Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning

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Lecture: 4

Matrices: Definition, Addition and Multiplication of Matrices

Hello, welcome to another module in this massive open online course in this module let us start looking at another very important component or another very important topic in linear algebra and that is matrices. So far, we have looked at vectors, now, let us start looking also at matrices, alright?

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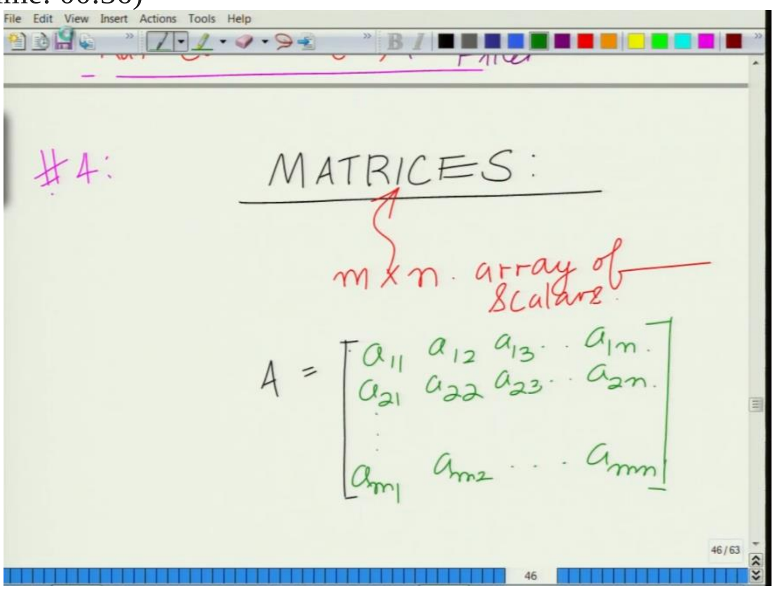

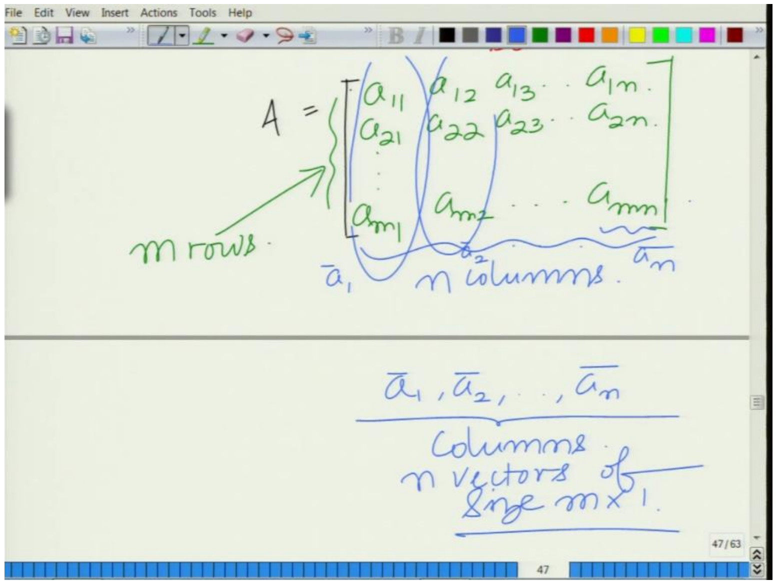

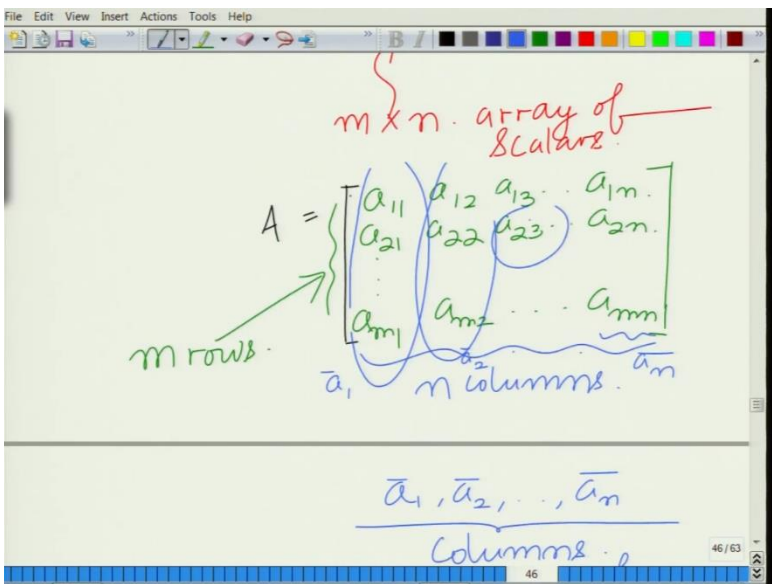

So, obviously when we talk about linear algebra which is essentially a lot of discussion about matrices. Sometimes linear algebra is also known as matrix algebra for the same reason. So, you have an $m \times n$ matrix, which is simply an $m \times n$ array of scalars So, of course, if n equal to one then it becomes a vector, but for a general $m \times n$ array it becomes a matrix.

