

These are the time domain samples which are eventually transmitted but before that we have to do something else. So the point is from that frequency domain symbols you get the time domain samples and this can be shown to be the principle here is the conventional transmission if you look at this conventional transmission. If you look at this conventional transmission, it can be shown that this corresponds to this is the frequency axis.

This corresponds to a single carrier transmission, that is the conventional transmission corresponds to a single carrier transmission, but in OFDM what is interesting is you move away from the single carrier transmission to essentially one that is multi carrier, so OFDM multi carrier.

So, OFDM has, so what is happening is this can be thought of as moving away from a single carrier transmission that you have in a conventional communication system which is centered at carrier frequency fc to an OFDM you move to a multi carrier system and that is the significant difference. So, OFDM is also known as an MCM system that is Multi Carrier Modulation. So this is also known as MCM this is equal to multi carrier modulation.

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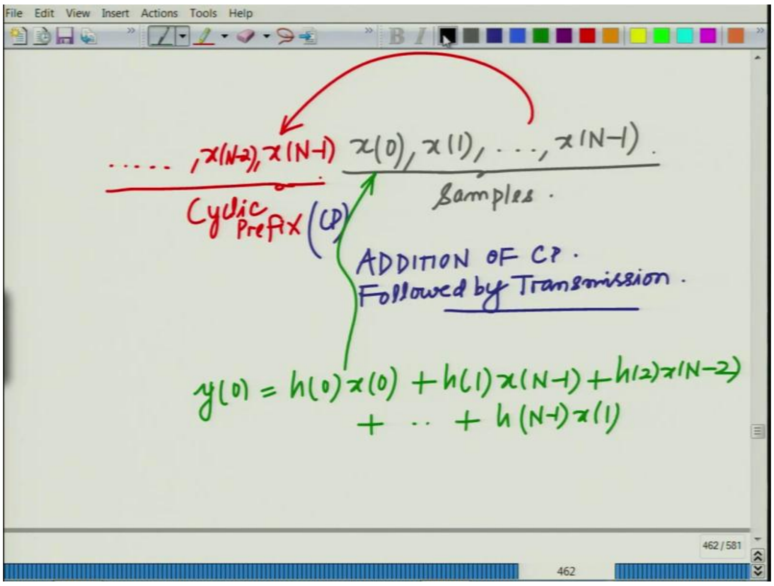

