Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
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Lecture – 34
MUSIC Algorithm for Direction of Arrival (DoA) Estimation

Hello, welcome to another module in this massive open online course. So, let us continue our discussion on the MUSIC algorithm. As you are probably aware by now, MUSIC stands for Multiple Signal Classification which is used to estimate the direction of arrival when you have multiple targets and an antenna array at the receiver and these multiple targets, they are either transmitting their signals or reflecting the signals that are initially transmitted by your radar and so on.

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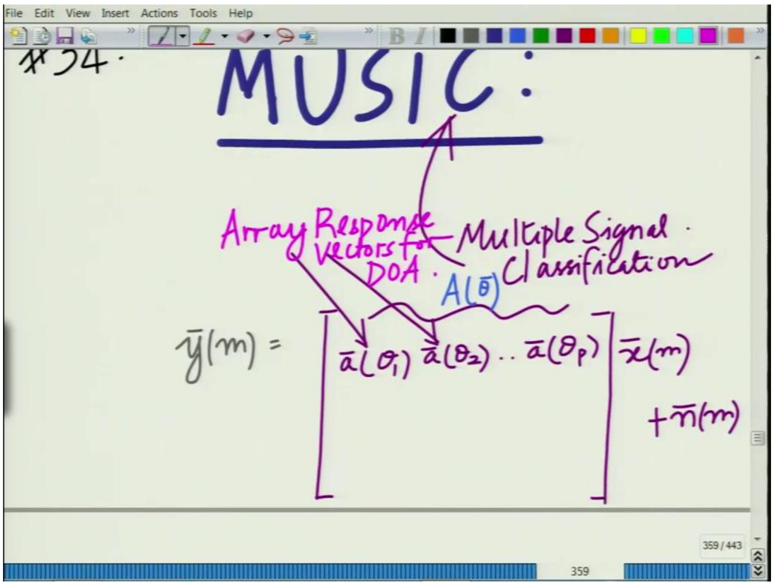

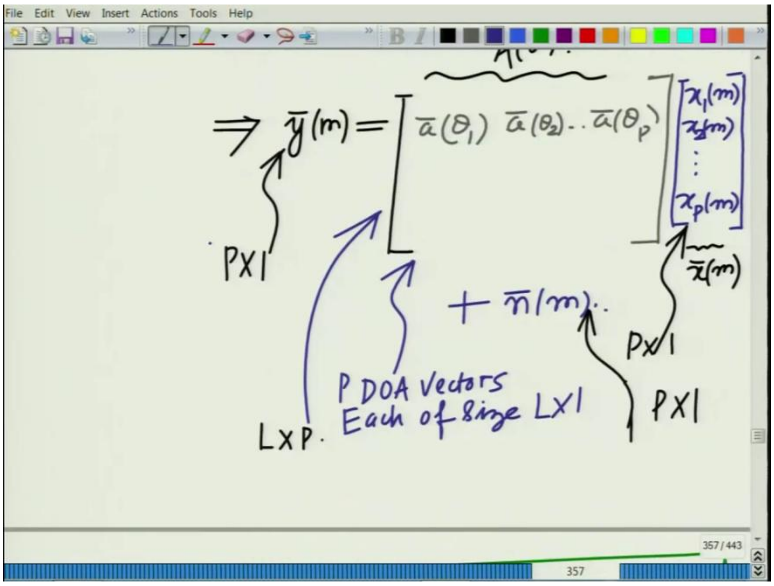

So, we have MUSIC algorithm which is considered to be 1 of the path breaking algorithms in array processing, significantly popular for direction of arrival estimation, this is known as multiple signal classification and we have seen the model for this is you have you have your well, you have your vector, if you go back and take a look at it, you have your y 1 m, y 2 m, you have your y bar m which essentially a bar theta 1.

So, you have your vector which is y bar m which is the vector or the matrix of DoA vectors that is you have a bar theta 1, a bar theta 2, a bar theta P times x bar m plus your noise vector, this is the matrix you can say a of theta or you can call it as a of theta bar which is the matrix you can see this is the matrix which contains the error response vectors corresponding to the DoAs directions of arrival. So, each of these these are your array response vectors, array response or array steering vectors you can also see holds the direction of directions of arrival of the targets.

