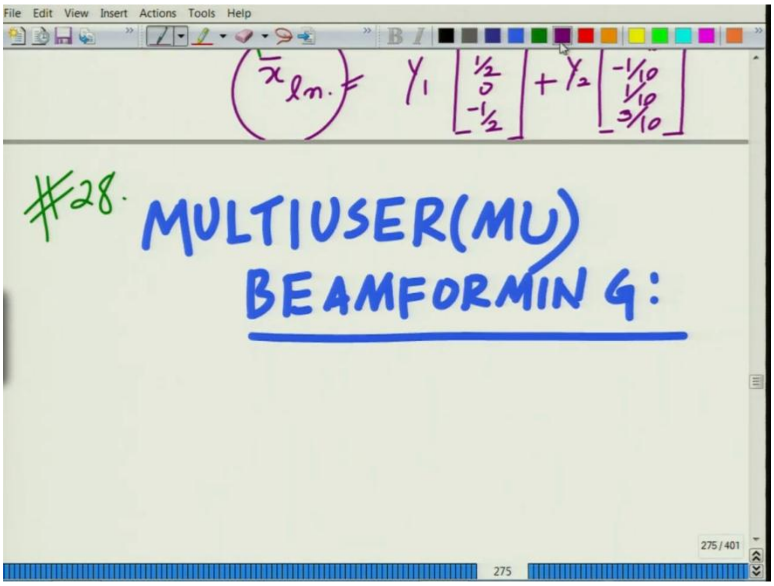
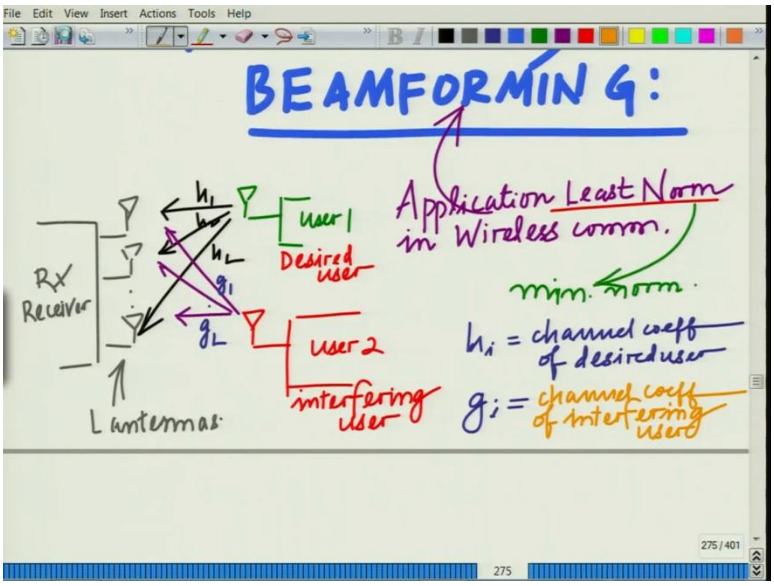
Applied Linear Algebra for Vector Processing, Data Analytics and Machine Learning
Professor Aditya K Jagannatham
Department of Electrical Engineering
Indian Institute of Technology Kanpur
Lecture 28
Wireless Application: Multi-user beamforming

Hello. Welcome to another module in this massive open online course. So we have looked at the least norm solution to an underdetermined system of linear equations. In this module, let us look at an interesting application, a very practical application of that, in the context of which is in the context of multi-user beamforming in wireless communication systems.

(Refer Slide Time: 01:00)



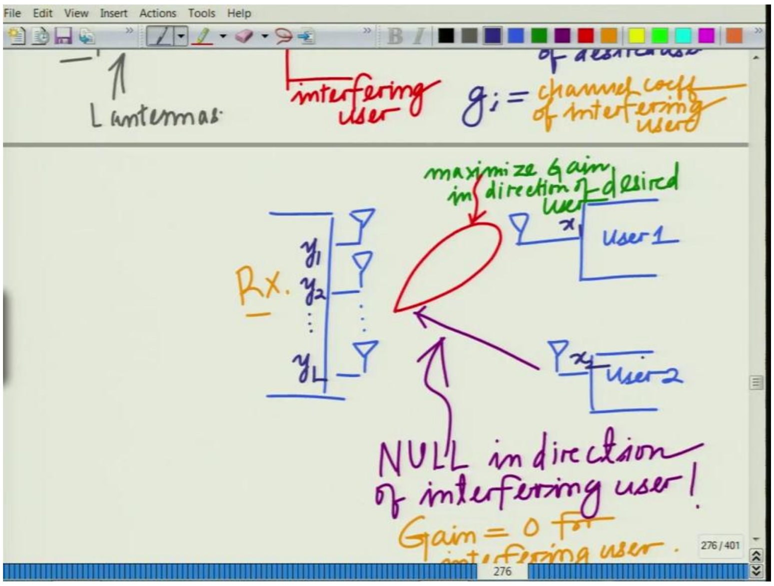


So, let us talk about multi-user or basically what we also call a multi-user beamforming, which is again a very interesting application in the context of, this is essentially an application of the least norm in wireless communication. And just to mention this, clarify this, this least norm, this is also sometimes known as the minimum norm. I think most can be used interchangeably, that is the minimum norm solution to a system of linear equation or the least norm solution to a system of linear equations.

