Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
Professor Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur
Lecture – 36
LMMSE Estimate and Error Covariance Matrix

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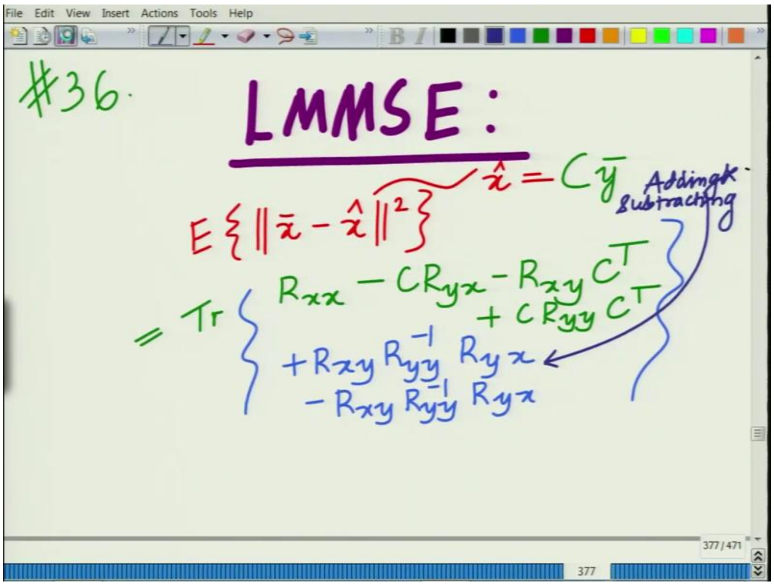

Hello, welcome to another module in this massive open online course. So, we are discussing about the LMMSE estimator that is the linear minimum mean squared error estimator that is given a quantity y bar how to estimate how to find the best linear estimate of another correlated vector x bar such that the mean squared error is minimized, so, that is the LMMSE principle, which as I have already told you has several applications in the context of estimation, so we are talking about the LMMSE estimator.

And what we have shown so far is that I have norm of expected value of norm of x bar minus x hat whole square where x hat remember this is a linear estimator, this is of the form C y bar, this can be expressed as, well this can be expressed as a trace of our x x minus trace of R x x minus C y x minus R x y z C transpose plus C R y y C transpose and to this what we are doing is we are adding and subtracting this quantity plus R x y R y y inverse R y x minus R x y R y y inverse R y x. So, we are adding and subtracting this quantity essentially,, and it minimizes the trace of this whole quantity, so adding and subtracting, so adding or subtracting this quantity.

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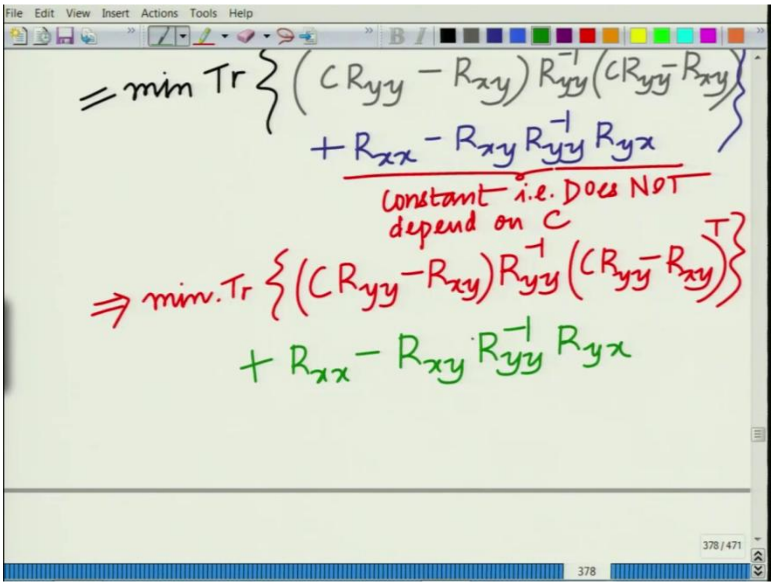

And therefore, now if we want to minimize this, that becomes equivalent to minimizing this quantity that we have just written over here. And this you can see essentially, if you examine this deeply, what we are trying to do is essentially we are trying to complete the square over here as you are going to see and you can write this you can check this, this can be written in a very nice compact form as minimize trace C R y y minus R x y into R y y inverse C R y y minus R x y transpose plus R x x minus R x y R y inverse R y x. So, this can be simplified in this fashion and now you can see that this quantity here if you look at this quantity, this is a constant in the sense that does not depend on C.