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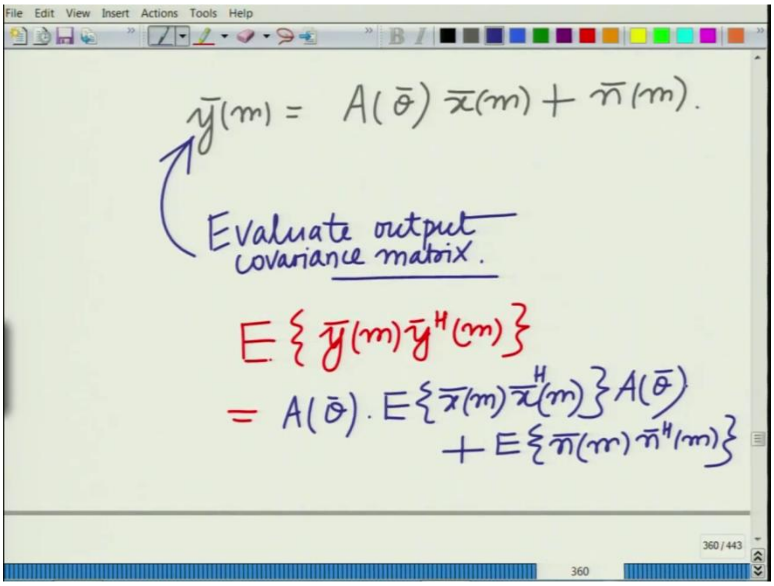

So, essentially we have y bar of m which is equal to your y bar of m which is equal to which is equal to, you can write this as we have seen above A theta bar into x bar of m plus n bar of m. And now I can estimate the output covariance. So, I can evaluate the output covariance evaluate evaluate evaluate the output covariance matrix.

So, this becomes your expected value of y bar m y bar hermitian m, which you can write as A theta bar times expected value of x bar m x bar hermitian m times A theta bar plus expected value of n bar m n bar hermitian m.

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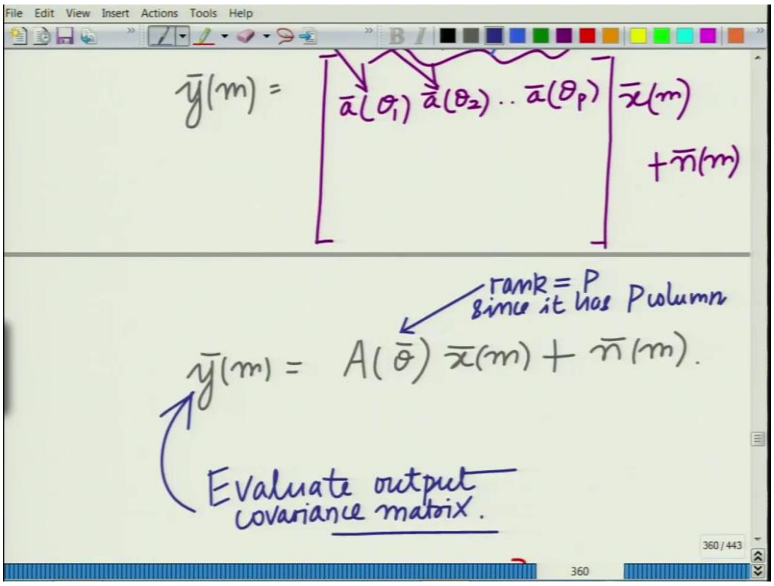

And the important thing to realize here is that if you look at the matrix A theta bar, the matrix A theta bar it contains P vectors, a bar theta 1, a bar theta 2, A bar theta p. So this has P columns therefore this is of rank P that is the important thing. So this vector A theta bar, the rank the important thing is this is a rank equal to P since it has P columns only P columns. So, the rank is not equal to L but the rank is equal to P.