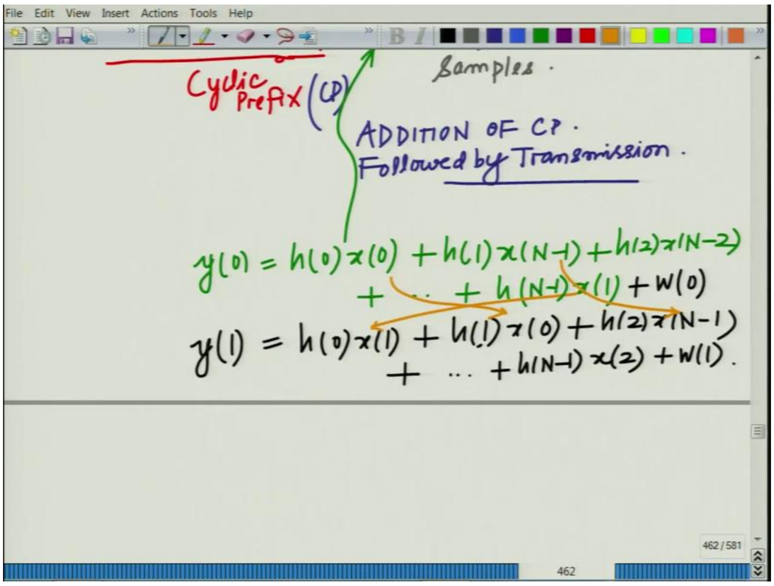

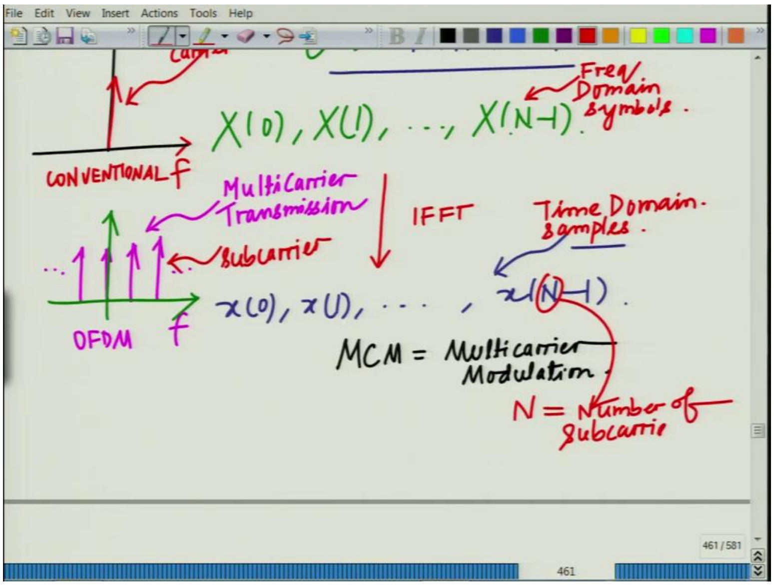

Now the point here is; now prior to the transmission of these samples, x 0, x 1 and in fact if you look at this figure N, this N has certain significance, this N is basically there is a number of carriers each of these is known as a sub carrier. So you have these multiple sub carriers now, N equal to the number of sub carriers, N denotes the number of sub carriers. So, N denotes the number of sub carriers and what is additionally done is this interesting thing where you have x 0, x 1 these are the time domain samples.

Now what is done is these are repeated again that is you take these things from here and you prefix them. This is known as cyclic, so samples of, this is a cyclical repeated so it looks gives the impression of a periodic sequence. So what you are taking is, if you are taking the samples, take some samples from the end put them again in the front so it looks as if it is a cyclically repeated sequence. This is an important step in OFDM. This process is known as the addition of a cyclic prefix.

This reputation itself is quantity that is prefixed with the head this is known as a cyclic prefix. So this process is known as the addition of a cyclic prefix. So this is essentially the addition so cyclic prefix is also abbreviated as CP. This is addition of CP and then follow; this is the addition of cyclic prefix followed by transmission. Now what happens when you transmit this? Now look at this.

Let us look at the output corresponding to this. So, we have y 0 equals h0 times x 0 plus h1 times the previous symbol, but what is the previous symbol? The previous symbol is x N minus 1 plus h2 times x N minus 2 plus so on. This would be hN minus 1 times x of you can see not very difficult to see hN minus 1 times x of 1. So, this is essentially what it would be plus of course, there is going to be noise, I am not concerned about that too much at this point that would be W of 0.

Now, let us look at one more y 1 equals corresponding to the next point y 1 equals h0 of x 1 plus h1 of the previous symbol but the previous symbol is x 0 plus h2 of x N minus 2 or rather N minus 1 that is a previous symbol to x 0 plus, plus hN minus 1 times x 2 plus W 1. Now, if you look at it all you can see all that has happened is x 0 has moved to the right, x N minus 1 has moved to the, everything has moved to the right, x 1 from the right has come to the leftmost top.

So, x 1 you can see what has happened, x 1 has come to the left and then x 0 has moved to the right. x N minus 1 has moved to the, so everything has moved progressively to 1 step to the right. So, you can see this is essentially a circular shift. So, you are taking x 0, x 1, x N minus 1 and you are circularly shifting this sequence. This is an important property.

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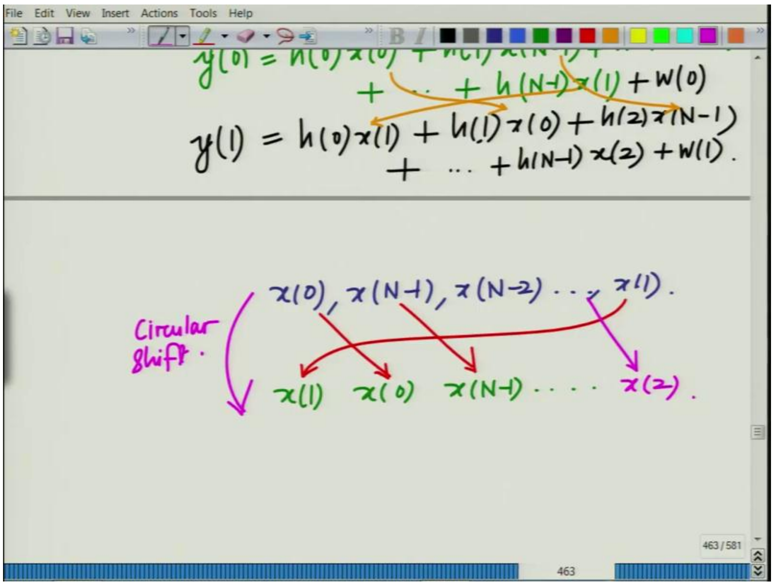