Remember we are trying to find the best C that minimizes the mean square error, constant that is does not this does not depends on C, implies I can write this as minimize trace of C R y y minus R x y times R y y inverse C R y y minus R x y transpose plus R x x minus x y times R y y inverse R y x which is a constant. So, this comes out of the minimization and you are left with this first step essentially, because the second term is essentially a constant that does not depend on the matrix C.

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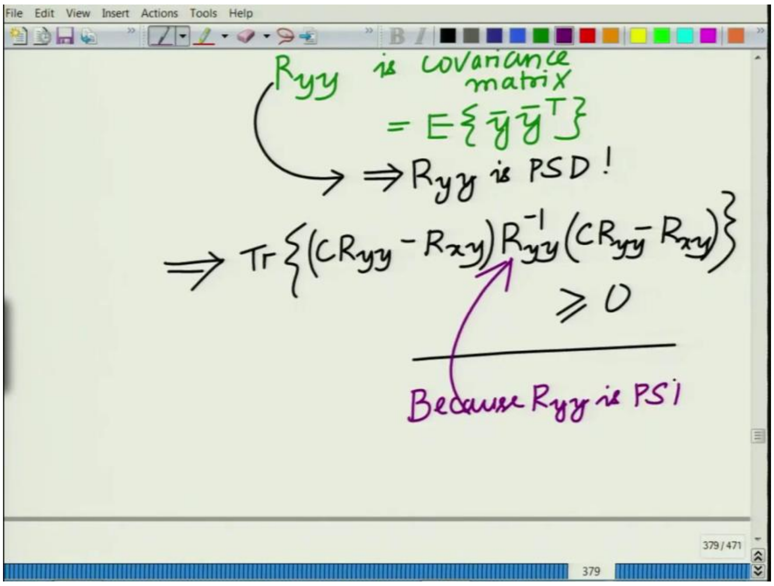

And now, if you look at this to minimize this quantity, note that this quantity if you look at this this R y y inverse this is interesting because R y y remember R y y is a covariance matrix. This is basically if you remember, this is basically expected y bar y bar transpose. So this essentially implies that remember any (posi) covariance matrix is positive semi definite, this implies that R y y. So, this implies that R y y is a positive semi definite matrix, this implies that this quantity that is trace of C R y y minus R x y into R y y inverse C R y y minus R x y, this quantity is always greater than or equal to 0 because R y y is PSD

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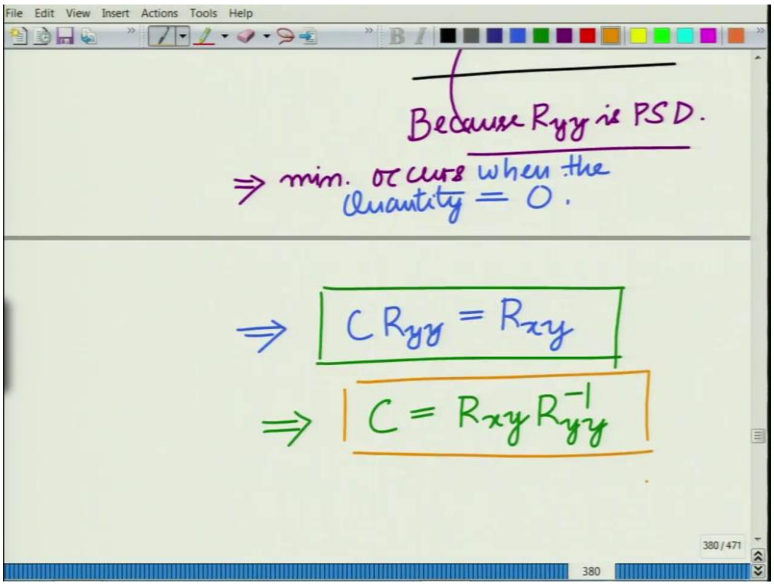