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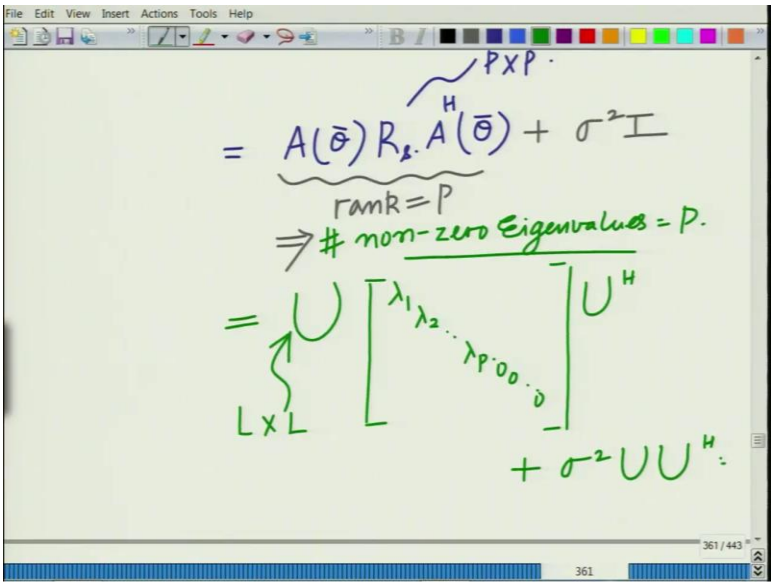

Now, that implies not let us look at this, I can always write this as A theta bar times R s times this is going to be A theta bar hermitian where this is a P cross P matrix, which is the covariance matrix of the input plus noise covariance which is sigma square, assuming the noise samples are independent identically distributed, noise covariance will simply be sigma square times identity.

Important point here is again once again this is this A theta bar times R s A theta bar hermitian. This is a rank P matrix, there are P targets and therefore, this will only have P nonzero Eigen values, this is a positive semi definite matrix and remember the number of nonzero Eigen values is equal to the rank of the matrix therefore, the number of nonzero Eigen values will be equal to P.

So, therefore, the important thing to note here is this rank overall rank equal to P implies the important observation here is the number of nonzero, the number of nonzero Eigen values is equal to P and therefore I can write this as you write U, the Eigen value decomposition will be this will be an L cross L matrix U sigma, this will be lambda 1, lambda 2, lambda P lambda P and the rest P minus L minus P Eigen values will be equal to 0 times U hermitian plus I can write sigma square identity I can write as U U Hermitian now a couple of things over here.

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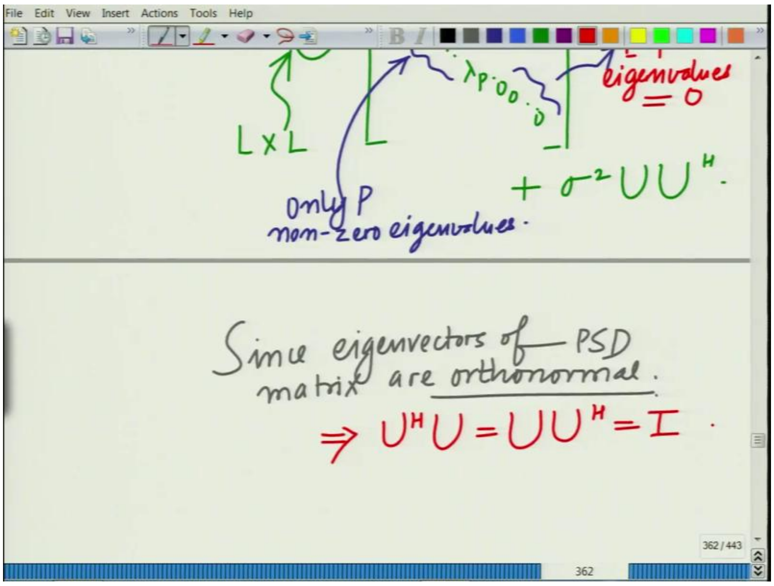

First as we have already said, only P nonzero Eigen values only P nonzero Eigen values and L minus P Eigen values are equal to 0, L minus P Eigen values equal to 0. Now, further if you look at these Eigen vectors remember the Eigen vectors of positive semi definite matrix are orthonormal, I can find an orthonormal set of Eigen vectors therefore, the matrix U will be such that U U Hermitian equal to U Hermitian U equal to identity so U is a unitary matrix.

So, since eigenvectors of PSD matrix a positive semi definite matrix these are orthonormal so this implies your U Hermitian U equals U U Hermitian equals identity.

