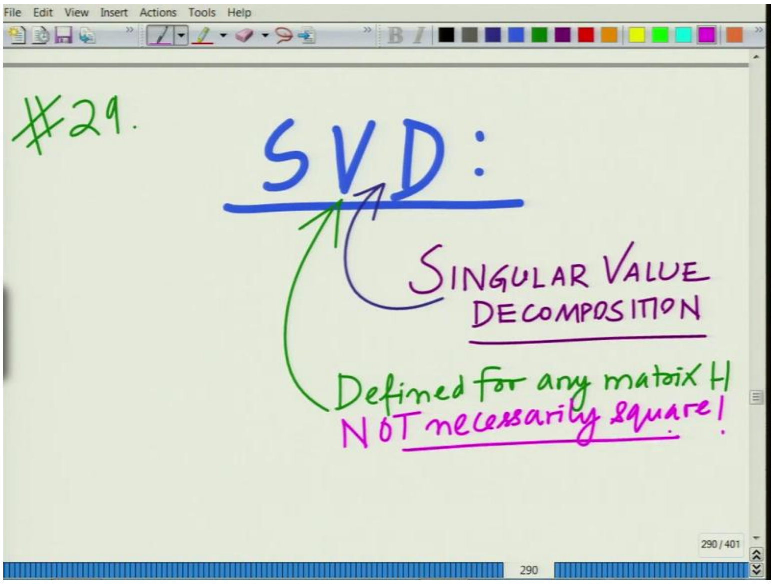
Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
Professor Aditya K Jagannatham
Department of Electrical Engineering
Indian Institute of Technology Kanpur
Lecture 29
Singular value decomposition (SVD): Definition, properties, example

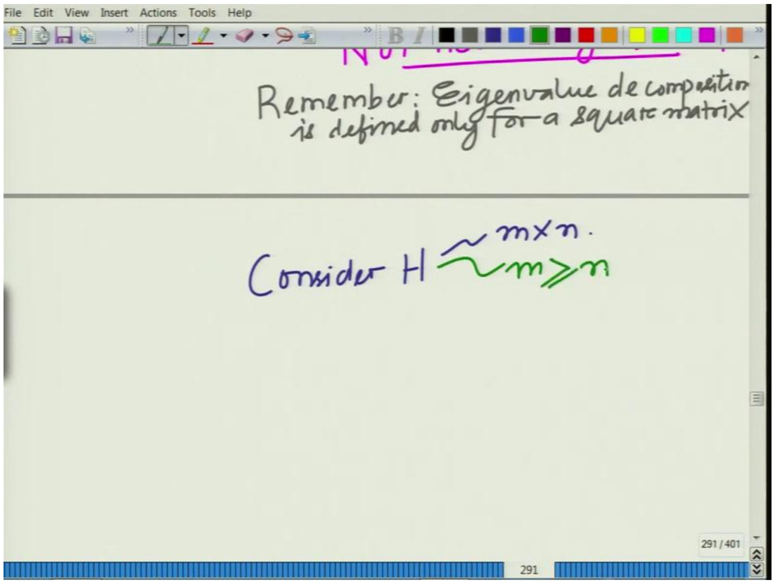
Hello. Welcome to another module in this massive open, online course. So in this module, let us start looking at another very important concept that is related to decomposition of any matrix and this is known as the singular value decomposition.

(Refer Slide Time: 00:32)



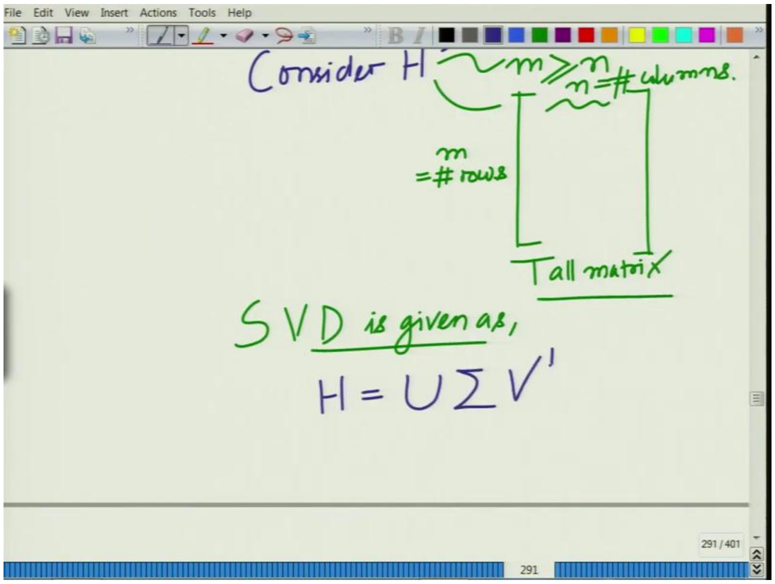
So in this module, let us start looking at a new and a very important concept and that is of the SVD, which basically stands for, the singular value decomposition, which is a very important decomposition. It is a very important concept and the SVD, the singular value decomposition is defined for any matrix, not necessarily square. This is an important thing to remember because the Eigen value decomposition is defined only for a square matrix but the singular value decomposition is defined for any matrix, any arbitrary m cross n matrix that is not necessarily square.

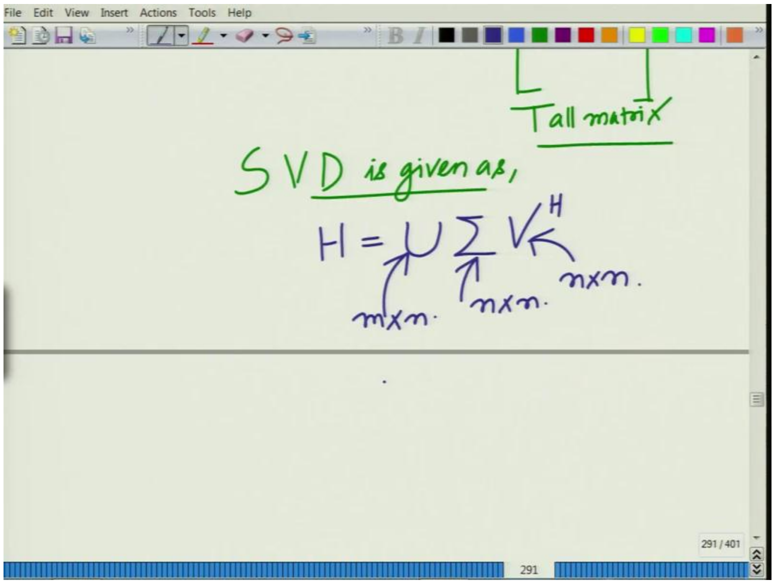
(Refer Slide Time: 02:07)



So consider H, which is an m cross n matrix. I think at this point, it is worth remembering that the Eigen value decomposition is defined only for a square matrix, that is, only for square matrices. So consider H to be any matrix. Consider H to be an m cross n matrix. For ease of visualization, we will consider m greater than or equal to n, and the case where m is less than n is similar. So let us consider a tall matrix, m cross n matrix, where m is greater than equal to n.