And what is this application multi-user beamforming, is essentially as follows. So, let us say once again you have multiple antennas and this is your receiver, let us say you have L antennas and you have, two users. Now you have, this is a multi-user scenario, so we have user 1, who is the desired user, and you have user 2 who you can think of. So this is your desired user and user 2 is the interfering user and you have a channel between each user and the receiver.

So, you have the channel. So, let us say channel coefficients of the desired user, these are denoted by h1, h2, so on up to hl, and the interfering user we denote by g1, up to gl. So, hi equals to the channel coefficient of the desired user that is between the desired user and antenna i of the receiver. And similarly, gi is a channel coefficient between the interfering user and antenna i of the receiver. So this is the channel coefficient of the interfering user. And now, what we want to do in multi-user beamforming is something that is interesting.

(Refer Slide Time: 04:33)

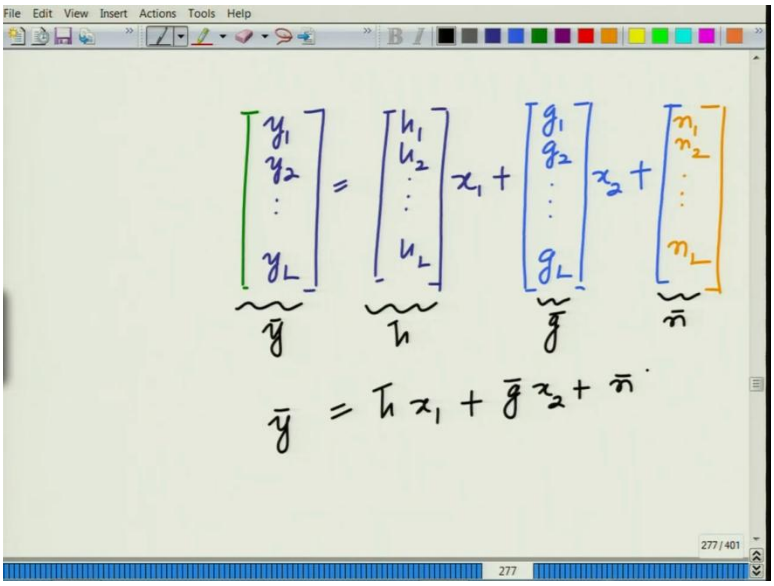


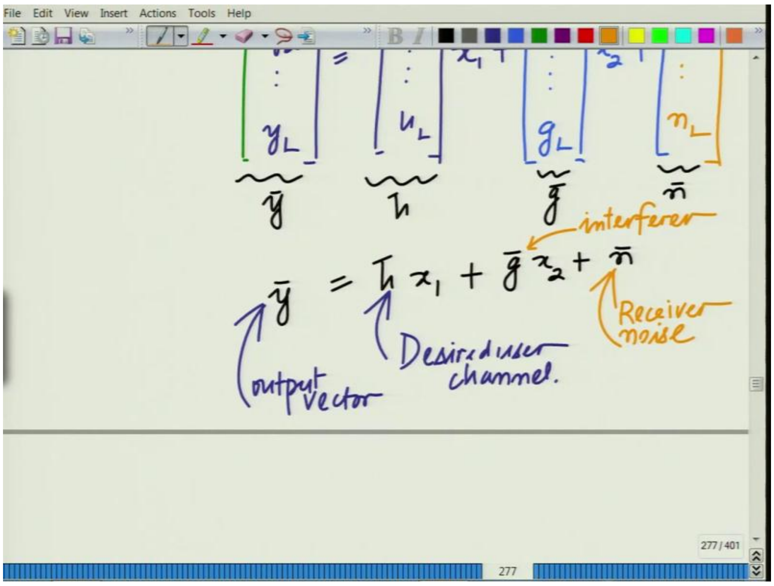
So you have this multiple antenna system. And that allows us to separate multiple users very interestingly to receive signal from the desired user while suppressing the interference from the interfering user as follows. So, we form a beam in the direction of the desired user. So we form a beam that is you maximize the signal gain in the direction of the desired user. You maximize gain in direction of the desired user, while in the direction of the interfering user there is no gain that is gain equal to 0. So there is a null in direction of the interfering user.

What this means is that is the gain for the interfering user, that is gain equal to 0 for the interfering user. So, this is your receiver. So, what this multiuser beam former is doing is essentially there are two users, one is the desired. So, you want to receive the signal from one user while you want to suppress the interference, the signal, from the other user, who is the interfering user. So, this multiple antenna system it allows you to do that.

It allows to direct your beam, the receive beam that is maximize the signal gain in the direction of the desired user while placing a null or zeroing the gain in the direction of the interfering user. So, that the signal of the interferer is essentially suppressed, so that is the essential. So this is your null place, it is known as null placement and on or interference suppression essentially. And therefore, the way this works is as follows. So, let us say we have the received signal, let us call this as y1, y2, yl, these are the outputs, and x1 is the transmit symbol of user 1, x2 is the transmit symbol of user 2.