So, you have

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

and therefore, this is an $m \times n$ matrix in the sense that this has m rows, you can see that this has m rows and the n columns. And you can essentially look at each column also as a vector which is going to be very helpful. So, we can write or represent this as

$$\mathbf{A} = [\bar{\boldsymbol{a}}_1 \quad \bar{\boldsymbol{a}}_2 \quad \dots \quad \bar{\boldsymbol{a}}_n].$$

So, you can think of this as putting together of n columns of our n vectors of m elements each resulting in the matrix **A**. So, you can also think of this as $\bar{\boldsymbol{a}}_1, \bar{\boldsymbol{a}}_2, \dots, \bar{\boldsymbol{a}}_n$, these are the n columns or these are the n vectors of size $m \times 1$. So, m is the number of rows and n is the number of columns and a_{ij}, for instance if you look at this entry a_{23}, it is the entry in the second row and third column.

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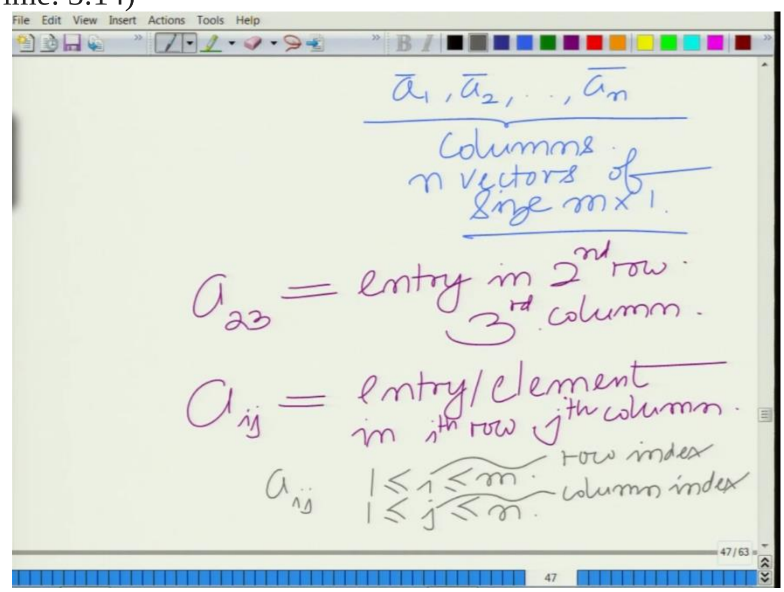

And a_{ij} in general equals the entry or coefficient whatever you call it, entry or element in ith row and jth column of the matrix **A**. So, you have

$$a_{ij}, \text{where } 1 \leq i \leq m, 1 \leq j \leq n.$$

So, this is essentially your matrix, this is an $m \times n$ matrix.

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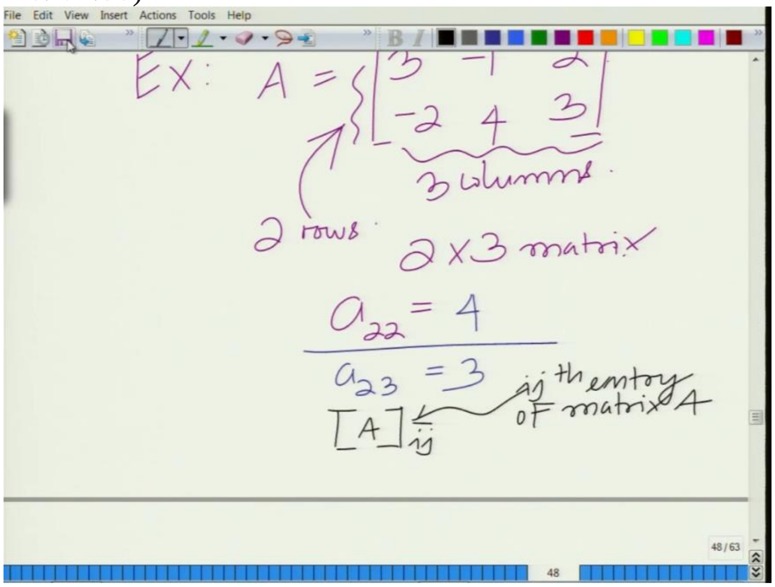

Let us take a simple example again, there are many examples, I mean most of you might have already encountered this many times. So, let us take a simple example

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 \\ -2 & 4 & 3 \end{bmatrix}.$$

So, this is naturally, this has 2 rows and 3 columns. So, this is 2×3 matrix. And if you look at the entry a_{22} that is second row second column, this is equal to 4 and so on and then a_{23} equals second row third column that entry is 3 and so on and so forth, right?