(Refer Slide Time: 03:18)

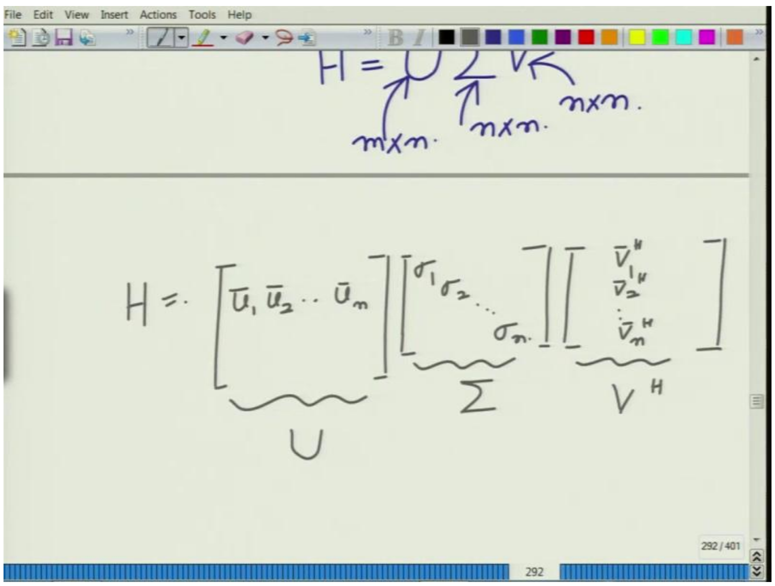




So I can write such a matrix as this. It looks like this. So m equals the number of rows and if you look at n, this is equal to the number of columns. Number of rows is greater than equal to number of columns which implies that this is a tall matrix. So that is our nomenclature.

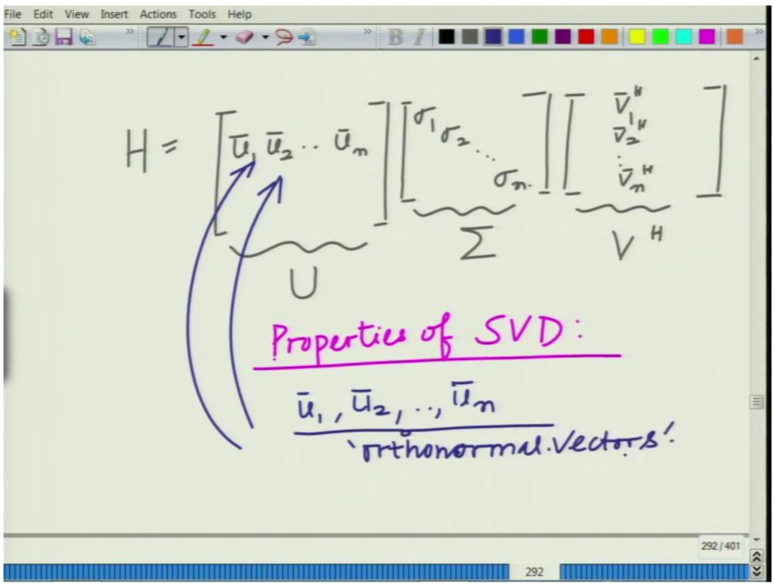
And the singular value decomposition, SVD is given as H equal to U Sigma V Hermitian where U can be written as an m cross n matrix, Sigma can be written an n cross n matrix and V Hermitian can be written as an n cross n matrix and this looks like as follows.

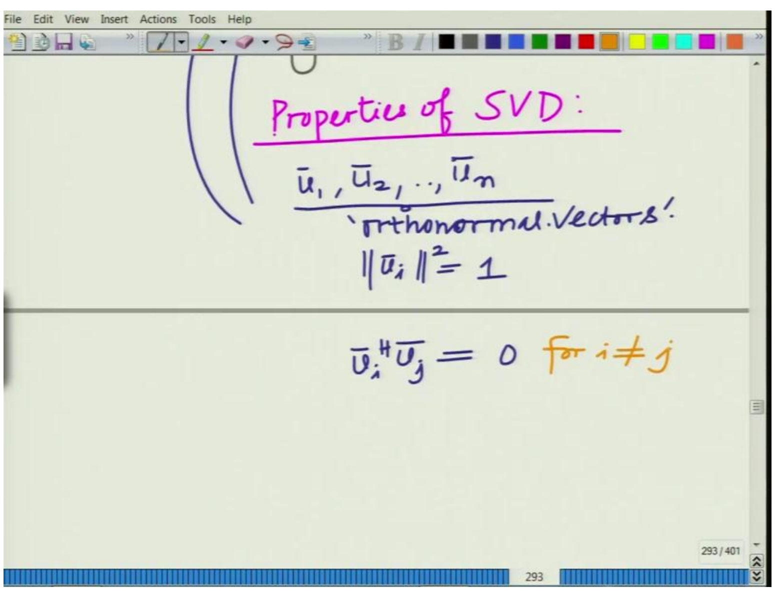
(Refer Slide Time: 04:26)



So H equals the singular value decomposition. So you have m cross n matrix, U, so that is u1 bar, u2 bar, un bar. So it has n columns. Then you have the diagonal matrix, Sigma 1, Sigma2, Sigma n which is n cross n, and then you have V Hermitian which comprises of the rows, v1 bar Hermitian, v2 bar Hermitian, vn bar Hermitian. So this is your matrix U, this is your matrix Sigma and this is the matrix V Hermitian. So these are the components of the singular value decomposition. So this is basically how the singular value decomposition looks for a tall matrix H, with m greater than equal to n. Now, let us now start looking at the properties of the SVD.

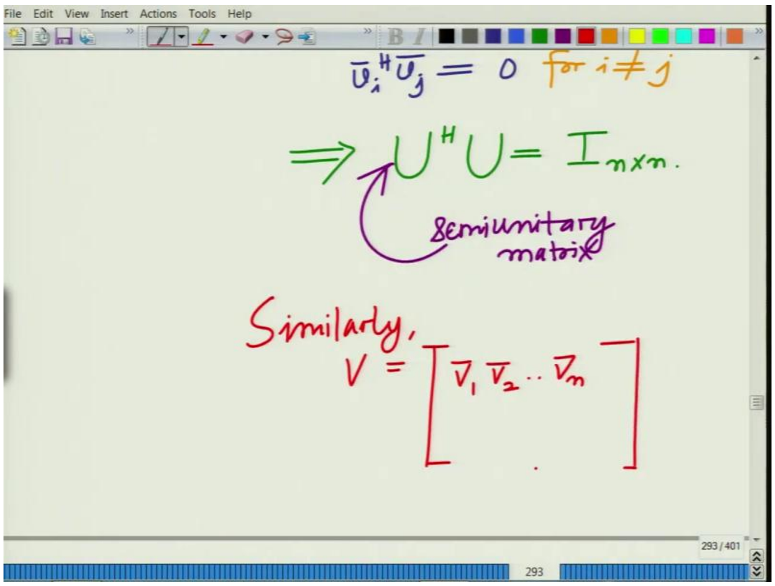
(Refer Slide Time: 05:29)