(Refer Slide Time: 08:32)

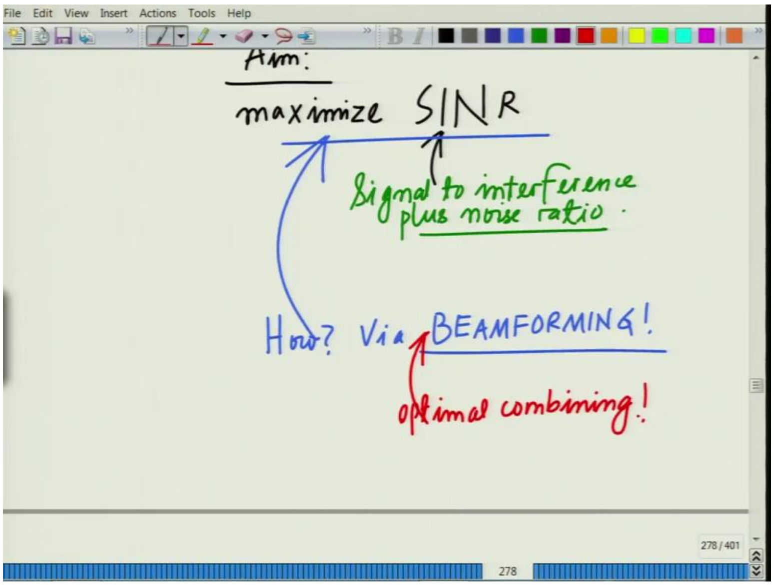




Then I can write this model as the system model for this multiple user system as y1, y2, yl, l is the number of antennas. This is the signal of the desired user which is the channel coefficients h1, h2, hl, times x1. The symbol transmitted by user 1 plus the channel coefficients of the interfering user g1, g2, gl, times x2 plus, n1, n2, nl, which is the noise vector. And therefore, this is y bar this is equal to h bar this is your channel vector g bar this is n bar. So, we have y bar equal to h bar x1 plus g bar x2 plus n bar. This is the system model where y bar is output vector and this is your desired user channel.

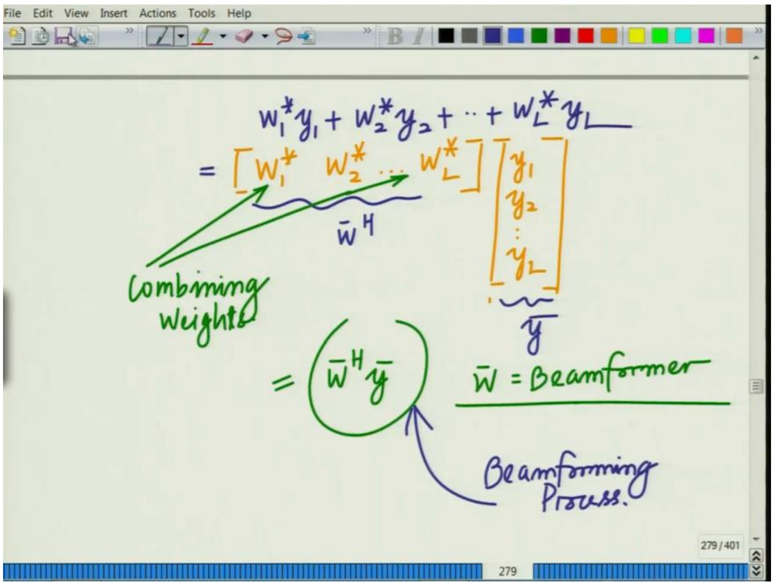
This is your undesired or interference channel and this is essentially your noise at the receiver. This is the receiver noise. Now, what needs to be done here is again as you might have seen before, we perform beam forming, remember that is what we do. Whenever we want to perform the optimal signal processing that is the maximize the signal to noise per ratio. And in this case maximize the signal to interference plus noise ratio is maximize the signal to noise per ratio and also suppress the interference.

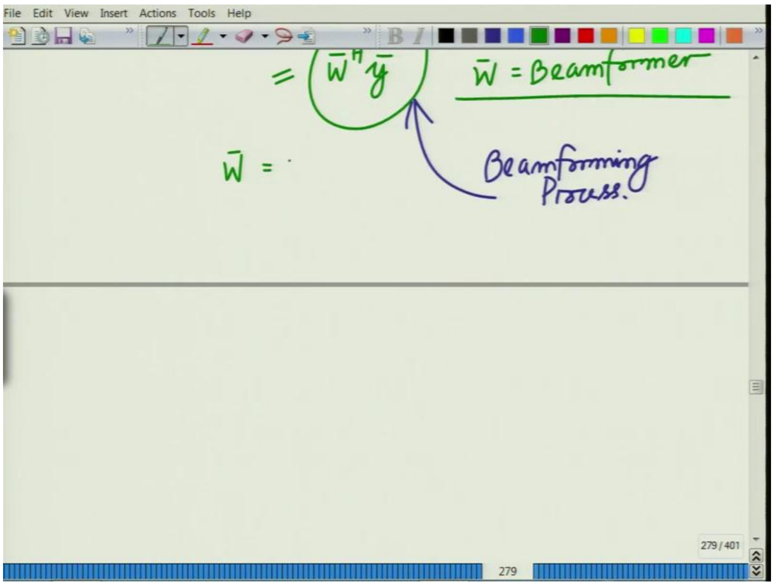
(Refer Slide Time: 10:48)



So, maximize the SINR, that is what we are doing in this case, because there is also interference. So, our aim is to maximize SINR. What is SINR? This is the signal to interference plus noise ratio. We want to maximize the signal to interference plus noise ratio. And so this is our aim. What is our aim? Our aim is to maximize the signal to interference plus noise ratio. And for that, and how do we do that? How to achieve this? We ask the question, how again, our answer is via good old beamforming, which is essential and nothing but the optimal combining. This is nothing but optimal combining.