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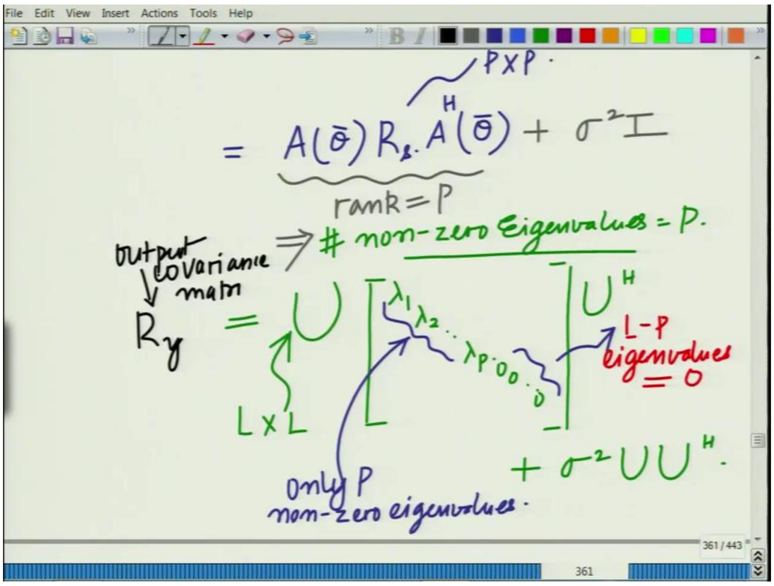

Therefore, if we look at this R y which is expected y bar y bar hermitian this is your output covariance matrix.

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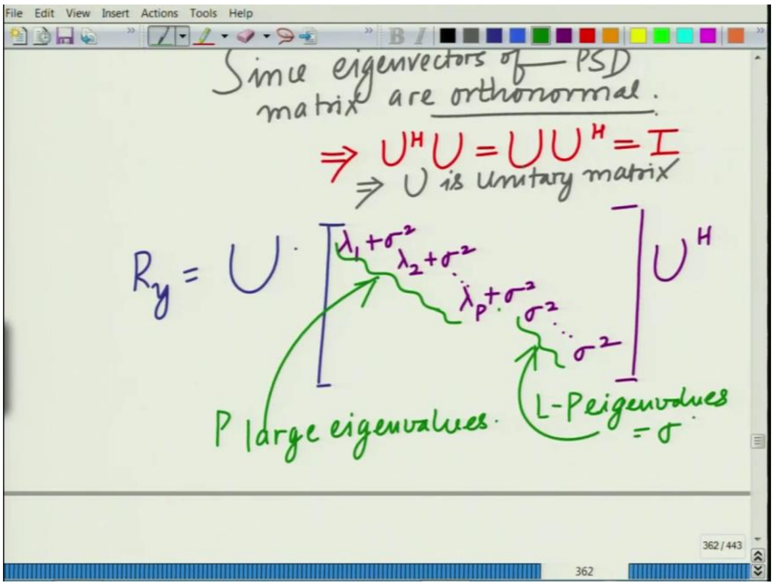

If you look at this therefore, I can write R y, I can now simplify this as U times, combine both these things I can write it as U. So, this is basically implies U is a unitary matrix, U is a unitary, U is a unitary matrix and therefore, I can simplify this as U lambda 1 plus sigma square, lambda 2 plus sigma square so on lambda P plus sigma square and the rest will only be sigma square.

So, the rest when you combine these 2, the rest L minus P will be sigma square will be some low value sigma square. So, these are the large, so P large P large Eigen values and rest L minus P are equal to sigma square.