So, if you look at it, what is happening is you have x 0, x 1, x 2 or let us write it this way rather. x 0, n N minus 1, x N minus 2 so on up to x 1. What is happening is next instant x 1 or you can say x 1 comes over here and x 0 goes to the next part, x N minus 1 goes to the next part. So, this is your x N minus 1 so on and ultimately here you have x 2.

So, this is essentially a circular shift from step to step and it is a circular shift, it is not a push out, it is not a linear shift because you are not simply pushing out the last one, but getting the last one bringing it back to the front that is in fact your x 1. So, this is a circular shift and this process is essentially what is known as a circular convolution. Now, when we represent it as a linear transform, we are going to observe something very interesting.

So, essentially this is a circular convolution. So, every step there is a circular shift which implies there is a circular convolution. This is not very important at this stage because we are going to look at it in terms; now if we represent this as a matrix now, this can be not let us look at what is going to be the matrix form of this that is going to be very interesting.

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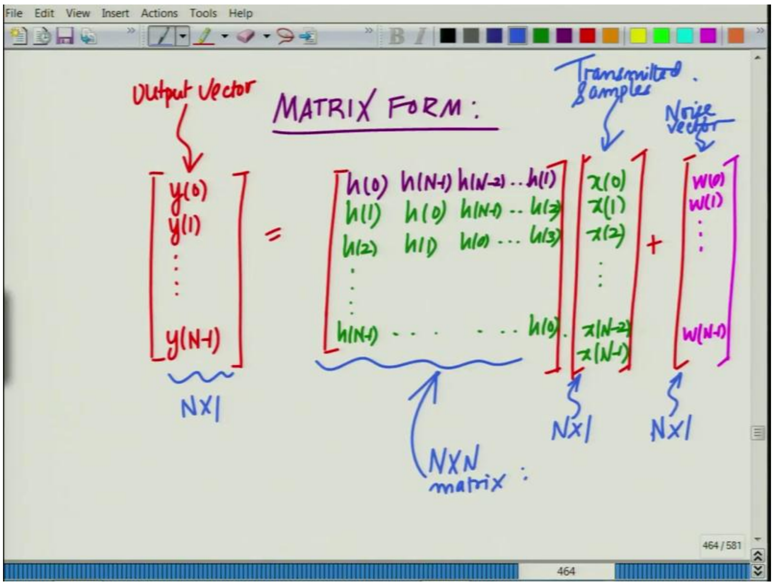

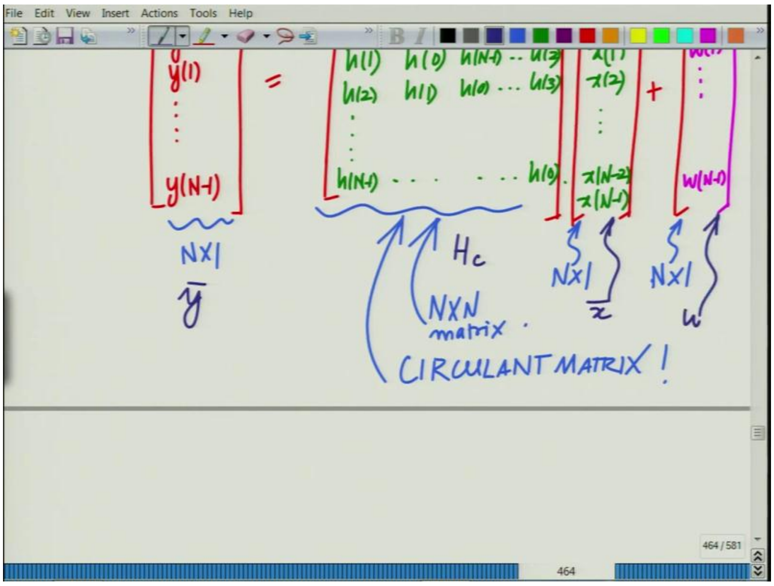

So, what is the matrix? The matrix form of this is very interesting, the matrix form if you write this as y 0, y 1 up to y N minus 1 times which is essentially this is the output vector. So, this is essentially your, this is equal to a matrix times the input vector plus of course, the noise vector that is your W 0, W 1, W N minus 1 and now if you write this input symbols, these are going to be your x 0, I am sorry, these are the samples x 0, x 1, x 2, x N minus 2, x N minus 1 and if you look at this matrix, this is going to be something that is very interesting.