(Refer Slide Time: 11:47)



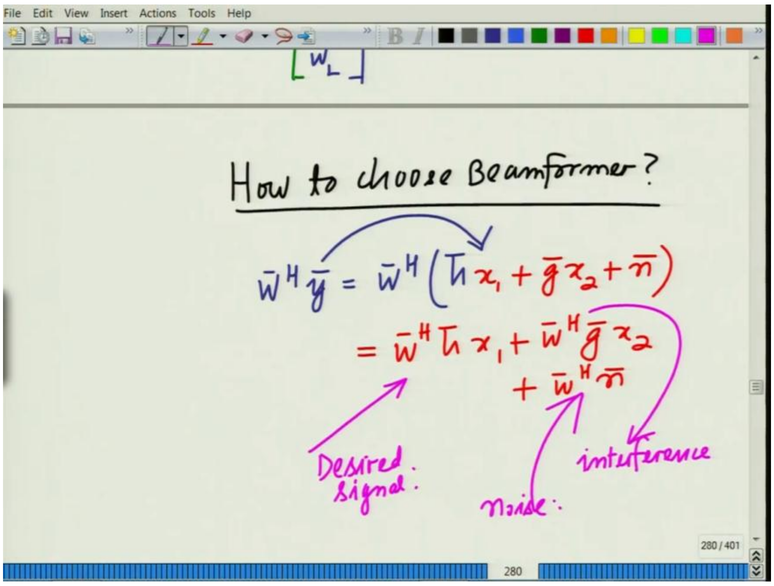


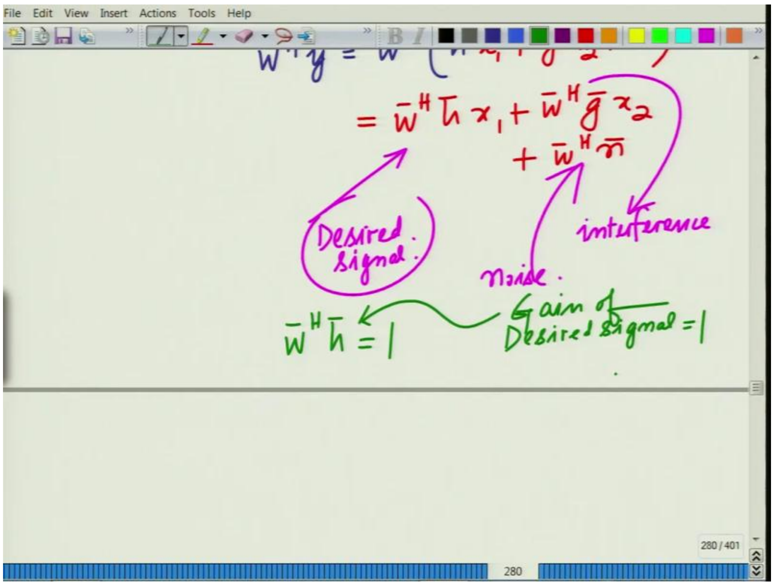
So, take the received vector, received output symbols and we have performed W1 conjugate, W2 conjugate, so on and we have combine the output symbols. That is essentially we are performing W1 conjugate y1 plus W2 conjugate y2 plus WL conjugate y1, which can be written as follows and this is your vector W bar Hermitian. This is your vector y bar.

And these are essentially your combining weights or these are essentially your beam forming, you can think of this as your beam forming weights. And therefore, I can write this as W bar Hermitian y bar, where W bar is now, this is your W bar is the beam former that we seek. So, W bar Hermitian y bar this is the beam forming process. W bar is the beam former, this is your beam forming process.

And so, therefore, we are combining the outputs y1, y2, yl across the antennas using the weights W1, W2, WL. And if we denote this by the vector W bar, we want, that is essentially what is our optimal combiner or this is the beam former. Because essentially we are trying to find a beam that is essentially what the beam former means. And thus vector W bar is the beamforming vector and we wish to find the optimal beamforming vector, which essentially maximizes the signal to interference plus noise ratio. So, that is the aim. So, your W bar equals W1, W2, WL. So, this is your beam former.

(Refer Slide Time: 14:30)