So, it is a very, and another way of representing this is, you can also represent this as the ijth entry is $[A]_{ij}$. So, this is another way of representing ijth entry of the matrix $\mathbf{A}$.

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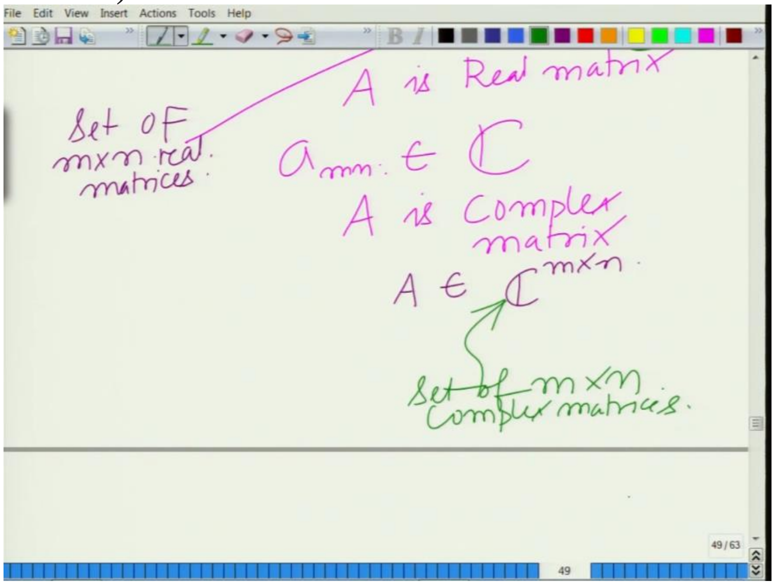

Also, equally apparent if a_{mn} is real that is all elements are real, then **A** is a real matrix. If a_{mn} is complex that is all elements are complex, then **A** is a complex matrix. In fact

$$A \in \mathbb{R}^{m\times n}, \text{where } \mathbb{R}^{m\times n} \text{ is the set of } m \times n \text{ real matrices.}$$

$$A \in \mathbb{C}^{m\times n}, \text{where } \mathbb{C}^{m\times n} \text{ is the set of } m \times n \text{ complex matrices.}$$

So, this is you have the set of $m \times n$ real matrices and you have the set of $m \times n$ complex matrices.

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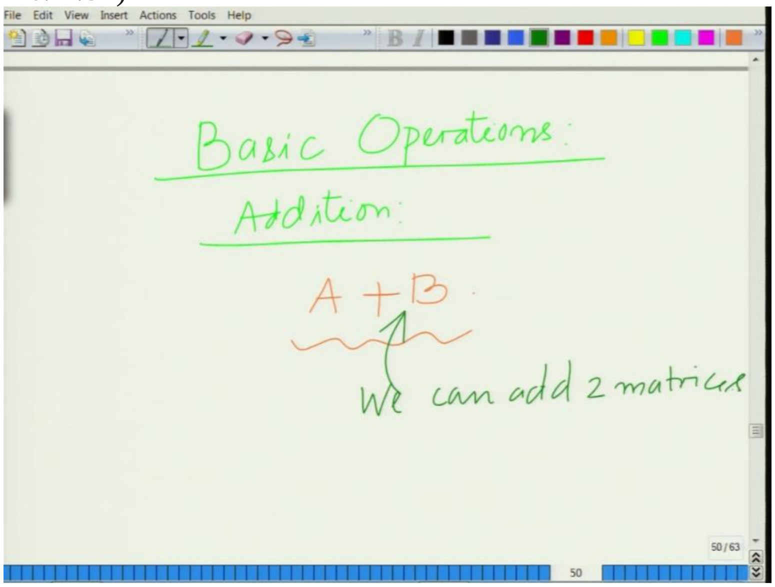

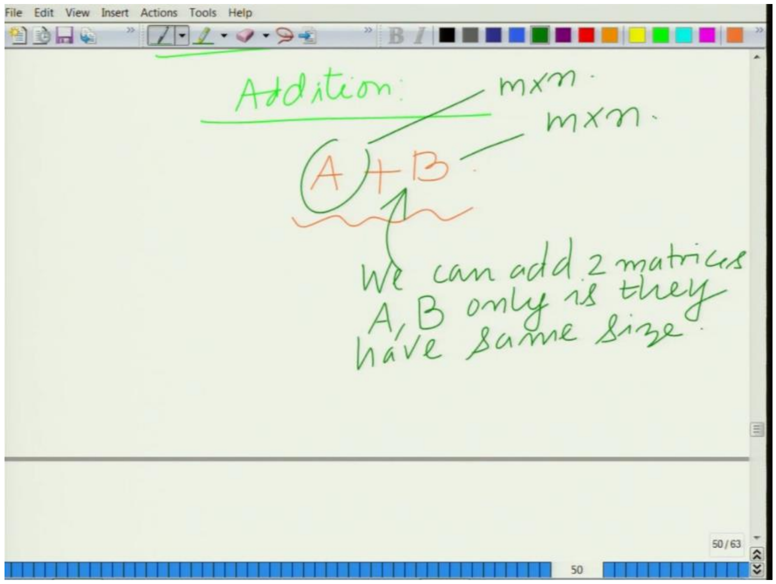

Let us look at some basic operations. Again, many of you might already be familiar with these basic operations. That is, for example, let us look at matrix addition what do we mean by matrix addition **A** plus **B?** when can you add two matrices?