Implies minimum implies minimum of this occurs for above occurs when the quantity equal to 0, when the quantity is equal to 0, that is obvious because it is always greater than equal to

0 so the minimum value of this is naturally this is equal to C. And therefore, now for this to be minimum that occurs when does the minimum occur, if you remember it is a positive semi definite matrix minimum occurs only when C R y y equals implies C R y y equals R x y which essentially implies that C equals R x y times R y y inverse implies C equals R x y into R y y inverse.

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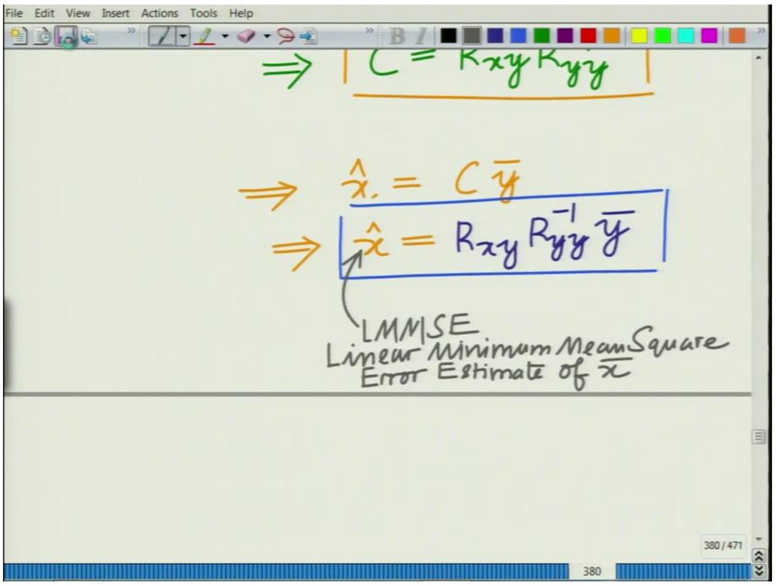

And therefore, this implies now that x hat equal to C y bar which implies x hat the best that is the linear minimum mean squared error estimate of x bar is R x y R y y inverse into y bar. This is the LMMSE estimator. So, it is very simple. It is a very simple structure, this is the LMMSE that is again linear Linear minimum mean squared error estimate of x bar. So that is the LMMSE estimate of x bar.

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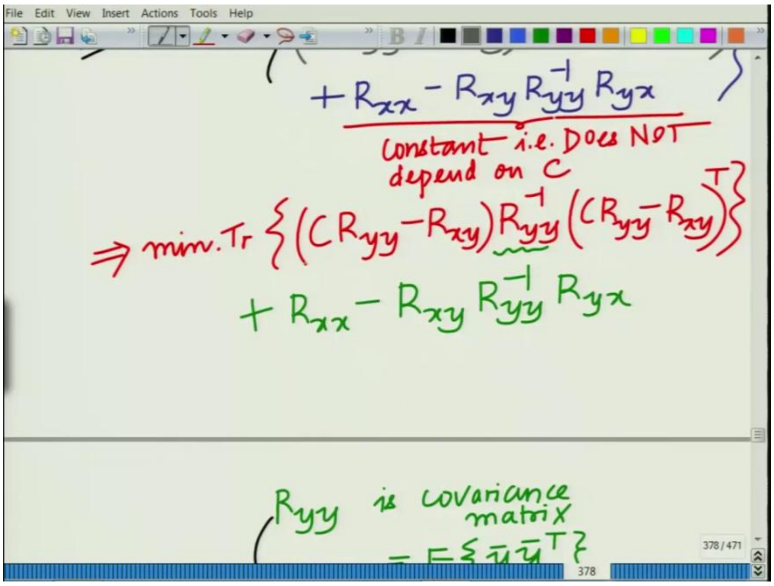

And what is the minimum error corresponding to this? The minimum error corresponding to this if you look at this remember the first term is reduced to 0 if you take a look at this, the first term this term can be reduced to 0, this term reduces to 0 implies error equal to the second term.