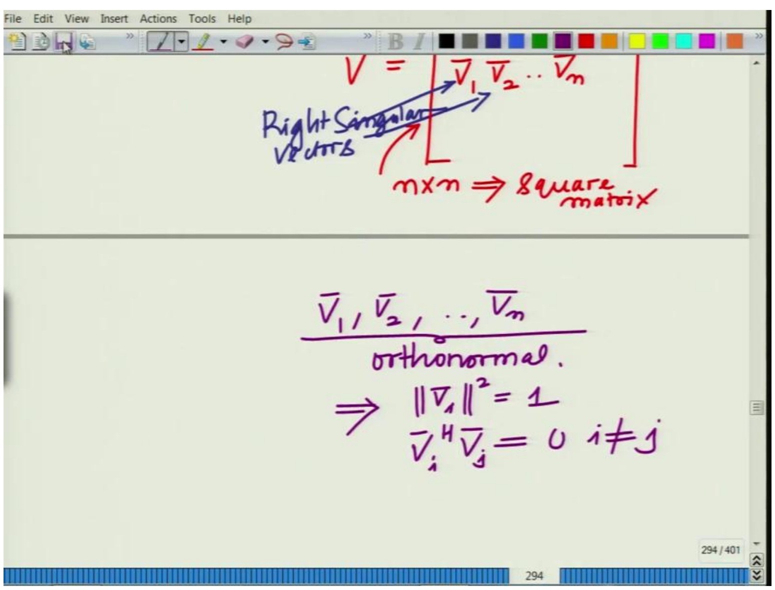


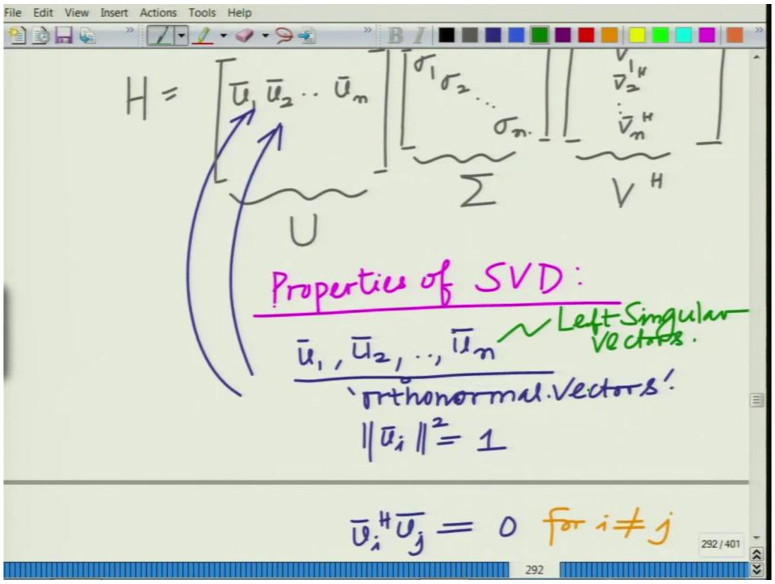


So what are the properties of the SVD? Properties of the SVD, that is, if you look at these vectors, u1 bar, u2 bar, so on, un bar, that is if you look at the vectors u1 bar, u2 bar, so on up to un bar, these are orthonormal vectors which essentially implies that each is unit norm. That is norm ui square equal to 1 and orthogonal which means ui bar Hermitian uj bar equal to 0 for i not equal to j. So this is an orthonormal set of vectors.

(Refer Slide Time: 06:54)







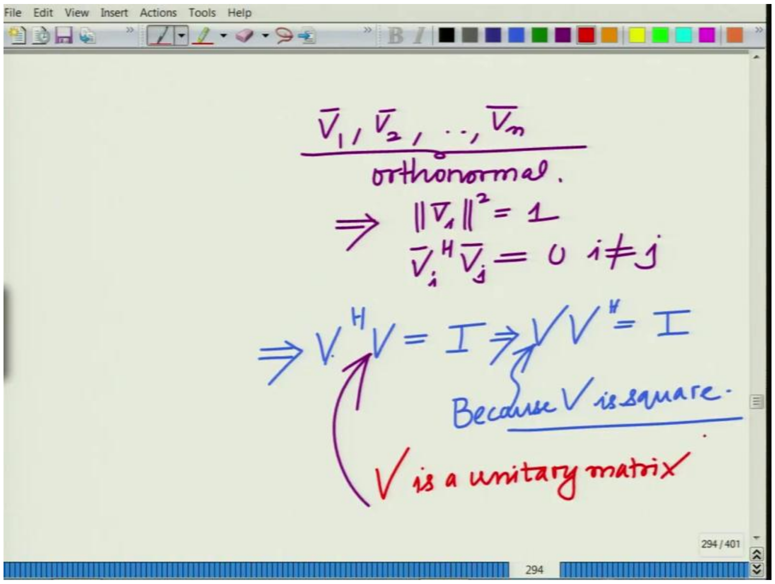
And therefore this implies, naturally, this implies that if you look at the matrix product U Hermitian U, this will be the Identity matrix of size n cross n. And U is therefore, this is known as a semi-unitary matrix. So the matrix U contains orthonormal columns u1 bar, u2 bar, un bar such that norm of each ui square is, ui bar square is unity and ui bar Hermitian uj bar equal to 0 if i is not equal to j and therefore the matrix U Hermitian U is Identity.

And these vectors are known as the left singular vectors. That is, these vectors u1 bar, u2 bar, un bar in the singular value decomposition which are in the matrix U on the left are known as the, it is self-explanatory, these are known as the left singular vectors. Now, what are these left singular vectors? It is not very difficult to see that if you look at H H, we will look at that again.

Now, let us come now to the property V. What about the matrix V? Now, similarly if you look at V, v1 bar, v2 bar, vn bar this is an n cross n matrix which implies this is a square matrix. And V can be written as, and V also contains, again, now these are known as the right singular vectors.

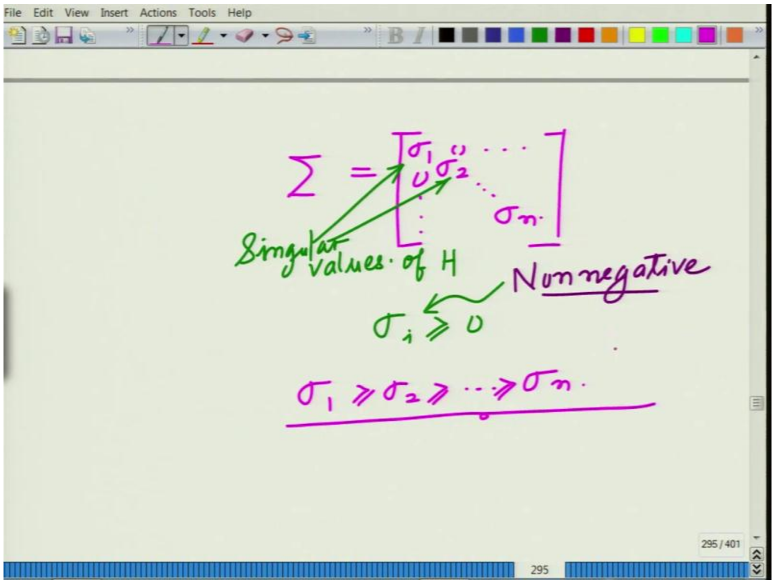
Naturally, these are part of the vectors of the right. These are known as the right singular vectors because these are in the matrix v bar which is on the right and these are also, if you look at v1 bar, v2 bar, vn bar, these are also orthonormal which implies norm Vi bar square equals Unity and Vi bar Hermitian Vj bar equal to 0 for any i not equal to j. So Vi bar Hermitian Vj bar, these are unit norm as well as orthogonal.