So, we can add two matrices **A** and **B,** only if they have same size or essentially same dimension that is **A** is $m \times n$ and **B** is also $m \times n$**.** So matrix addition is only defined when both the matrices are of the same size otherwise, you cannot add the matrices and the addition is simple, you add them element wise. So, we add the $[\mathbf{A}]_{11}$ with $[\mathbf{B}]_{11}$ that gives you the $(1,1)$th element of the addition as $[\mathbf{A}+\mathbf{B}]_{11}$.

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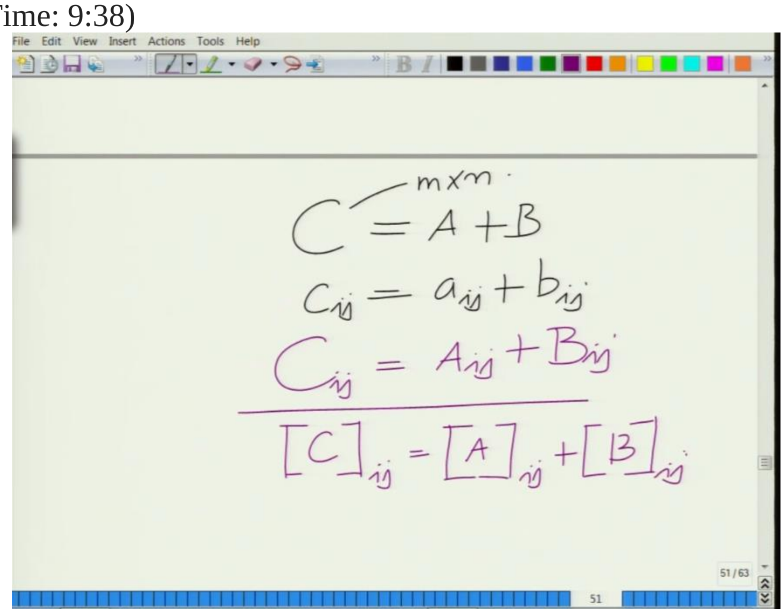

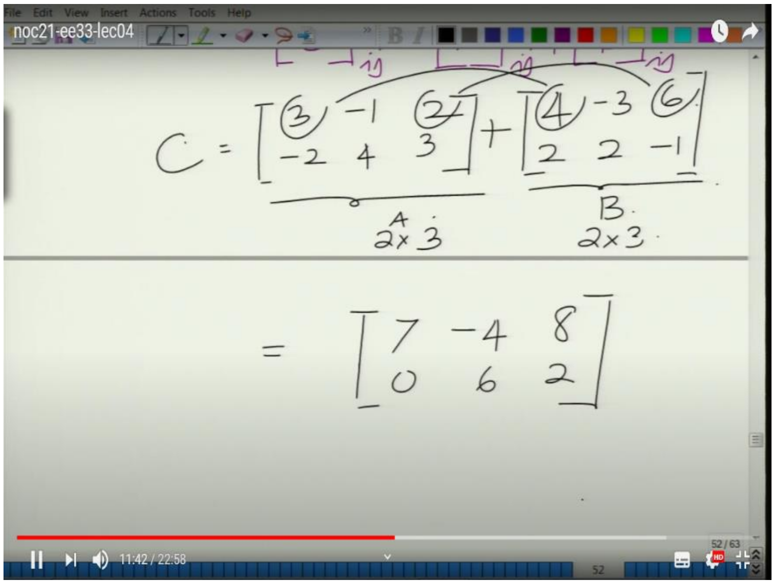

So, the way to present this is $\mathbf{C} = \mathbf{A} + \mathbf{B}$, and the matrix $\mathbf{C}$ will also be $m \times n$. And you can simply say, each element c_{ij} equals ijth element of $\mathbf{A}$ plus ijth element of $\mathbf{B}$ or you can also write as $c_{ij} = a_{ij} + b_{ij}$ or $\mathbf{C}_{\mathrm{ij}} = \mathbf{A}_{ij} + \mathbf{B}_{ij}$.