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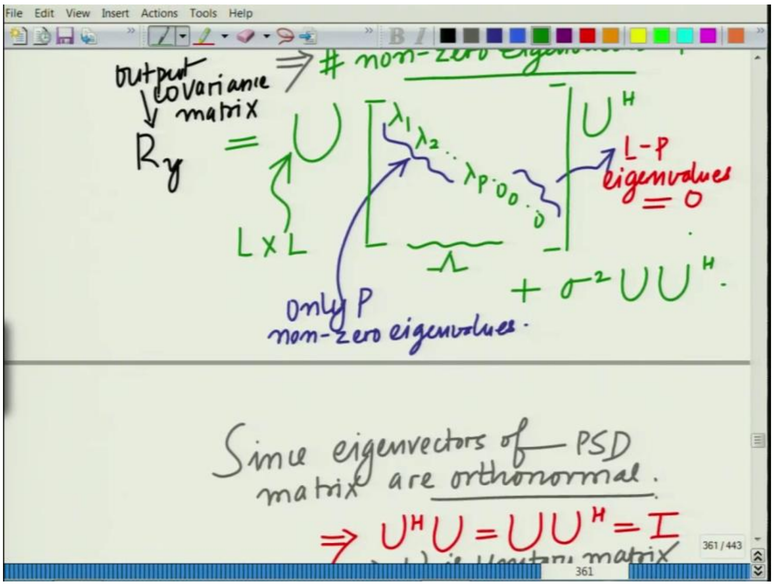

So, L minus P Eigen values that is if you look at this over here, when you combine these 2 components that is the U this matrix let us call this matrix as lambda U, lambda U Hermitian plus sigma square U hermitian.

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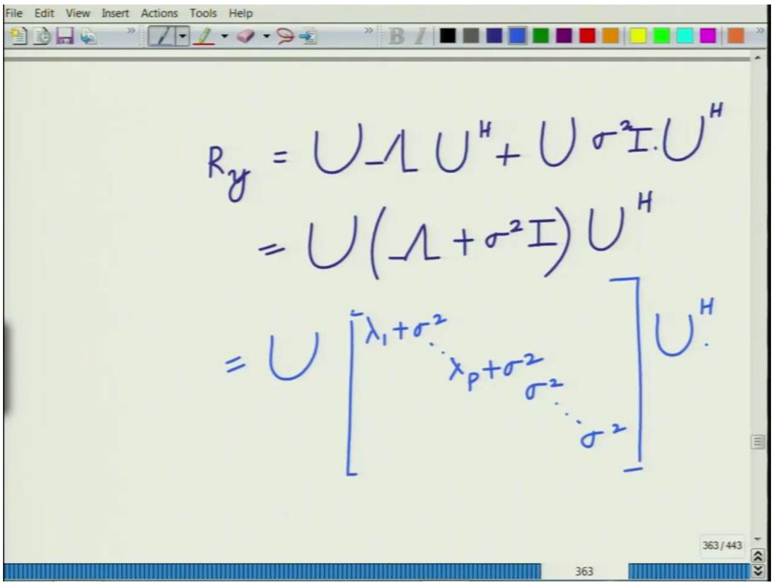

So, the way we are writing this is if you look at this you can write this as R y equals U lambda U Hermitian plus U sigma square identity times U Hermitian. So, this will be equal to U lambda plus sigma square identity times U Hermitian which is now if you write this, this will be equal to U and this matrix lambda plus sigma square times identity, this is essentially your lambda 1 plus sigma square until lambda P plus sigma square and the rest L minus P identity entries will be sigma square.

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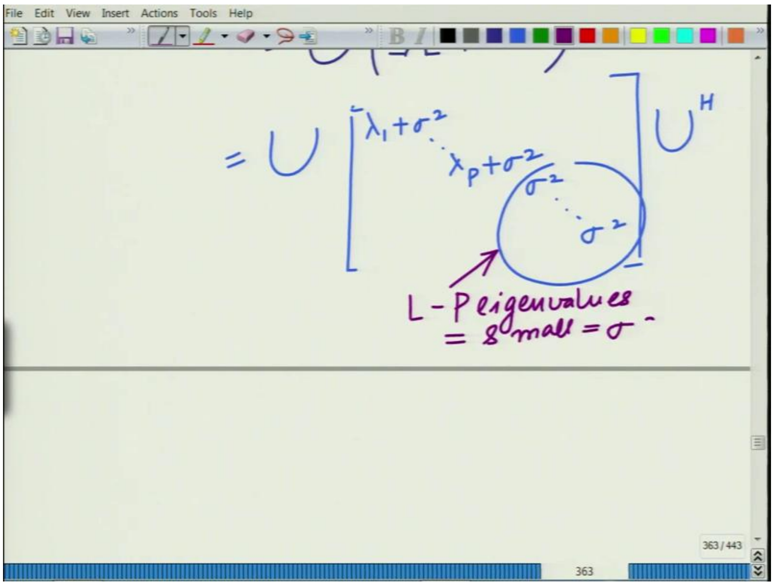