(Refer Slide Time: 10:07)



And therefore, the matrix V satisfies the property, V Hermitian V is Identity which also implies, V V Hermitian equals Identity because V is a square matrix. The reason being, if for a square matrix B, the inverse is unique, the left inverse is the same as the right inverse. That is, if AB is Identity, then BA is also identity. So V V Hermitian is Identity implies that V Hermitian into V should also be Identity and therefore, V is Unitary matrix. Because it is a square matrix, V is a Unitary Matrix.

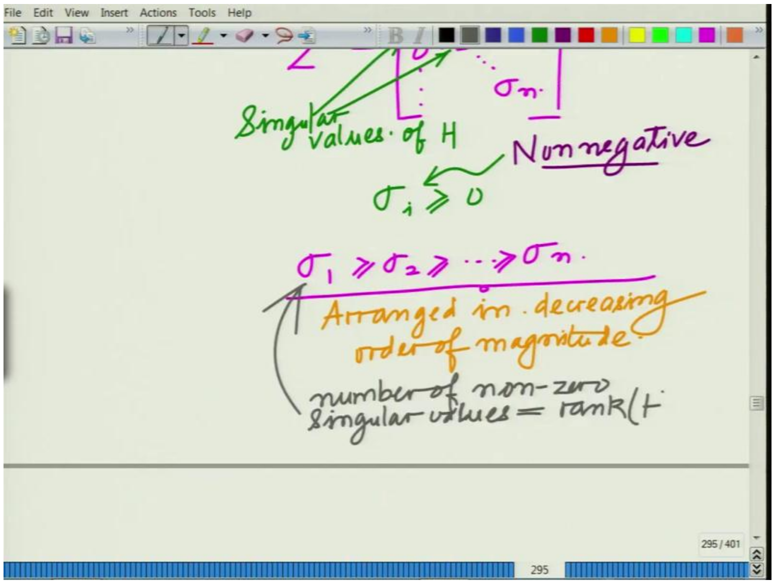
(Refer Slide Time: 11:22)

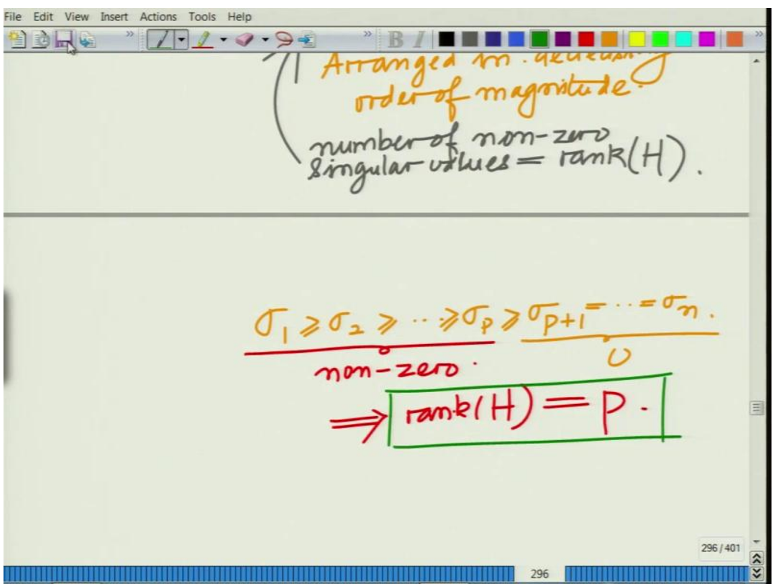


And the matrix Sigma has an interesting structure. Now, we come to Sigma which is essentially, you can write this as Sigma equals Sigma 1, this is a diagonal matrix. Now, these are terms the Sigma i's these are terms the singular values of H, just like the Eigen values. We have the singular values of H. These Sigma i's are all greater than equal to 0. These are real and greater than equal to 0.

So Sigma i's are non-negative, that is, they can be 0 but they cannot be less than 0. And also important to note, is that the singular values have to be arranged in the decreasing order of magnitude. Sigma 1 greater then equal to Sigma2 greater than equal to Sigma n. So singular values are arranged in decreasing order of magnitude.

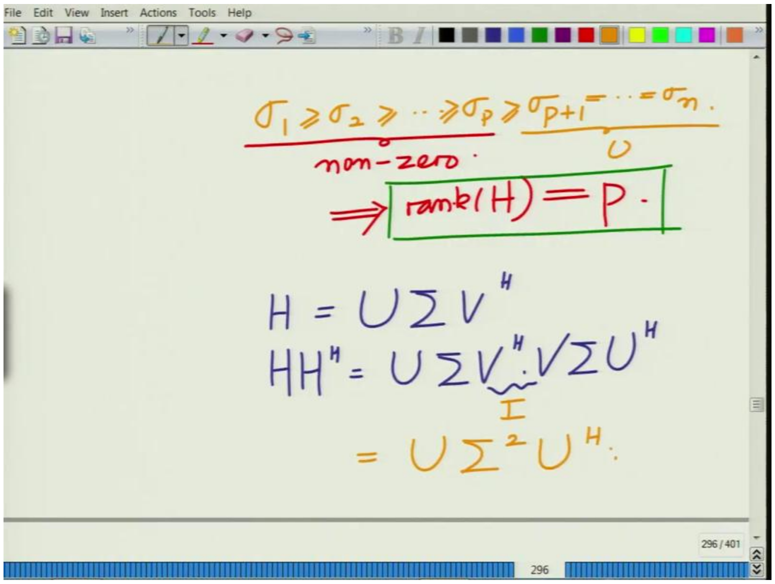
(Refer Slide Time: 12:38)

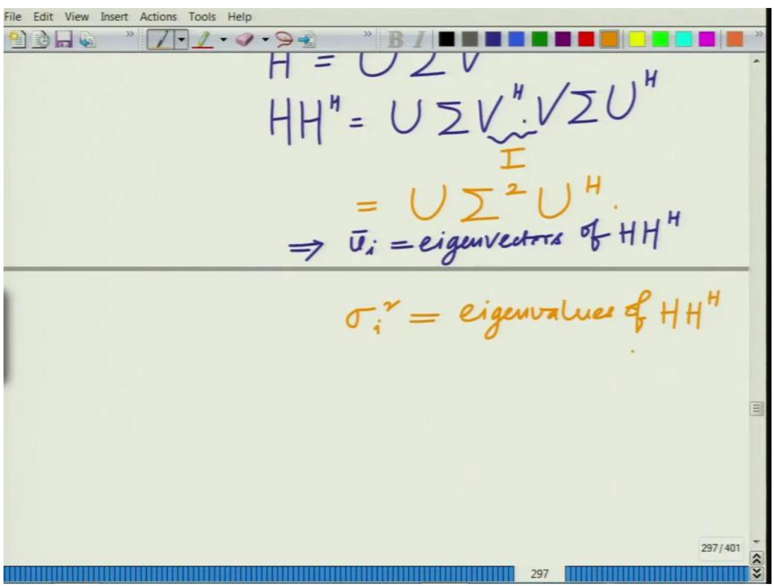




These are arranged in decreasing order of magnitude and the other important thing here, is the number of non-zero singular values. It equals the rank of the matrix H. That is, if you have the singular values Sigma 1 greater than equal to Sigma2, greater than equal to so on, Sigma P and Sigma P plus 1 onwards, if they are 0, that is if you look at Sigma P plus 1 equal to and these are, let us say zeroes, these are essentially non-zero. This implies that rank of H equals, this implies that the rank of the matrix H is equal to P, number of non-zero singular values of H. Number of non-zero singular values of H is equal to, basically, the rank of the matrix H.