I mean, all of these are, depending on how you define the notation, the content, I mean, all of these are valid. I mean, as long as it is clear what you are talking about, all of these are valid. And you can also write this as $[\mathbf{C}]_{ij} = [\mathbf{A}]_{ij} + [\mathbf{B}]_{ij}$.

Let us take a simple example, we have our matrix $\mathbf{A}$, which is essentially $\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 \\ -2 & 4 & 3 \end{bmatrix}$, plus our other matrix B, which is let us say $\mathbf{B} = \begin{bmatrix} 4 & -3 & 6 \\ 2 & 2 & -1 \end{bmatrix}$.

So, both of these are essentially you can see these are both 2×3 matrices. So, you have $\mathbf{C}$, which will be called to add them element wise. And this is essentially your matrix $\mathbf{C}$

$$\mathbf{C} = \begin{bmatrix} 3 & -1 & 2 \\ -2 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 4 & -3 & 6 \\ 2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -4 & 8 \\ 0 & 6 & 2 \end{bmatrix}.$$

So you are essentially adding the elements of $\mathbf{A}$ with the corresponding element of $\mathbf{B}$ and you get the matrix $\mathbf{C}$, which is of the same size as $\mathbf{A}$ and $\mathbf{B}$.

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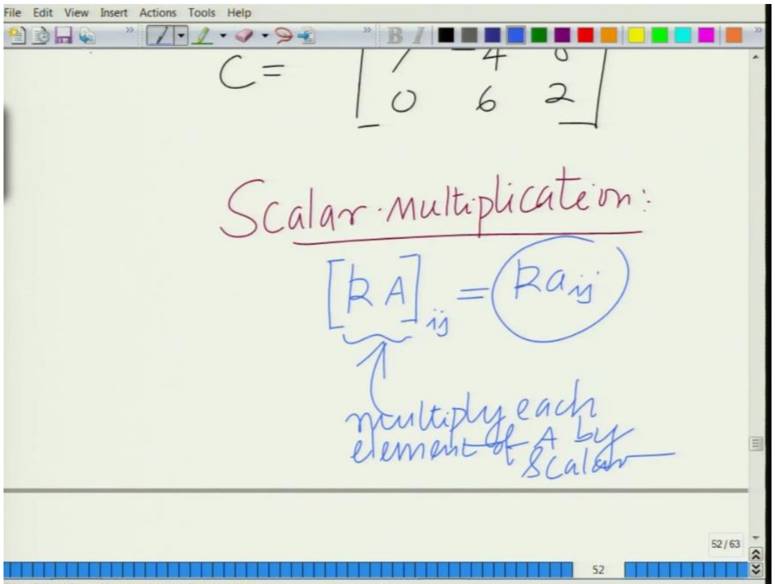

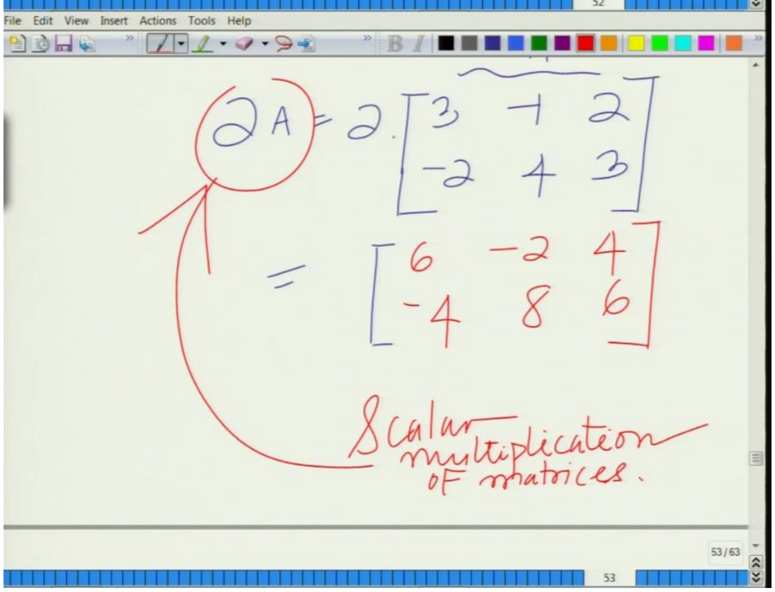

Scalar multiplication again very simple. So, you have the concept of scalar multiplication that is similar to vectors. Once again, if you multiply a matrix **A** by a scalar k, essentially it means that if you look at its ijth element. That is simply k times a_{ij} that is take each element of matrix multiplied by the scalar, right? So, what this is doing basically is that it multiplies each element of **A** by the scalar k.

Let us take a simple example, two times **A** that is

$$2\mathbf{A} = 2\begin{bmatrix} 3 & -1 & 2 \\ -2 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 4 \\ -4 & 16 & 9 \end{bmatrix}.$$

So, you take each element of A multiplied by that scalar quantity, which in this case is two so that is the property of scalar multiplication of matrices.