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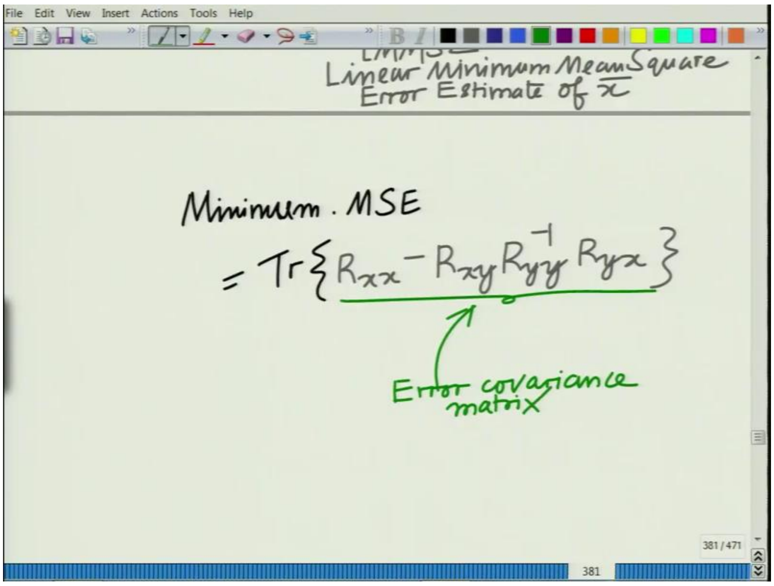

So, therefore, the minimum error if you very easy to deduce the minimum MSE becomes trace of trace of R x x minus R x y R x x minus R x y times R y inverse R y x this is the minimum MSC and in fact if you look at this, this is nothing but the error covariance matrix, matrix for the LMMSE.

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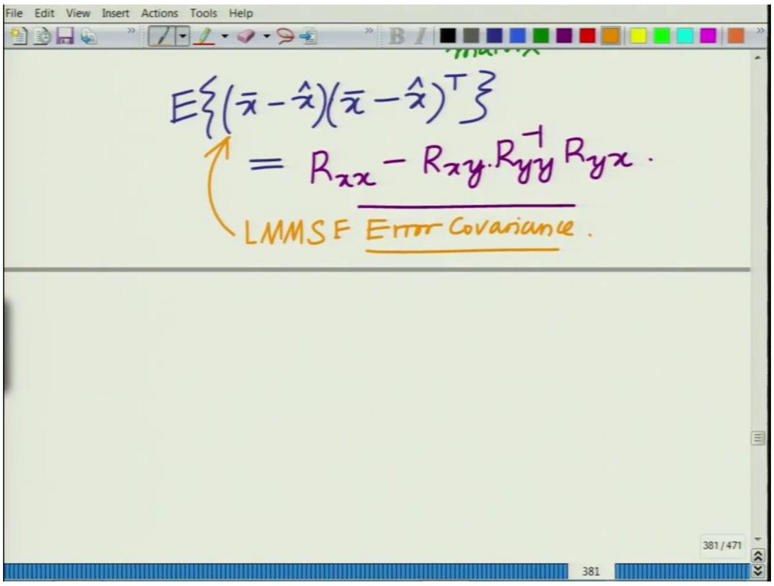

That is if you ask the question what is the expected value of X bar minus x hat into x bar minus x x transpose that is going to be given as R x x that is essentially what the error covariance is that is your R xx minus R x y times R y y inverse into R y x so this is your LMMSE error covariance matrix, this is the LMMSE error covariance. And so that is essentially that answers both the questions, what is LMMSE of estimator of x bar in terms of the correlated vector y bar and what is the what is the covariance error covariance of the LMMSE estimator.