(Refer Slide Time: 14:50)

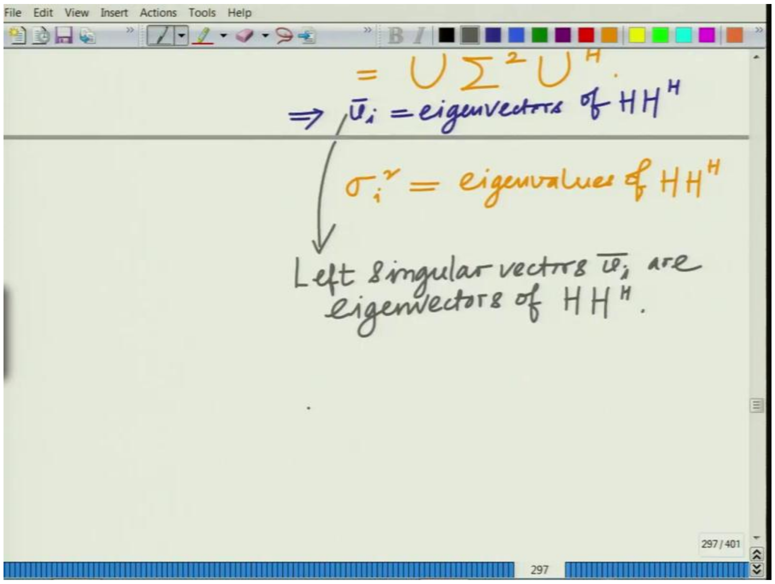


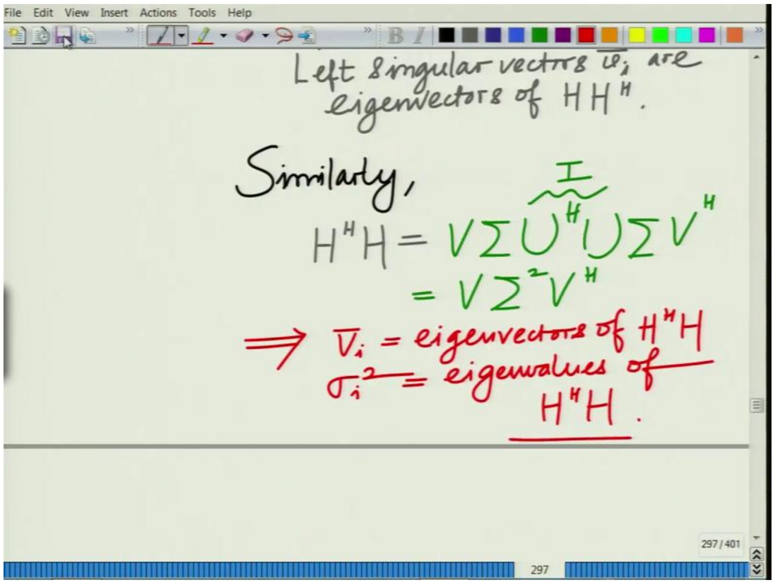


And, well, what is the relation between the singular vectors and singular value decomposition and the Eigen value decomposition? That is also interesting. So if you look at H equals U Sigma V Hermitian. So consider H H Hermitian, that is U Sigma V Hermitian times V Sigma U Hermitian, V Hermitian V, this is equal to Identity so this reduces to U Sigma square into U Hermitian.

So this implies that the ui bar are Eigen vectors of H H Hermitian. You can clearly see that this is the Eigen value decomposition and Sigma square, that is Sigma square, these are the diagonal elements of the matrix Sigma square, these are the Eigen values or H H Hermitian. That is, the singular value of H, the square of the singular values of H are the Eigen values of H H Hermitian or you can also say that the square root of the Eigen values of H H Hermitian are essentially the singular values of the matrix H.

(Refer Slide Time: 16:17)



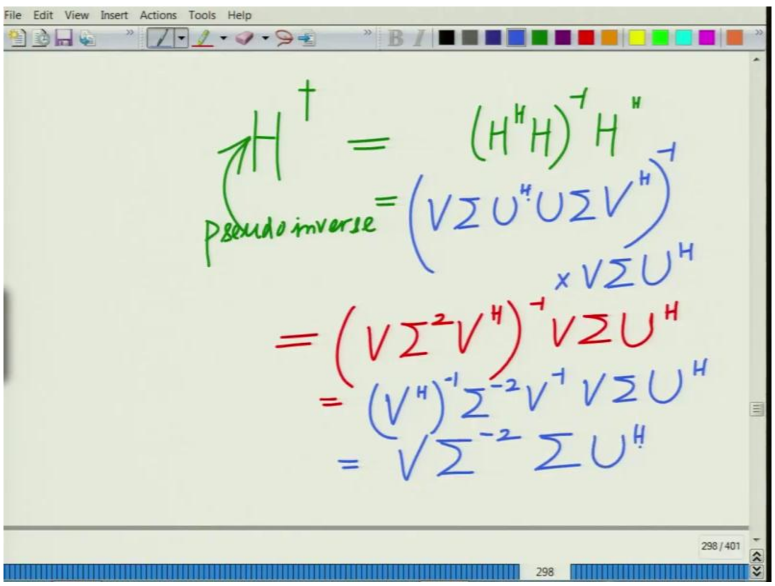


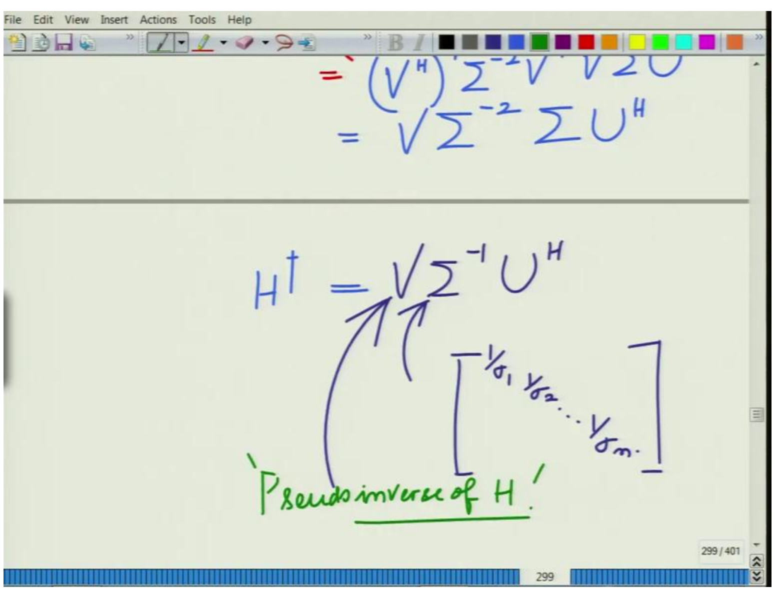
Similarly, if you consider H Hermitian, so what this shows is the left singular vectors ui bar are Eigen vectors of H H Hermitian. Now, similarly, consider H Hermitian H. This is essentially V Sigma U Hermitian U Sigma V Hermitian which is V Sigma square V