So, this is essentially your this matrix what we have written over here is essentially your lambda plus sigma square times identity,. And now, if you look at these are the key. So, if you look at these L minus P Eigen values which are sigma square now, these correspond to the Eigen vectors of, these correspond to the eigenvectors which are which correspond to this which correspond to 0 Eigen values of your matrix A theta bar R s A theta bar hermitian. So, let me explain, so this sigma square so, when you look at these L minus P Eigen values, these are small these are small and these are equal to sigma square.

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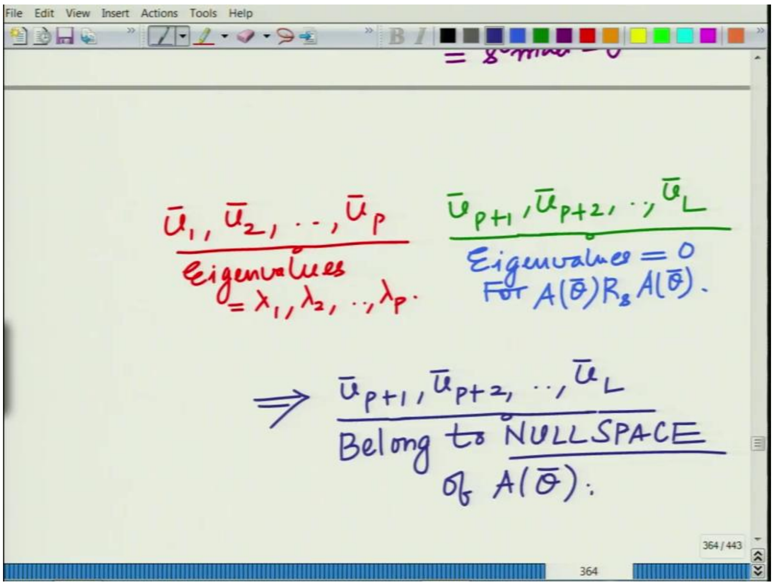

And now, where are these arising from if you go all the way back and if you trace these things, you can divide the eigenvectors into into 2 states, U 1 bar, U 2 bar, U p bar, these Eigen values are lambda 1, lambda 2, up to lambda p, and U bar P plus 1, U bar P plus 2, U bar L, these L minus P vectors U bar L, these correspond to Eigen values equal to 0 for the matrix for, remember A theta bar R s which is your source code expected value x bar x bar hermitian X m X m bar Hermitian times A theta bar, times A theta bar. So, these correspond to the Eigen values of A theta bar.

So, U bar P plus 1, U bar plus 2, U bar L, these L minus P Eigen vectors, these correspond to these are essentially the eigenvectors of the matrix A theta bar R s into A theta bar hermitian corresponding to 0 Eigen values. And therefore, this essentially implies that this U bar P plus 1, U bar P plus 2 so on U bar L these belong to the null space is these are basically Eigen values corresponding Eigen values 0 belong to the null space, this is an important property, these belong to the null space of A theta bar.

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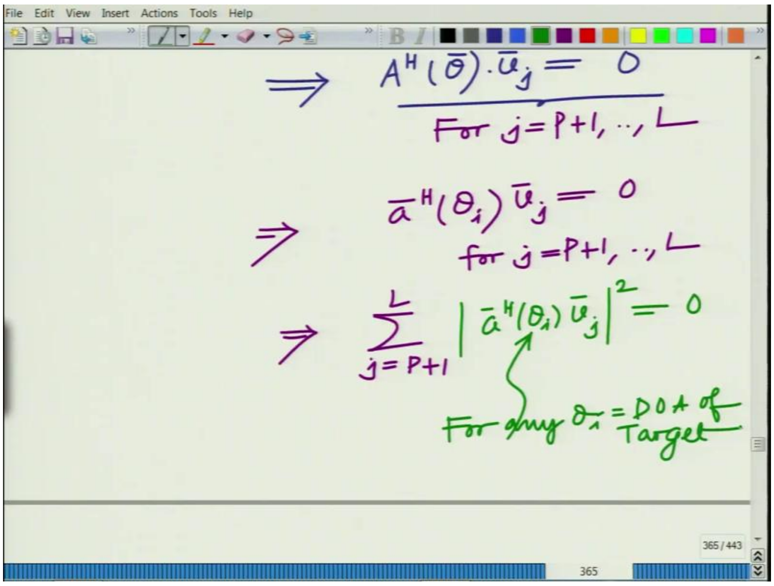