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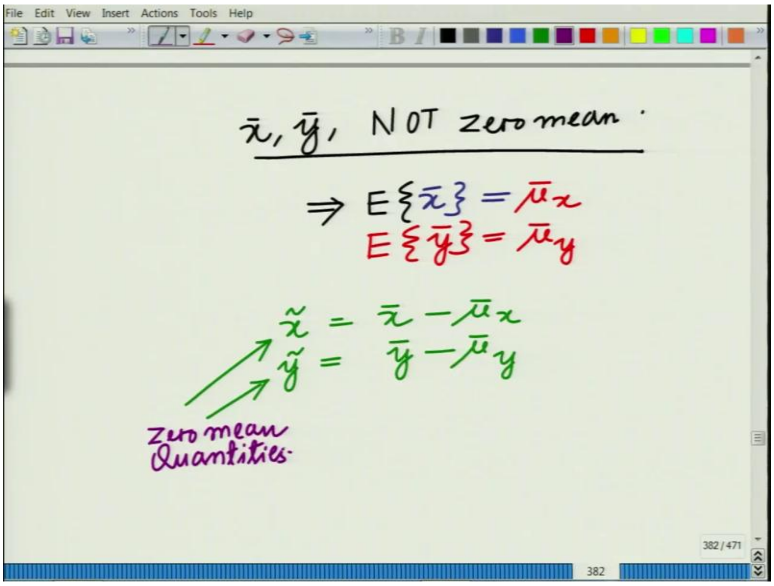

And now, let us look at the other case that is generalized this when x bar comma y bar are not, when x bar y bar are not 0 mean this implies that you have expected value of x bar is equal to 0, expected value of y I am sorry expected value of x x bar equal to Mu bar x which is not necessarily 0, this is Mu bar y not necessarily 0. Now, what we do is very simple, you subtract the mean for the 0 mean quantities x bar minus Mu x bar form y tilde equals y bar minus Mu bar y. Now, note that these are 0 mean quantities now you have subtracted the mean so you have got, these are 0 mean quantities.

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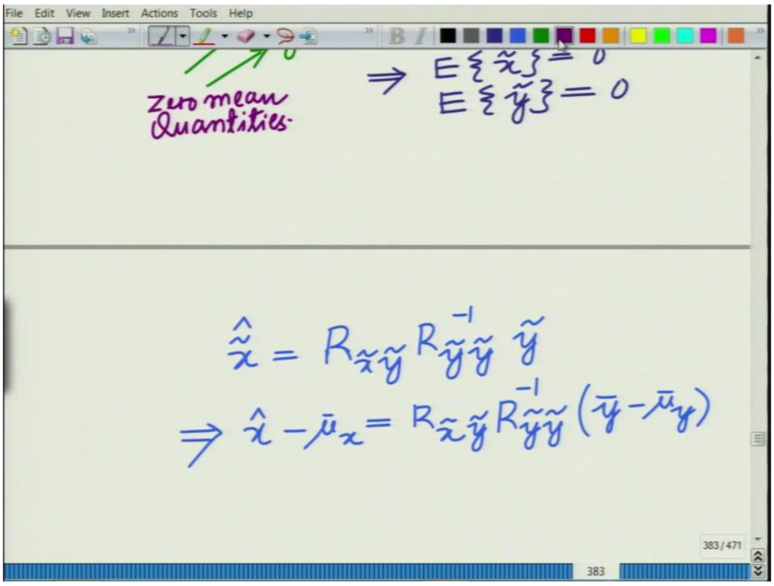

Essentially we have expected value of x tilde equal to 0 and expected value or y tilde equal to 0. So, we can use our we can use our LMMSE estimator that we have just been derived for the 0 mean quantities. And therefore, I can write the estimate of x tilde hat, this is going to be R x tilde y tilde times R y tilde y tilde inverse times y bar, which is nothing but remember now, writing this as x hat minus Mu bar x equal to R x tilde y tilde, sorry this will also be y tilde, R y tilde y tilde inverse times y bar minus Mu bar y.