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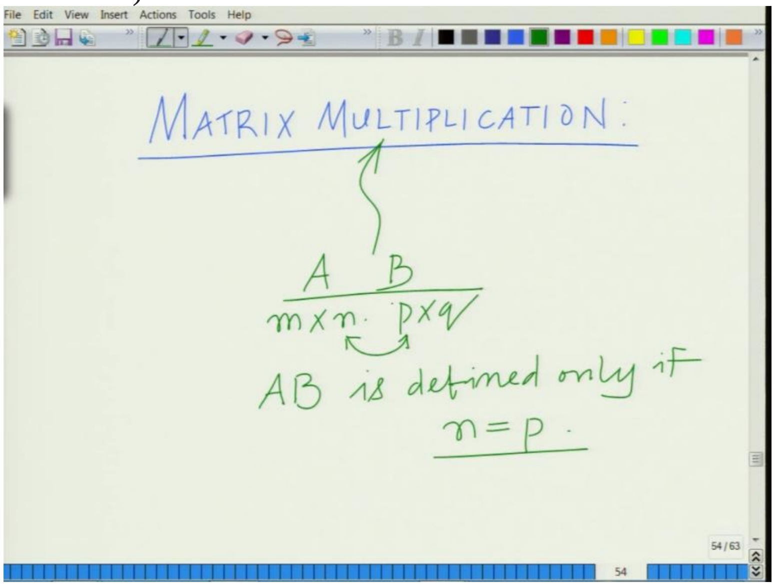