Implies this implies if you take any this implies that essentially if you take A theta bar hermitian A hermitian theta bar times U bar j, this is equal to 0 for j equal to P plus 1 up to L. This implies that if you take any vector A bar Hermitian theta i times U bar j this is equal to 0 for j equal to P plus 1 up to L. This implies that if you take the summation j equal to P plus 1 to L magnitude A bar Hermitian theta i U bar j whole square this will be equal to 0. For any theta i which is equal to DoA of target.

So, the interesting point here is that if you take any theta i which is the direction of arrival of the target for one of the targets i equal to 1 to P, i equal to 1 to up to P then that must satisfy this equation that is summation j equal to P plus 1 to L magnitude A bar Hermitian theta i U bar j the magnitude square of that summation, magnitude square of that must be equal to 0.

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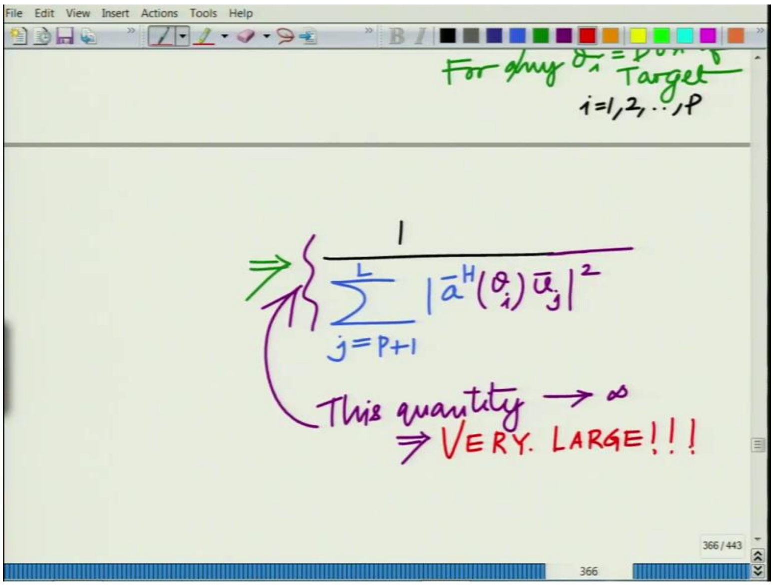

Which implies that for any theta i, which implies where i equal to 1 up to P, which implies if I look at the reciprocal of this, that is if I look at this quantity that is the summation j equal to people plus 1 to L summation magnitude A bar Hermitian theta i U bar square, since the denominator is small, this quantity is going to be very large, since the denominator is 0, this quantity tends to infinity, implies this quantity is very large essentially.

Therefore, how do you now determine the directions of arrival theta i? Very simple, plot this function f of theta, call this a function f of theta, plot it as a function of theta, when that theta equal to theta i, this will be equal to 0 and therefore. I mean this denominator will be equal to close to 0 therefore, the overall quantity which is the reciprocal of this will be very high tending towards infinity and therefore, you will have peaks in this spectrum, and those peaks will correspond to the directions of arrival.