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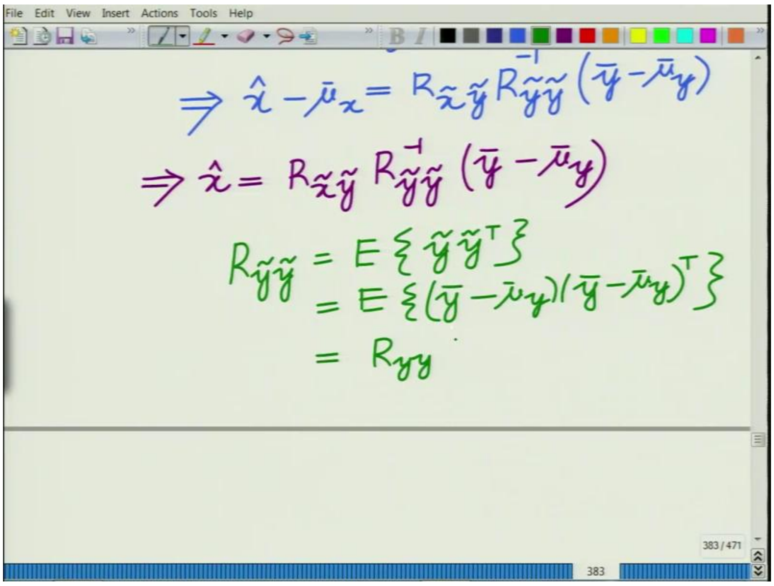

This implies that x hat equals R x tilde y tilde R y tilde y tilde inverse time times Y bar minus Mu bar y. Now let us ask the question what are these quantities R x tilde R y tilde y tilde and R x tilde y tilde, there is a covariance matrix of y tilde and the cross covariance between x tilde and y tilde.

And it is very easy to see that these quantities are nothing but R y tilde y tilde equals expected value of y tilde y tilde transpose which is nothing but expected value of y bar minus Mu bar y into y bar minus Mu bar y transpose which is nothing but R y y, covariance matrix of y bar.

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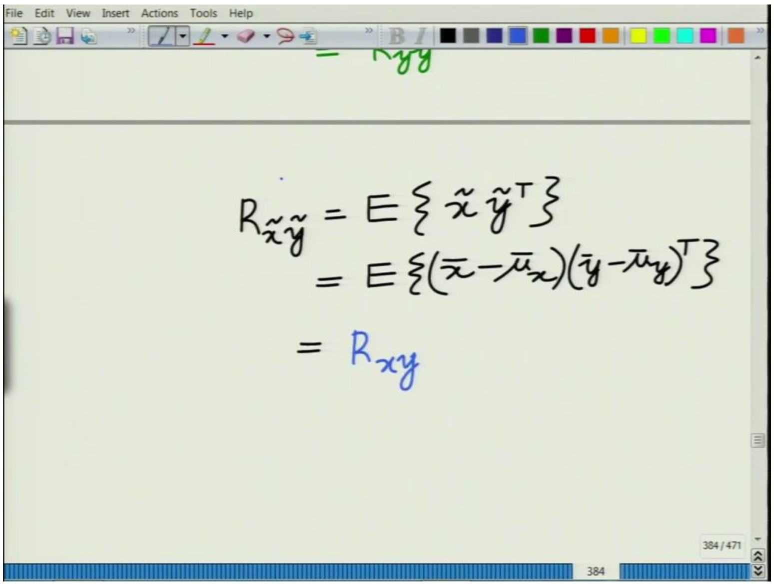

Similarly, R x tilde y tilde this is equal to the expected value of x tilde minus or expected value of x bar, this is expected value of x tilde y tilde transpose which is nothing but expected value of x bar minus Mu bar x into y bar minus Mu bar y transpose which is equal to R which is equal to R x y. So, we have R x tilde y tilde equal to R x y and therefore finally, we can write this implies that we have a very simple expression for the LMMSE estimator x hat equals R x y R y y inverse y bar minus Mu bar y correct.