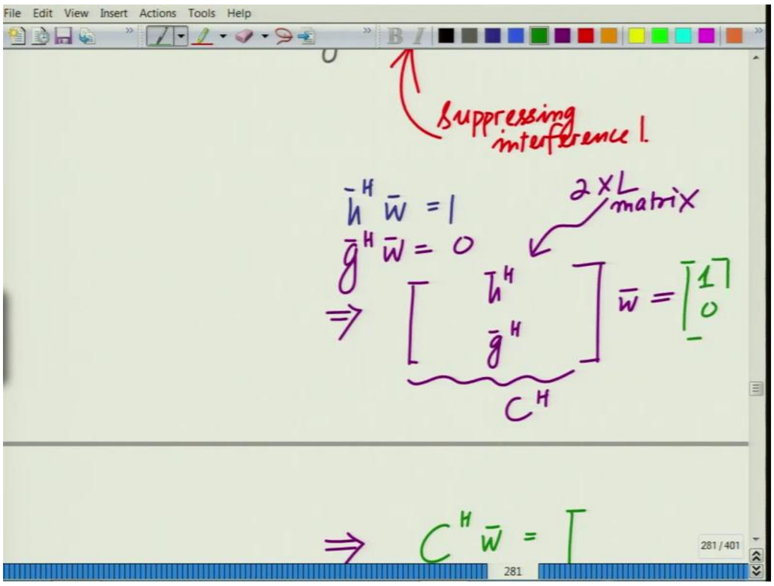


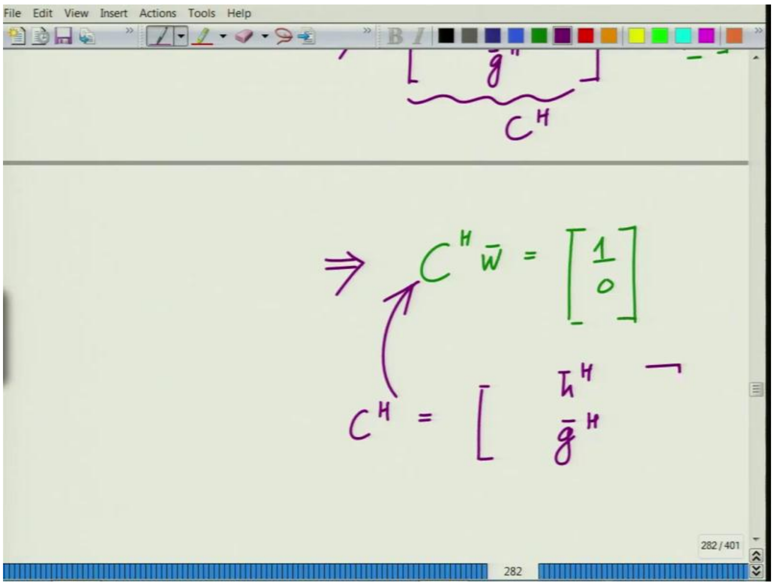


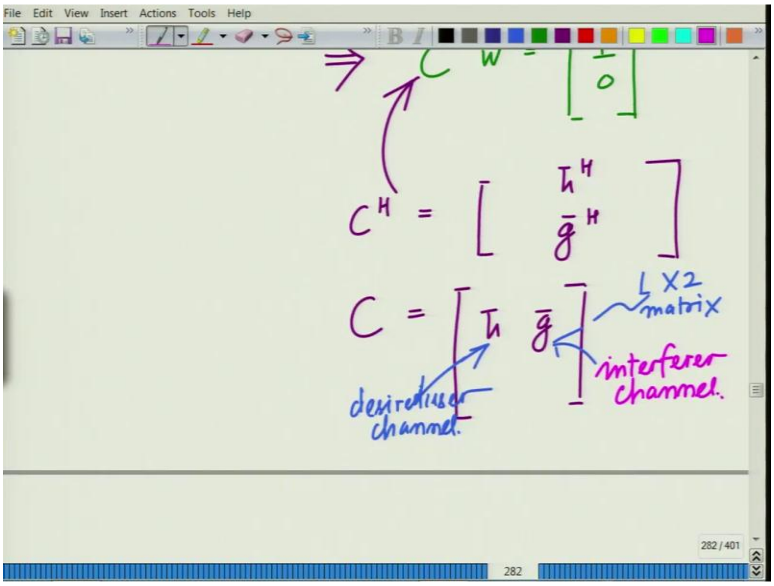
And now, how to choose the beam former? Now the question that we want to ask is, how to choose the beam former? How does one choose the beam former? And the solution to that is the following thing. We want to perform W bar Hermitian y bar, which is substituting for y bar, y bar is nothing but h bar, if you go back and take a look at this, this is h bar times x1 plus g bar times x2, which is equal to W bar Hermitian h bar times x1 plus W bar Hermitian g bar times x2 plus W bar Hermitian times n bar.

And now, there are three components. One is the desired signal, the other is the interference and this is the noise. And you want to maximize the signal to interference plus noise ratio, what we will do is we will start with the desired signal. For the desired signal, what we are going to do is we are going to set W bar Hermitian h bar equal to 1. So this implies that the gain of the desired signal equals unity. So we are fixing the gain of the desired signal to be unity.

(Refer Slide Time: 16:59)