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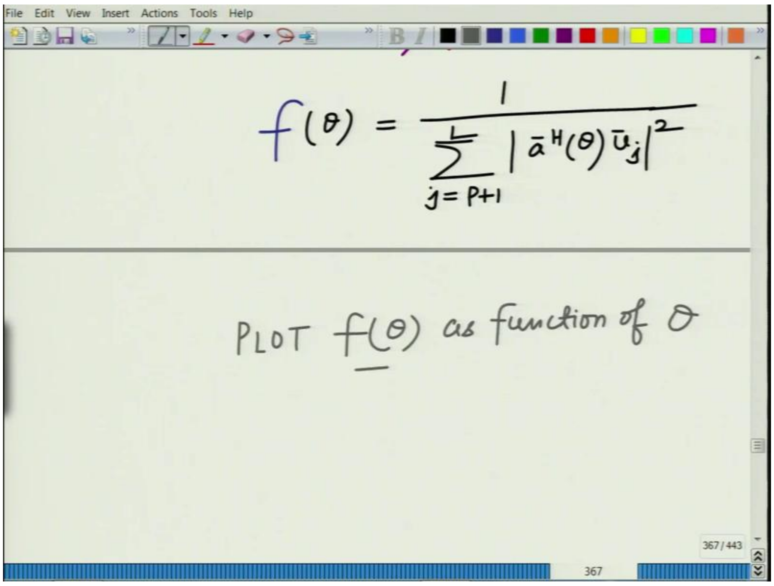

So, basically call this as your f of theta call this as your f of theta which is equal to 1 over, which is basically your 1 over summation j equal to P plus 1 to L magnitude A theta bar hermitian A bar Hermitian theta times U bar equal square. So plot this as a function of theta, plot f of theta now the step final step is plot f of theta as a function of theta, plot f of theta as, plot F of theta as a function of theta and where you have the peaks.

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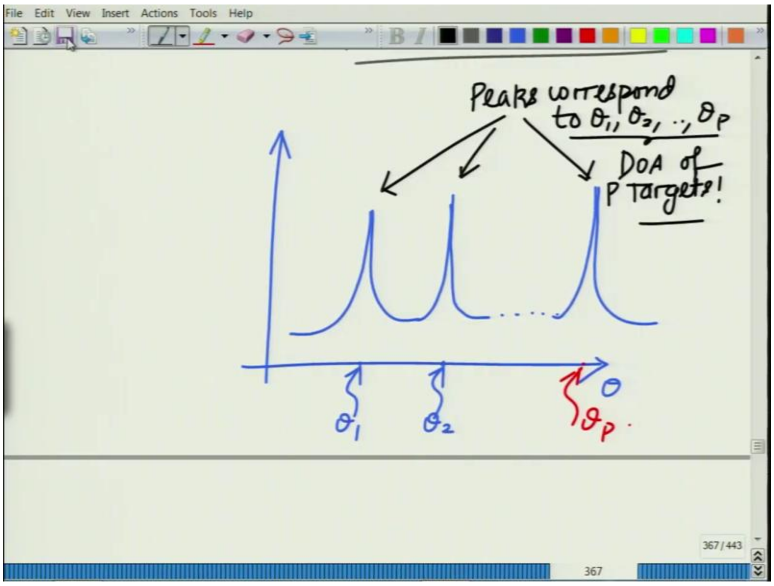

So, these are your axes plot this as a function of theta and your peaks and your peaks. So, this peak will correspond to this will be your theta 1, this will be your theta 2 so on and so forth, this will be your theta P. So the peaks correspond to, so the peaks and these are your DoA of the, and these are the DoA of the P targets. So, essentially what you are doing is very interestingly you are estimating the output covariance, from that output covariance you are looking at the Eigen vectors which essentially correspond to the smallest Eigen values which are essentially sigma square.

These correspond to, and the important property is these Eigen vectors correspond to the R in the null space of A theta bar, which essentially corresponds of the angles of arrival of your, which corresponds to the angle of arrival of the peak target, and therefore, we construct this cos function f of theta, plot it the peaks essentially correspond to your angles of arrival of the target. So, that is the interesting property and this is how the MUSIC algorithm operates.

So, you construct a square output covariance, from that extract the Eigen vectors corresponding to the smallest Eigen values, construct this function f of theta, plot the spectrum, plot the spectrum and the peaks from the peaks, you can essentially find out where the targets are. So, it is a very fast, simple and efficient algorithm, as I have already told you is one of the most considered a breakthrough algorithm and one of the most popular algorithms for direction of arrival estimation.

And this is yet another very, very interesting and exciting application of the principles of linear algebra, Eigen vectors, Eigen values and so on in signal processing, direction of arrival estimation which can have so many applications. So, let us stop this module here and let us continue looking at other aspects in the subsequent modules. Thank you very much.