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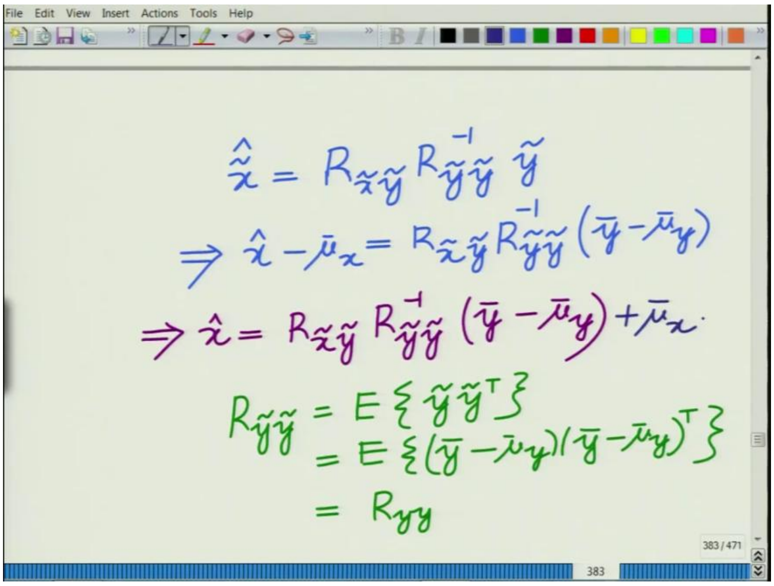

I am sorry here you have to have Mu bar x that comes from the left Mu bar x.

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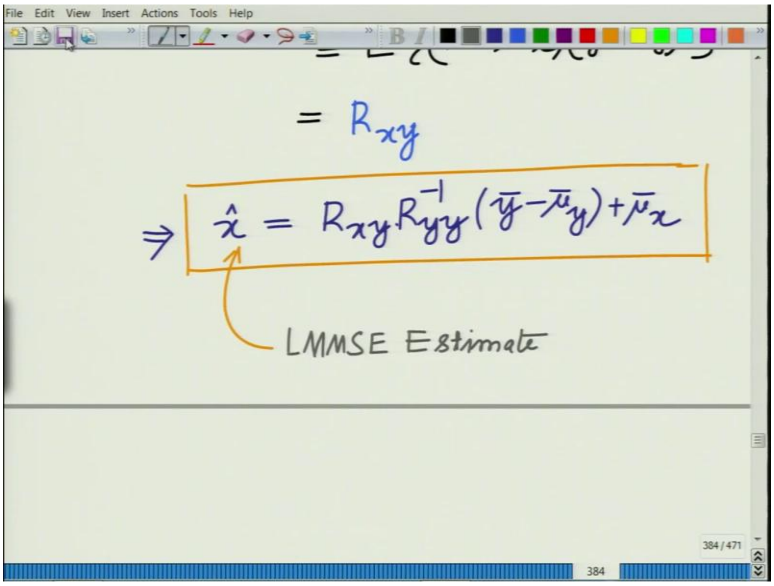

So, therefore, you have plus, you have this is essentially the expression that one obtains for the LMMSE estimate, once again this is the LMMSE estimate of, once again this is basically your LMMSE, this is basically your LMMSE estimate in terms of the this thing. So that essentially completes our derivation of the LMMSE estimate both for of course, first for the 0 mean quantities x bar and y bar and where the quantities are general that is the mean is not equal to the sum. And you can in fact observe something very interesting. So, what happens here is if x bar and y bar are uncorrelated.

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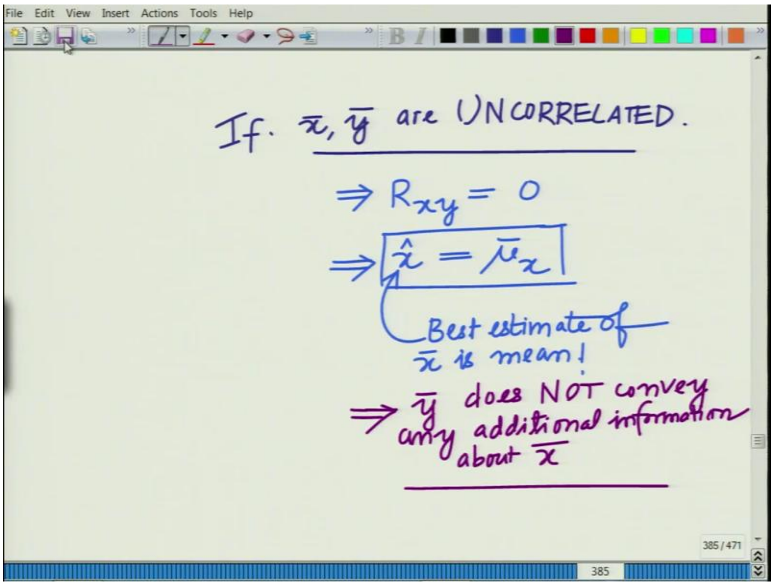