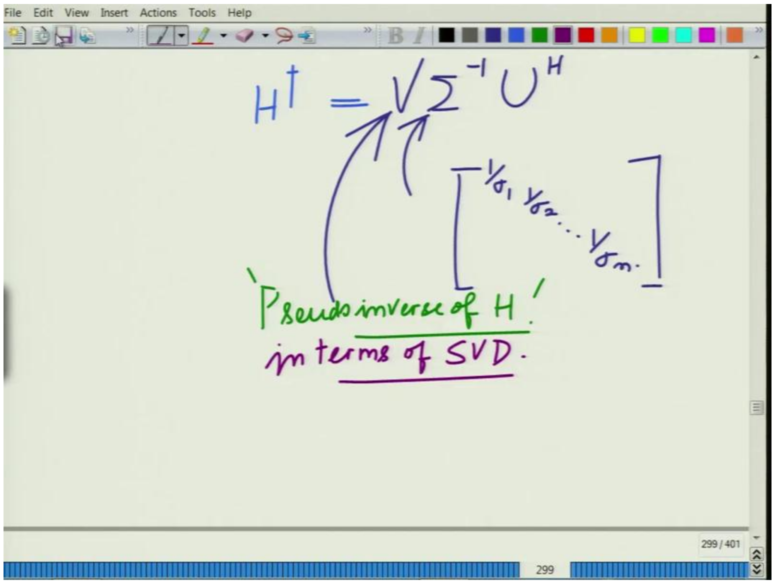
Hermitian because U Hermitian U is Identity which implies that Vi bar these are Eigen vectors of H Hermitian H and Sigma square are the Eigen values of H Hermitian H.

So Sigma square are also the Eigen values of H Hermitian H. So the ui's, the vectors ui bar are the Eigen vectors of H H Hermitian, Vi bar are Eigen vectors of H Hermitian H and Sigma square are essentially Eigen values of both, H H Hermitian and H Hermitian H.

(Refer Slide Time: 18:13)







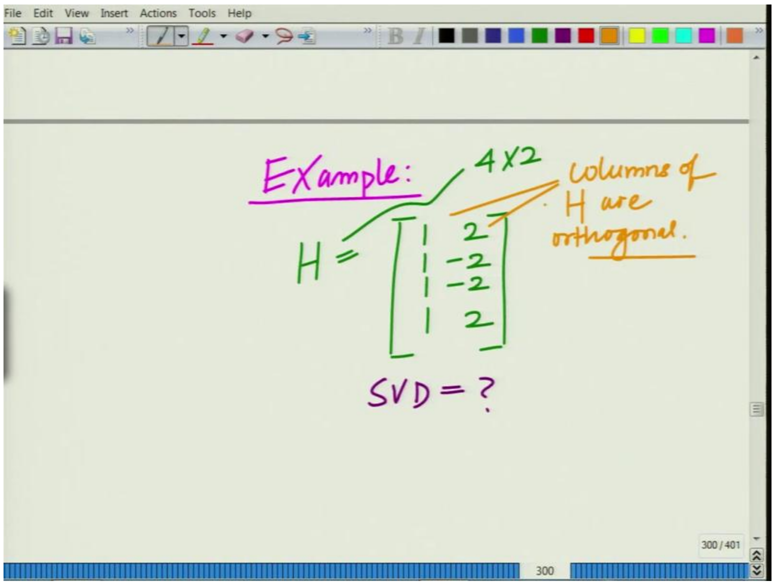
And let us look at some other interesting things about the SVD. Let us now compute the pseudo inverse in terms of the SVD. So we have the pseudo inverse. As we know, for a tall matrix this is given as H Hermitian, I am sorry, H Hermitian H inverse H Hermitian. Substitute for H in terms of the singular value decomposition, this is V Sigma U Hermitian U Sigma V Hermitian inverse times H Hermitian that is V Sigma U Hermitian.

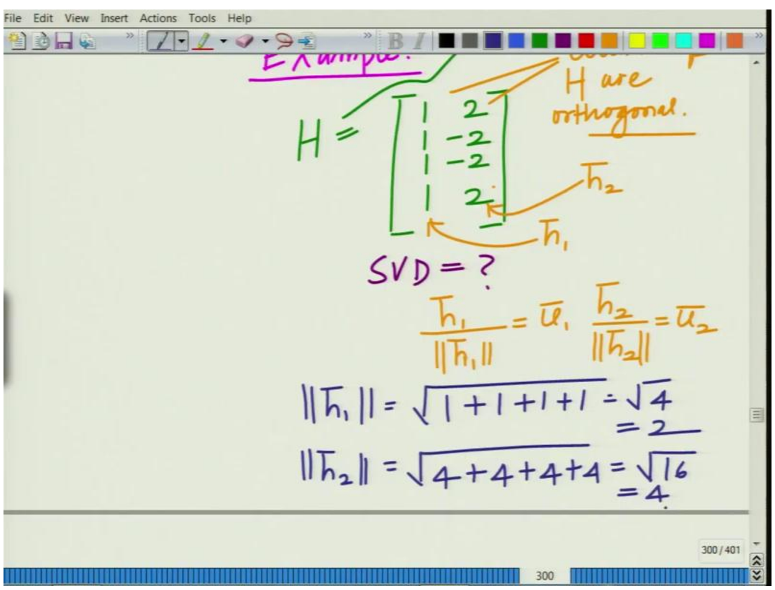
So this becomes, if you look at it, this becomes V Sigma square V Hermitian inverse V Sigma U Hermitian. So this becomes V Sigma square V Hermitian, so this is V Hermitian, Sigma square inverse is Sigma raise to minus 2 because V Hermitian inverse is V, so that becomes, you can write this as follows. Just write an additional step to make it clear. V Hermitian inverse Sigma square inverse is Sigma raise to minus 2, this is V inverse times V Sigma U Hermitian, but V Hermitian inverse is V.

So this is V Sigma raise to minus 2, V inverse V is Identity, times Sigma times U Hermitian which is essentially equal to, so H pseudo inverse is essentially equal to V Sigma minus 1 U Hermitian, where Sigma minus 1 naturally, is a diagonal matrix with the elements as 1 over Sigma 1, 1 over Sigma 2, 1 over Sigma n. So this is essentially the pseudo inverse of H, the pseudo inverse of H in terms of the SVD, in terms of the singular value decomposition, so the inverse of H in terms of the singular value decomposition.

So this is basically, if you look at it, this is basically the structure and the properties of the singular value decomposition or rather the different matrices, the different component matrices of the singular value decomposition. Let us now look at a simple example to understand this. Kind of a paper and pen example of the singular value decomposition.