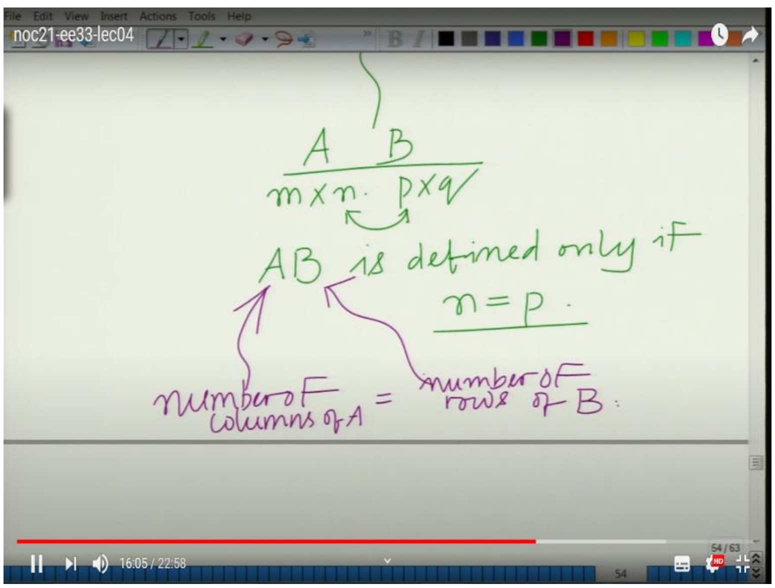

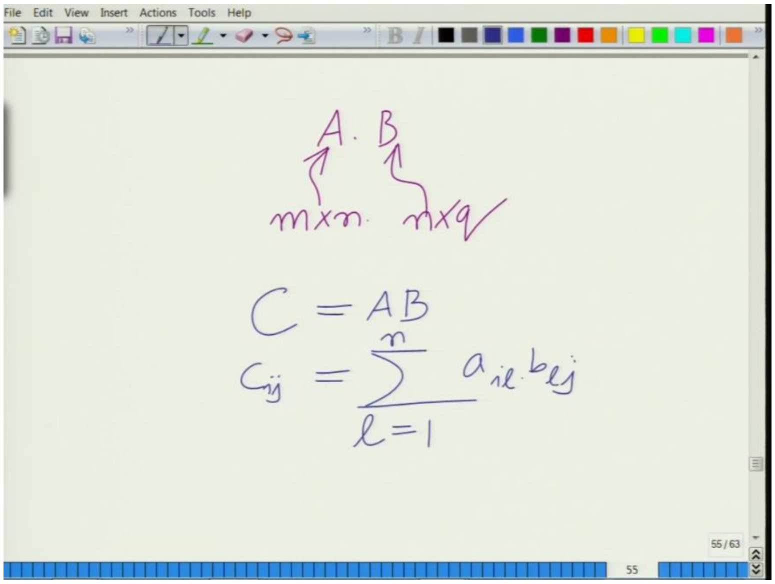

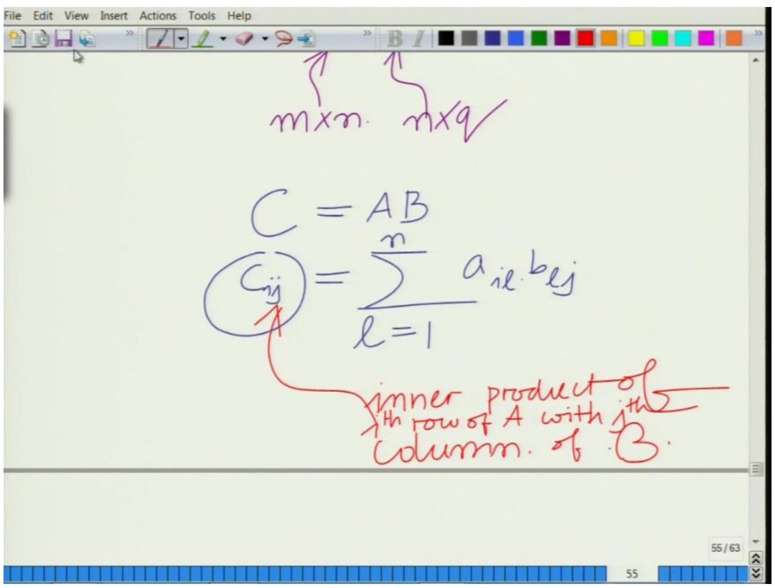

Let us now consider matrix multiplication. How and when can you multiply two matrices. So, this is and let us consider again two matrices, **A** is $m \times\ n$ and **B** is $p \times\ q$. Now, you can multiply **A** and **B** only if this n is equal to p. So, **A B** or **A** times **B** is defined only if n is equal to p that is number of columns of first matrix should be equal to the rows of the second matrix. So, that is number of columns of **A** should be equal to number of rows of **B**. So, that is the important condition that need to be satisfied. So, now we have **A** and **B**, as we said **A** is $m \times\ n$, and **B** is $n \times\ q$. So, number of columns of **A** is equal to number of rows of **B** which is n, then the product **C** equal to **AB** is defined as follows, that is

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}$$

So, essentially what you can see is that the c_{ij}, the way to look at this is, this is the inner product inner product of ith row of **A** with the jth column of **B**.

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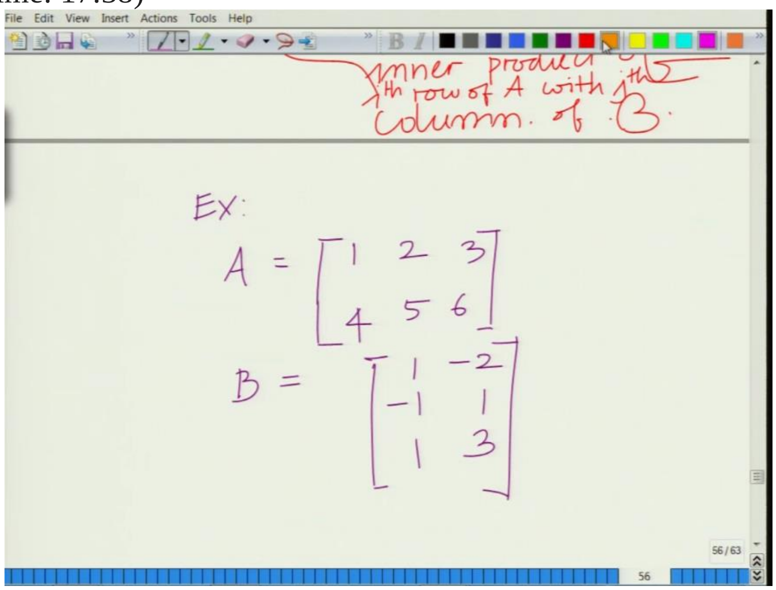

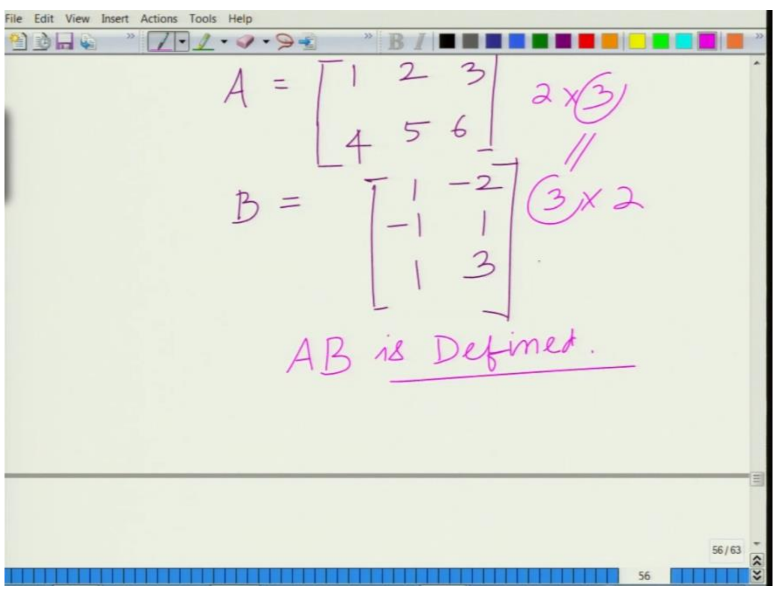

Let us take a simple example to look at this. For instance, let us say we have **A** equals the matrix that is your $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 3 \end{bmatrix}$, these are two matrices. And now you can see **A** is 2×3 and **B** is 3×2. So, the number of columns of **A** equals the number of rows in **B**.