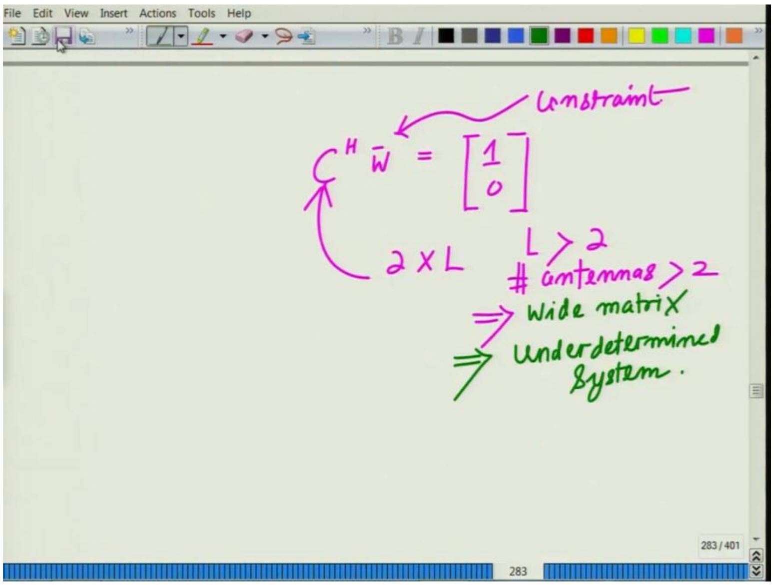


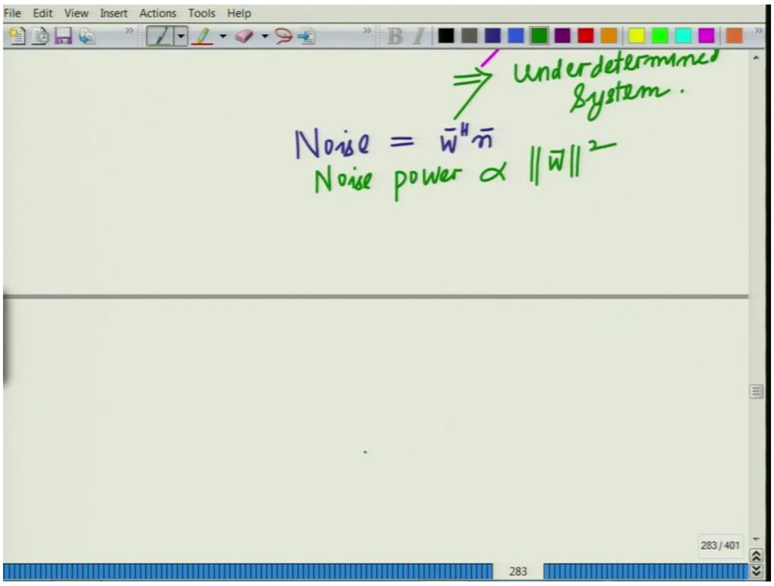
And then, now, the other important condition here is essentially what happens to the interference, remember, we want to suppress the interference. We want to place a null that is we want to reduce the interference to 0, which means the second quantity W bar Hermitian g bar this must be equal to 0. That is if you look at this quantity W bar Hermitian g bar this must be equal to 0 for this is the gain of interference.

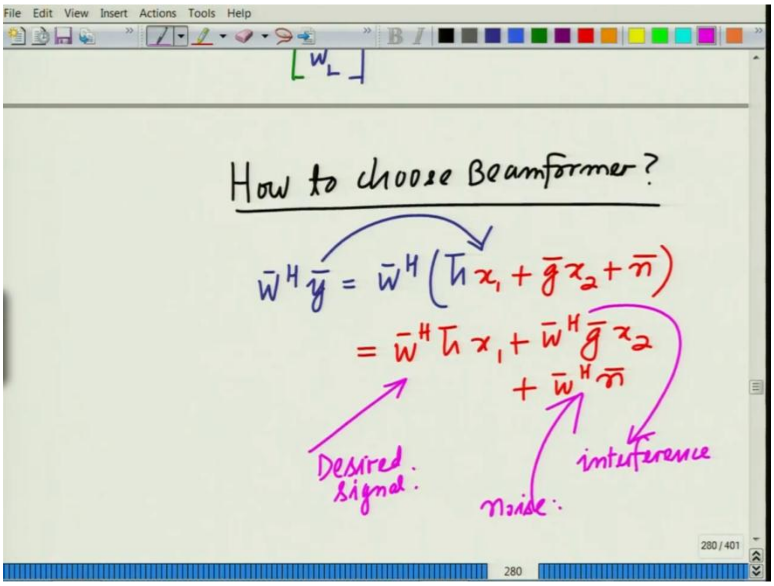
Now by setting it equal to 0 what you are doing is, you are suppressing the interference. W bar Hermitian g bar equal to 0, you are suppressing the interference. And therefore, now you put these two things together, you have, or let me write it slightly differently, you have h bar Hermitian W bar equal to 1, we are taking the Hermitian of that. And you have g bar Hermitian W bar equal to 0, which implies, now putting this together as a matrix, you have h bar Hermitian and g bar Hermitian times W bar equal to 0. And basically this is a 2 cross L matrix and let us call this matrix as C Hermitian.

So this implies that we have, from these two conditions you have C Hermitian W bar, I am sorry, this has to be equal to 1, 0, because h bar Hermitian W bar equal to 1, g bar Hermitian W bar equal to 0. So this is equal to 1, this is equal to 1, 0. Where what is the matrix C? C Hermitian, remember is h bar Hermitian g bar Hermitian, which implies C. If you look at C, the matrix C is the desired channel vector, which I am looked to desired user channel vector of interfering user. So this is going to be your L cross 2, matrix of size L cross 2. What is h bar? h bar is desired user channel and this is your interfering user, interferer channel.