(Refer Slide Time: 21:34)





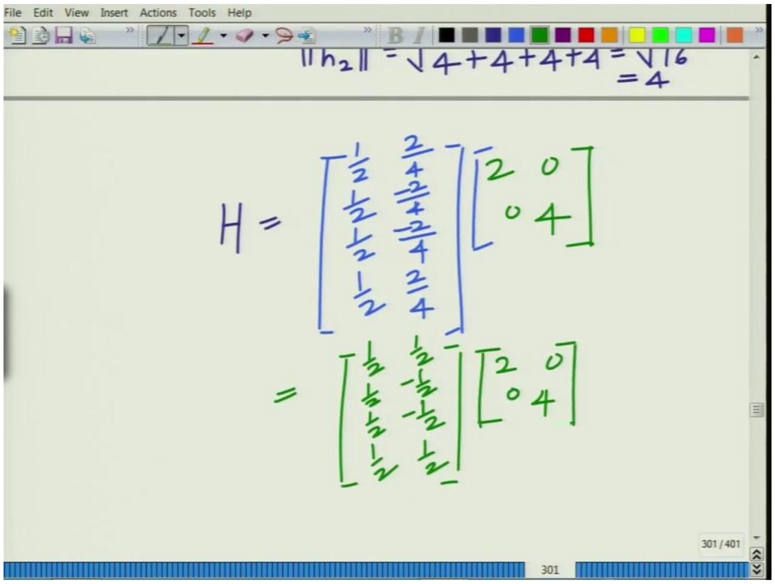
So let us look at, of course the singular value decomposition, typically is not easy to compute by hand, but let us look at a simple, let us try to do this via a simple example to illustrate this. So you have H. Let us consider H which is the following matrix. So H is this 4 cross 2 matrix. So it is a tall matrix, 4 cross 2. Now, you can see, now, let us ask the question, what is the SVD of this?

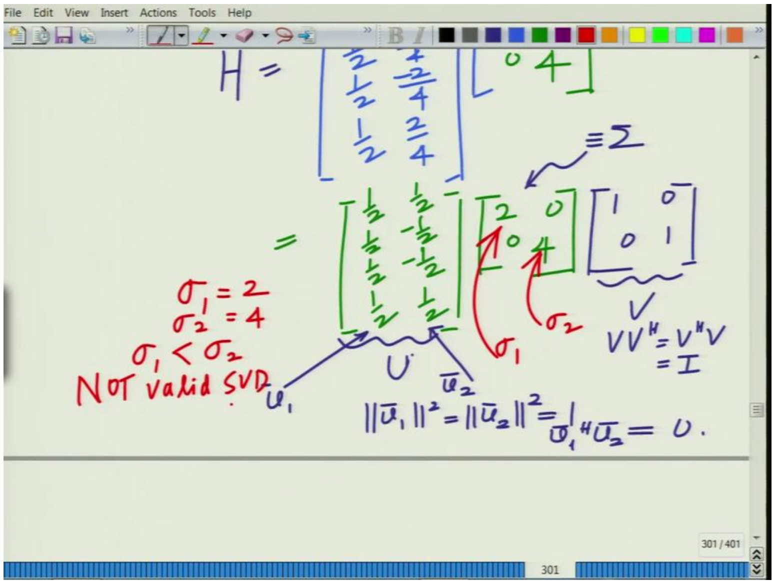
Now, you can see, ordinarily, it is not very easy to evaluate the singular value decomposition but in this case, there is an interesting property. If you look, observe closely, the columns of H are orthogonal. And therefore, it has, it is almost similar to the structure of the matrix U. So, we can take advantage of that to evaluate the singular value decomposition of this in a rather easy fashion.

So, what is the point here, the columns of H are orthogonal and therefore it is similar to U but except, columns of U are orthonormal. So, we have to normalize the columns of H. So, you look at this, this is your h1 bar and this is your h2 bar. We have to perform h1 bar divided by norm h1 bar and that will become our u1 bar and h2 bar divided by norm h2 bar, that will become our u2 bar.

So what is norm of h1 bar? Norm of h1 bar is under root of 1 plus 1 plus 1 plus 1 equal to under root of 4, equal to 2. Similarly norm of h2 bar equals under root of 2 square which is essentially 4 plus 4 plus 4 plus 4, equal to under root 16, which is equal to 4.

(Refer Slide Time: 24:27)





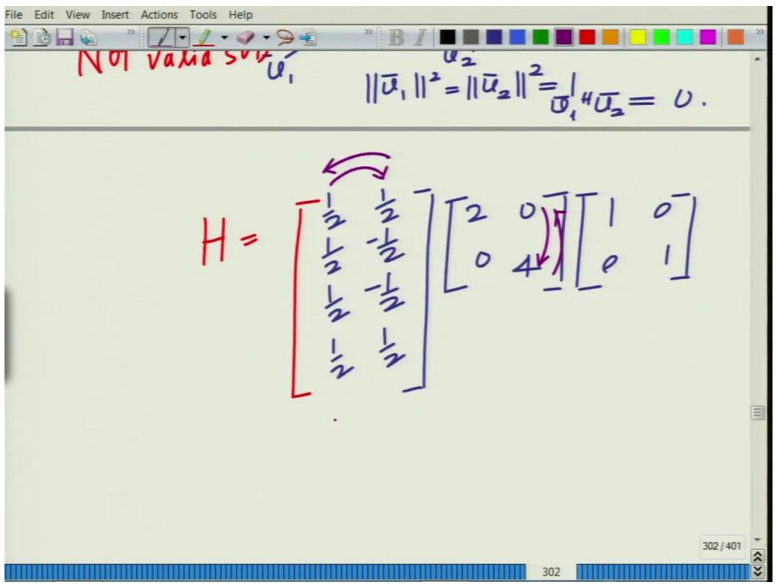
And therefore, I can write H, after normalizing the columns as, well, that will become 1 divided by 2, 1 divided by 2, 1 divided by 2, 1 divided by 2, 2 divided by 4, minus 2 divided by 4, minus 2 divided by 4, 2 divided by 4 times now, the diagonal matrix. First column has to be multiplied by the norm 2, second column has to be multiplied by the norm 4 which is essentially equal to, now simplifying this, half, half, half, half, half, minus half, minus half, half and times 2, 4, 0, 0.

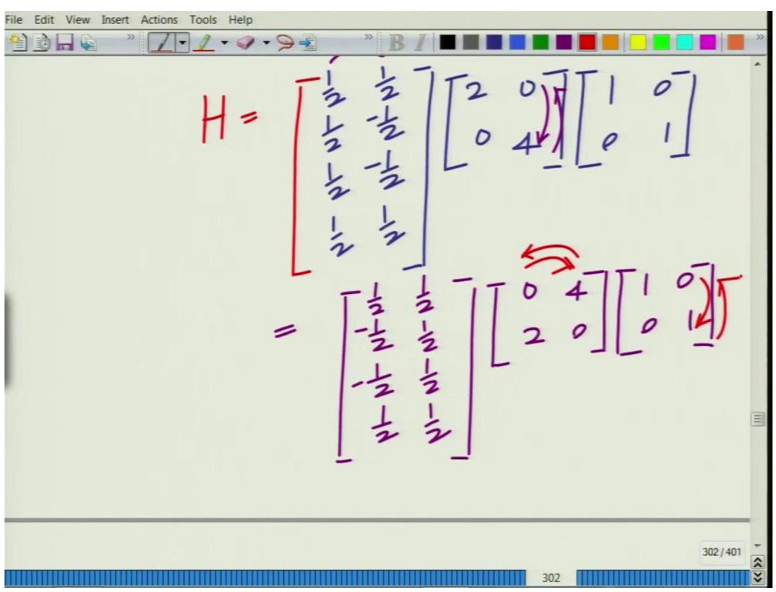
And now, you can see this is your matrix U and this is your column u1 bar and this is your column u2 bar and you can see they are orthonormal. That is u1 norm, u1 bar square equal to norm u2 bar square equal to 1, u1 bar Hermitian u2 bar equal to 0. And now you can see this is very similar to Sigma. I am not saying this is equal to Sigma but this is similar to Sigma. So all I need is V. I can insert the V as follows.