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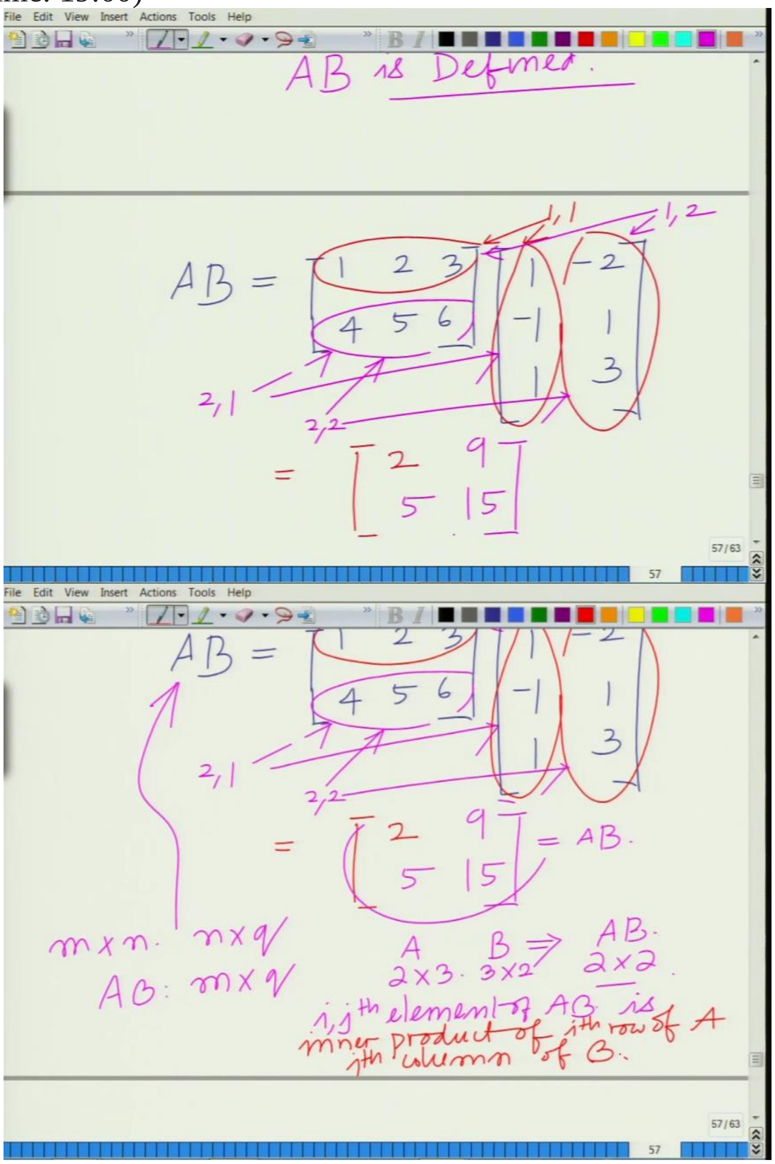

And then, now, let us look at how to compute **A** times **B**. so this will be equal to, essentially we take so for the (1,1) element, we take the inner product that is the first row of **A** with the first column of **B**. So, if you put these two together inner products of these two that gives the (1,1) element would be $1 \times 1 + 2 \times -1 + 3 \times 1$, so that would be equal to 2.

Now, the (1,2) element would be the inner product of first row of **A,** and the second column of **B**. So, that would be $1 \times -2 + 2 \times 1 + 3 \times 3 = 9$. Now, the inner product of the second row of **A** and the first column of **B** that gives you the $(2,1)$th element. So, this gives you $4 \times 1 + 5 \times -1 + 6 \times 1 = 5$. So, that is 5 and finally, the second row of **A** and the second column of **B** that gives you the $(2,2)$th element, that is, $4 \times -2 + 5 \times 1 + 6 \times 3 = 15$. So, **A** is $m \times n$ and **B** is $n \times q$, so **AB** is of size $m \times q$. So, **A** in this example, is 3×2, **B** is 2×2, this implies **AB** is of size 2×2. So, this is essentially your **AB**.

And as we have seen each ijth element of AB is the inner product of ith row of **A** and jth column of **B**. So, we will stop. So, in this aspect, we have looked at the basics of matrices, introduction, properties of matrices, addition scalar multiplication, multiplication of matrices and so on. So, let us stop this module here. And we will continue in the next module. Thank you very much.