(Refer Slide Time: 20:03)



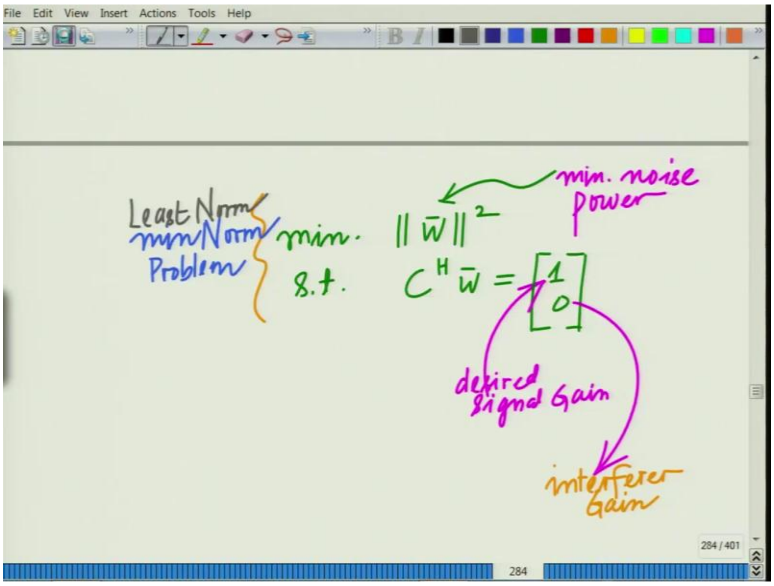


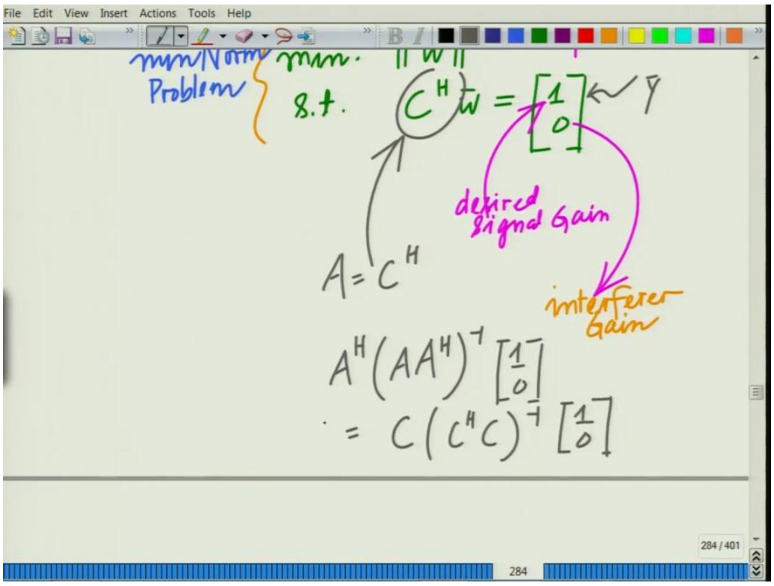


And Therefore, we have once again C Hermitian W bar equal to 1, 0, this is our constraint. And you can see, if C, now C Hermitian is of size 2 cross L, which means if L is greater than 2 that is number of antennas implies this becomes a wide matrix. For a number of antennas greater than 2, this implies wide matrix, which implies this becomes your underdetermined system. So, for L greater than 2 that is when you have a number of antennas greater than 2 at the receiver you have 4 or 3, 4, 5 and so on, this becomes an underdetermined system.

To constrain the solution now, of course, we have another aspect that is the noise power. Turns out that the noise power, now if you look at the output noise power recall the noise is basically if you look at this, if you go back and take a look at this. This is our W bar Hermitian n bar that is your noise power. So that is your noise power. So, the noise is W bar Hermitian n bar and turns out that the noise power is proportional to norm of W bar square.

(Refer Slide Time: 22:12)



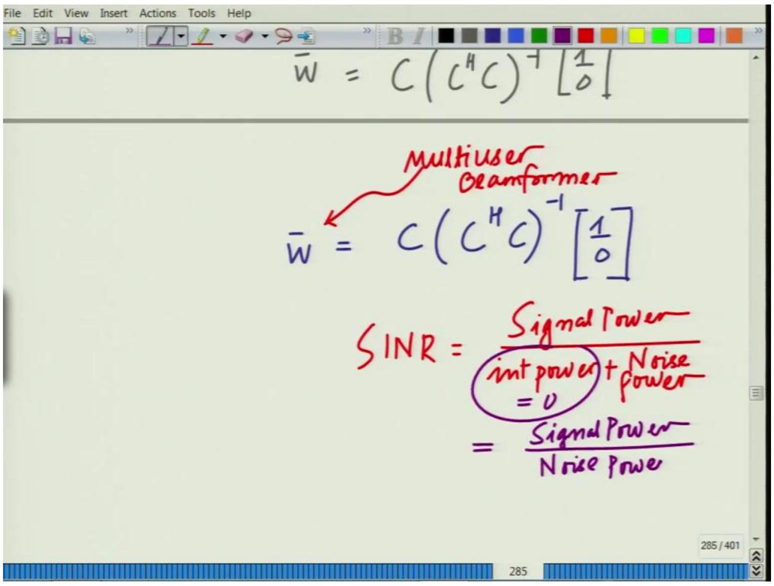


So, essentially, the pertinent optimization problem for this multi-user beamforming that can be formulated as, minimize the noise power. So the relevant optimization problem is minimize norm W bar square subject to the constraint C Hermitian W bar equals this quantity that is 1 comma 0. And this is now essentially, so this is basically minimize the noise power.

This is desired signal gain. This is essentially your interferer. And therefore, this if you look at it this multi-user beamforming problem this is now your, this is now a minimum norm problem. That is a minimum norm or the least non problem. You can also, I mean it is the same thing this is the least norm problem. And therefore, the only difference here is that you have A equal to C Hermitian.