I can write this as the Identity matrix. So this is Unitary. Now, this is V. Because V into V Hermitian equal to V Hermitian V equal to Identity and Sigma is a diagonal matrix, the elements, the diagonal elements are non-negative. The problem here is, the diagonal elements are not arranged in decreasing order. So therefore, this is not a valid SVD because if you look at the diagonal elements, Sigma 1 is 2, Sigma2 is 4, so Sigma 1 is less than Sigma2. That is the only problem here.

Except for that, so Sigma 1 and this is Sigma2, so Sigma 1 equal to 2, Sigma2 equal to 4, Sigma 1 less than, so if you look at this, we have Sigma 1 equal to 2, Sigma2 equal to 4 and therefore, Sigma 1 less than Sigma2 implies this is not a valid SVD. So all we have to do is, essentially we have to, if somehow we could switch the diagonal elements, instead of 2, in the place of 2 if we had 4, in the place of 4 if we had 2, this would be an SVD. And we can do that right away by doing a simple trick, by switching the columns and rows and columns alternatively.

(Refer Slide Time: 27:42)

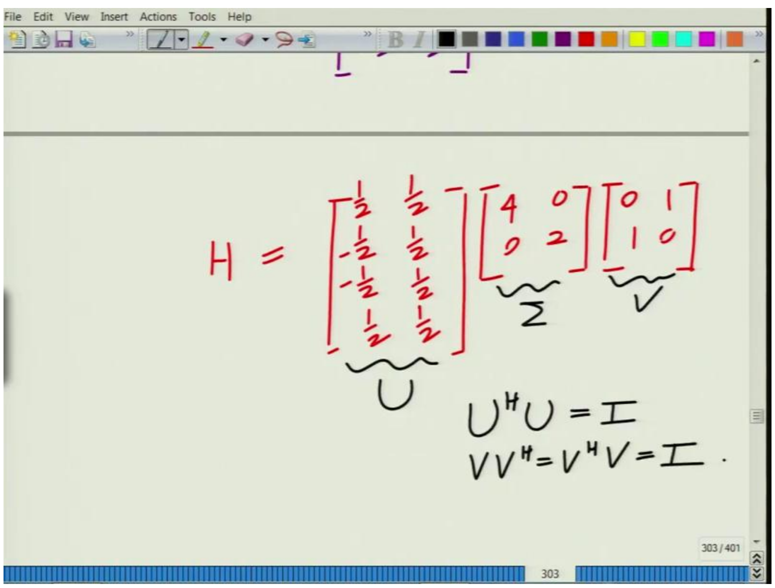


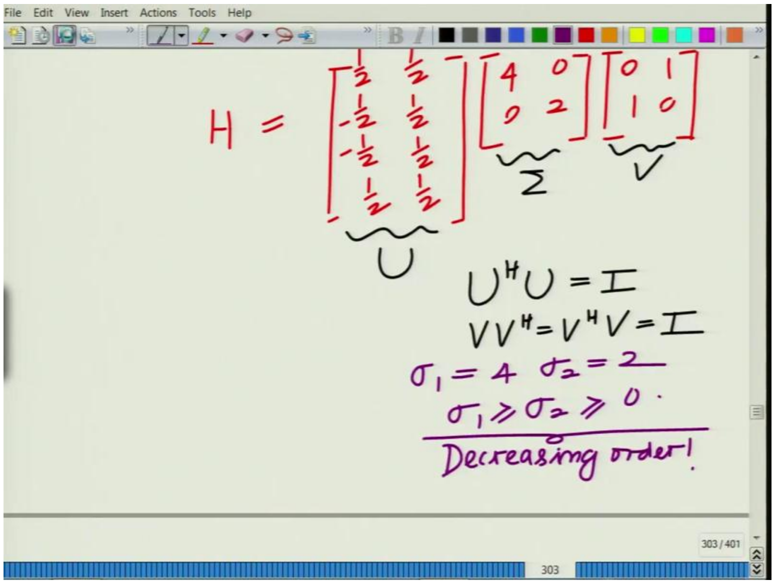


So I have H equals, let me write this once again and let me illustrate what can be done here. It is simply a permutation of the columns and rows. So you have half, half, half, half, half, minus half, minus half, half, times, you have, 2, you have 4, 1, 0, 0, 1. Now, what I am going to do is, I am going to switch the columns of this and correspondingly switch the rows of this. That will leave the product unaltered.

So this is going to become, switch the columns of the first matrix and the rows of the second matrix, half, minus half, half, minus half, half, half, half, half and here, I am going to write 0, 4, 2, 0, 1, 0, 0, 1. Still not in the form, in fact, it is worse because the diagonal have become off-diagonal elements. Now, all I have to do is perform switch once again. The columns of the middle matrix and the rows of the last matrix.

(Refer Slide Time: 29:21)





Now, this is going to become your half, half, half, minus half, half, half, half, half, half. Switch the columns, this becomes 4, 0, 0, 2 and switch the rows, this becomes 0, 1, 1, 0. And now, you can see this is a valid singular value decomposition. So this is your matrix U, this is your matrix Sigma, this is your matrix V. We have U Hermitian U is Identity. V V Hermitian equals Identity.

And we have, Sigma is diagonal with Sigma 1 equal to 4, Sigma2 equal to 2 and they are arranged in decreasing order. So these are in decreasing order. So that is a simple example. As I already told you, in general, it is not easy to evaluate the singular value decomposition via hand but in this case because H has an interesting structure, we have exploited that to rather easily come up with a singular value decomposition.

So in essence, once again, the singular value decomposition, just like the eigen value decomposition, in fact, it is more useful, I can say in certain ways than the Eigen value decomposition because the Eigen value decomposition is defined only for square matrices whereas the singular value decomposition is defined for any matrix H of an arbitrary size m cross n.

In particular, we have shown this for the scenario where m is greater than equal to n but you can readily also see what the counter part is going to be if m is less than n. And we have described the components that is U Sigma V Hermitian, the properties of each of these components, then U matrix, V matrix and the Sigma matrix and finally we looked at a simple example to understand this better.

So, please, as I have already told you, singular value decomposition is very significant and has many, many, many applications throughout in practice, talk about signal processing, wireless communication, machine learning, data analysis. I mean, there are a large number of applications and it is in fact, I would like to say, one of, along with Eigen value decomposition, it is probably one of the most widely used tools in modern scientific analysis. So please go through this again and try to understand and appreciate this concept better and understand it thoroughly. Thank you very much.