The first row is going to be h0 times x 0, hN minus 1 times x 1 hN minus 2 and so on, last entry is going to be h1 times x N minus 1, then your next row is going to be h1 comes to the left as we have seen, then h0 moves to the right, hN minus 1, h2 then you are going to have h2, h2 will come from the right, then you will have h1, then you will have h0, you have h3 so on and so forth. If you look at this, the last very last row will be hN minus 1 anyway, you do not need to worry too much about this and this is going to be h0.

Now, you see this matrix this is, now let us look at the dimensions. This is an N cross 1 vector, this is an N cross 1 vector, this is an N cross 1 vector, this is output vector, these are the samples, these are the transmitted samples and this is the noise vector. Now, if you look at this N cross N matrix, you will observe something very interesting. You will observe that in this matrix there is something very interesting.

Now, first column look at this is h 0, hN minus 1, hN minus 2 so on. Second column h1 comes from the right, every other element h1 comes from the right to the left most part. Every other coefficient is pushed one spot to the right. Then the next row h2 comes from the right, the right most to the leftmost position and every other coefficient is pushed one spot to the right. So, if you look at it each subsequent row is obtained by circularly shifting the previous row, in fact, same thing can be said for the column.

First column is h 0, h 1, hN minus 1 push it down by one coefficient, bring hN minus 1 back to the top next one is hN minus 1, h0, h1 next one be hN minus 2, hN minus 1, h0, h1, h2 so, each column is obtained by circularly shifting the previous column and therefore, this matrix has a unique structure this is known as a circulant matrix. This is a very interesting aspect. So, this is known as so, I am going to this kind of a matrix is known as circulant matrix. This is a very interesting aspect. So, let us call this as your y bar, let me call this matrix as H c, let me call this as x bar, let me call this as w bar.

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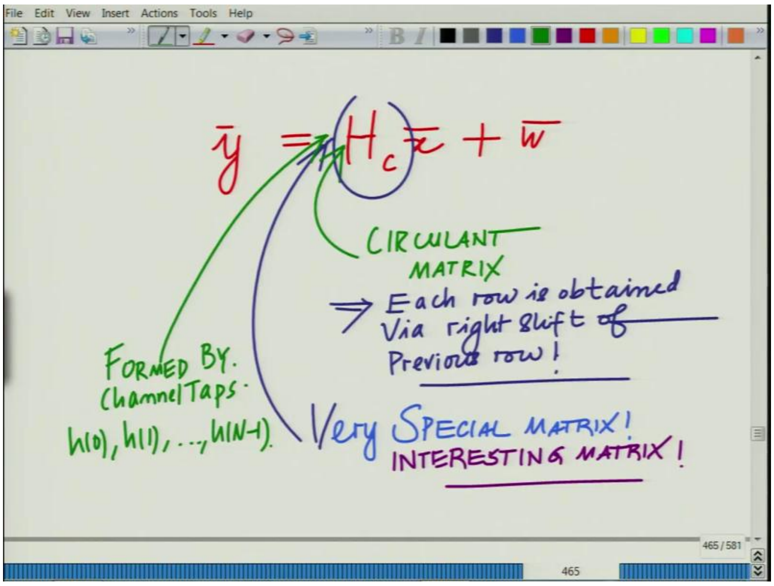