Now, let us look at an interesting case, if x bar and y bar are uncorrelated. Now, this implies that this implies that R x y equal to 0, if you look at this if R x y equal to 0, x hat is simply equal to Mu bar x. So this implies very interestingly, if you look at this, this implies x hat equals Mu bar x that is it, very interestingly x hat reduces to Mu bar x that is the best estimate of x is simply the mean because y bar does not convey any additional information about x bar because the cross correlation are x y equals 0. So, this also implies that y bar, observing y bar y bar does not convey any additional, y bar does not convey any additional information about x bar.

So, therefore, the best way is to observe y bar your knowledge about x bar or your estimate of x bar does not change and the estimate of x bar is simply the mean that is Mu x bar, which you had set anyway, if there were no there was no observation to begin with. So, if you have a random vector, and there are no observations, and if someone asks you what is the best estimate that you can or the best prediction that you can make of a random quantity, without any observation is essentially that the best estimate would be the mean of that random quantity.

Now, because x bar and having observed y bar if x bar and y bar are correlated, you would expect to make a better guess or a better estimate of x bar, but unfortunately if x bar and y bar are uncorrelated, observation of y bar does not help and the best estimate of x bar is still the mean, which also goes to show that the better the correlation the stronger the correlation between x bar and y bar the better is your estimate of x bar going to be and that is in fact, if you look at this expression, that is in fact reflected in the error covariance.

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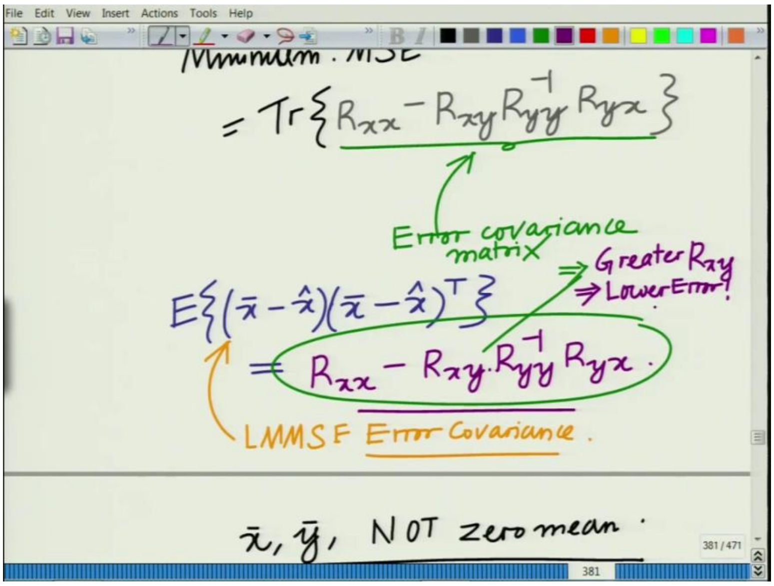

If you examine this again you will notice that this error covariance is R xx minus R x y R y y inverse implies this implies greater R x y, greater the cross correlation of course, these are matrices So, you cannot say one is greater one is lesser but intuitively greater R x y implies more error.

So, what you can see is that if R x y if the correlation is strong, the resulting error will be lower because your prediction of x bar is much better. So, these are essentially the very interesting things that followed from analyzing this LMMSE estimator and the LMMSE estimation principle in general. So, let us stop this discussion here and we will continue in the subsequent modules. Thank you very much.