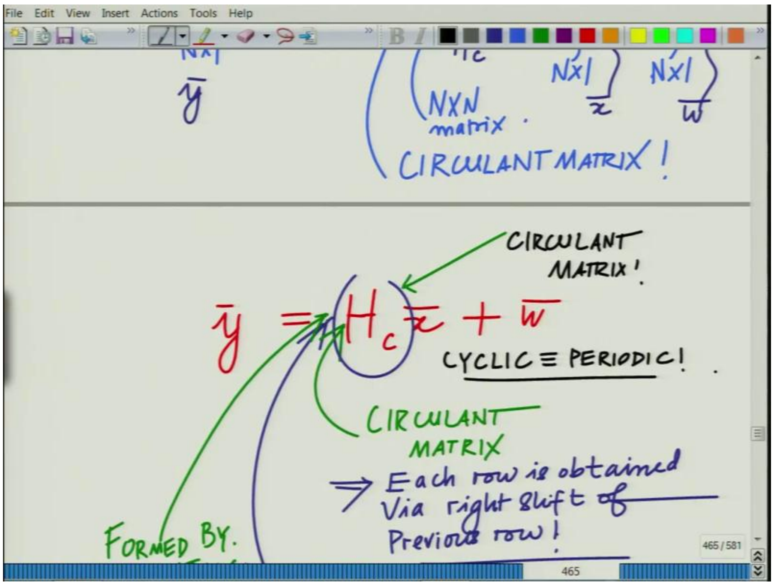

And now, if you look at this I can write this as y bar. In fact, I can write this as y bar equal to H c times x bar plus w bar and H c this is a circulant matrix, which means that each row is obtained by shifting, right shifting the previous row. Each row is obtained via, this is a very special matrix. We are going to discover its properties, but I am going to write this right away. This is very, very special matrix and I would like to say, as a mathematician, as an engineer a very interesting matrix.

You would agree it is not every day that you see a matrix like this, does not arise very naturally. You have every row circular the shift of the previous row. So in that sense, there is only one unique row, every other row is circularly shift of the previous row. So second row is circulation to the first row, third row you get by self. So if you know the first row you can write down, in fact, if you know any row you can write complete the entire matrix.

In fact, if you know any columns for the same logic, you can complete the entire matrix and that is true because all the entries are essentially formed from the channel coefficient. So if you know the channel filter, that essentially gives you the matrix. So, this is formed from the channel taps. So this is formed from the channel taps and has a very interesting, formed by the channel taps, which are y 0, y 1 up to I am sorry, these are not the channel taps, these are the output so these are the channel taps are h0, h1.

Now, the point is other minor point here is what happens if the number of channel taps is not equal to N, the number of sub carriers typically the normal channel taps is much smaller, the rest are replaced by 0, so you simply 0 pad. So in case you are wondering if you know a little bit already about OFDM and if you are assuming that, you know the number of sub carriers is usually about the order of 100s, channel taps might be of the order of 10s that the rest of the channel taps can always, I can always replace them by zeros, so it is known.

So, to make things simple, I have simply assumed that the number of channel taps is equal to the number of sub carriers but this is in no way a limiting assumption. This is essentially something that is I would like to say a generalized or a simplified depending on which way you would like to or which viewpoint you would like to take. Now, this matrix, but our bone of contention or admire or gaze at this point, rests or gaze is transfixed on this matrix, this wonderful matrix, I would like to say which is and I cannot talk enough of this.

This is this circulant matrix I think I have already said it a couple of times, because it has indeed if you look at it a very, very interesting structure and this structure has a, and naturally you see, when you see things that are cycles, when you see things that are periodic like cyclic means nothing but essentially cyclic has essentially some relevance to this notion of periodic. The moment you say cycles, remember what is cycles?

Cycles is nothing but a unit of frequency. Cycles per second something is cycling. So, cyclic naturally means there is something that is periodic and periodic naturally must inspire in you as an engineer, no matter which field of engineering or science management you are from, the Fourier part of your brain, the thing that talks about sinusoid, sines and cosines because we think of periodic waveforms, the fundamental I mean, the fundamental waveforms that come to your mind are basically sines and cosines.

In fact, every other periodic signal can be expressed as a linear combination of these sines and cosines. That is the wonderful thing that Fourier theory or harmonic theory tells us. The moment you have something that is cyclic, something as periodic, so naturally, the IFFT and FFT have a very important role to play in this system. Now, what is that roll that is something that I would like to keep aside for the next module.

So, we will stop here, look at this structure of this cyclic matrix. Appreciate it, admire it. Try to assimilate it, try to understand it, of course we are going to dissect it and we are going to scrutinize it at much greater depth in the subsequent modules and all this naturally ties into OFDM. Thank you. Thanks very much.