And the solution therefore is given as, remember the solution was A transpose but for complex matrix you can replace the transpose by Hermitian. So this becomes A Hermitian AA Hermitian inverse times y bar, which is in this case 1 comma 0. So this is your vector y bar. Now, replacing A by C Hermitian, this become C, C Hermitian, C inverse 1 comma 0. And this is basically your desired multi-user beam former.

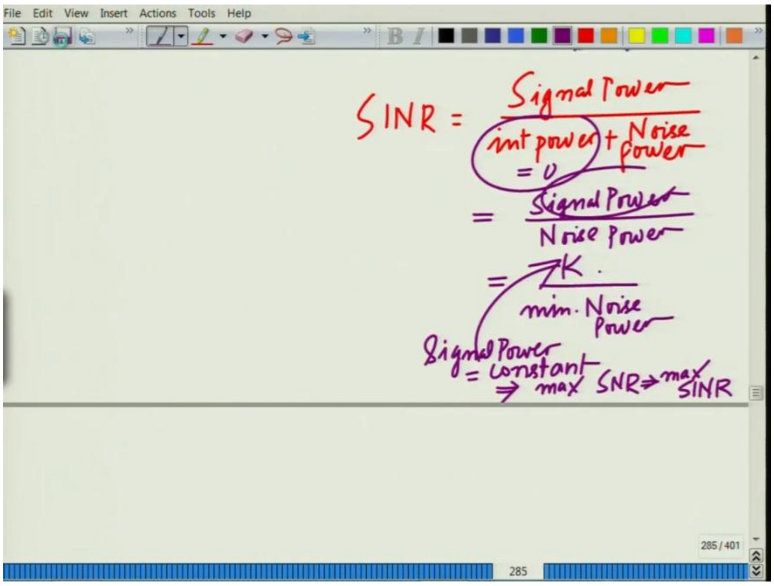
(Refer Slide Time: 24:47)

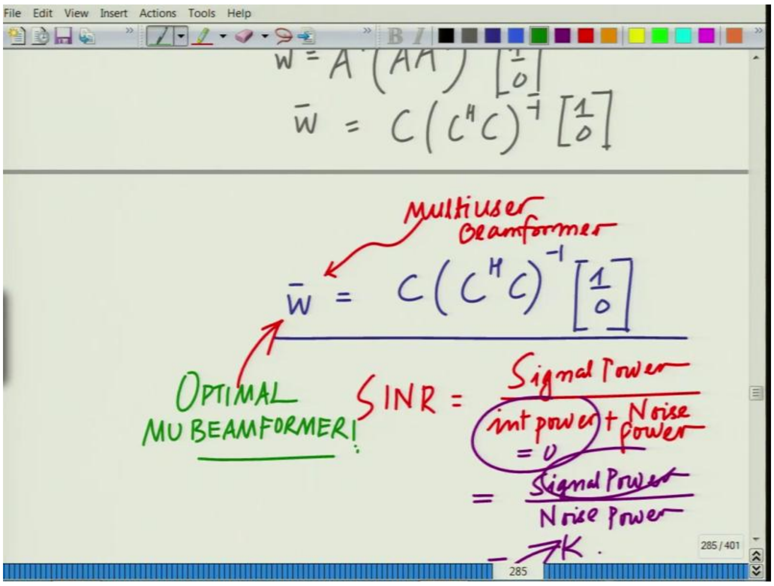


So the optimal multi-user beam former is C C Hermitian C inverse 1 comma 0. So this is essentially your, this is basically the multi-user beam former. And what is this beam former doing? Fixing the signal gain, minimizing the noise power. So it is basically maximizing the SINR signal noise power ratio.

At the same time, interference is 0. So you can say that is maximizing the signal to interference plus noise ratio, because interference plus noise is simply the noise. So, if you look at SINR that is essentially your signal power by interference power plus noise power. Now in this formulation, our interference power this is equal to 0. So this reduces to signal power by noise power.

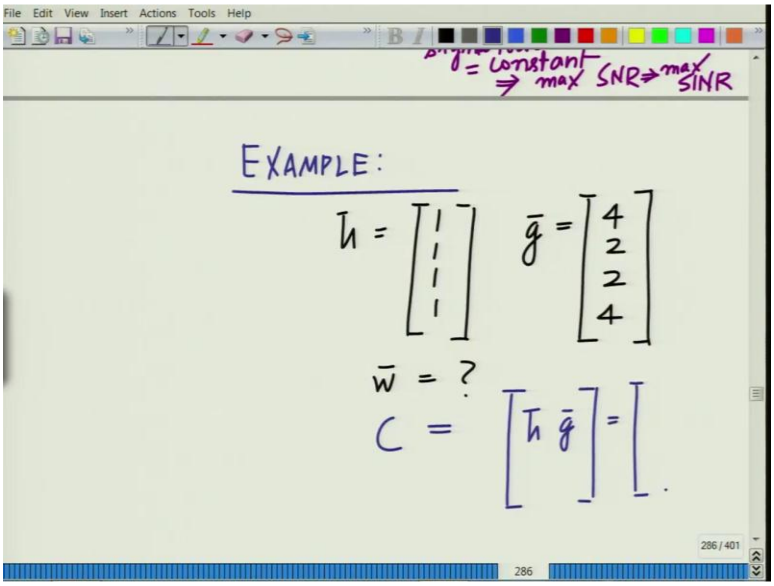
(Refer Slide Time: 26:21)

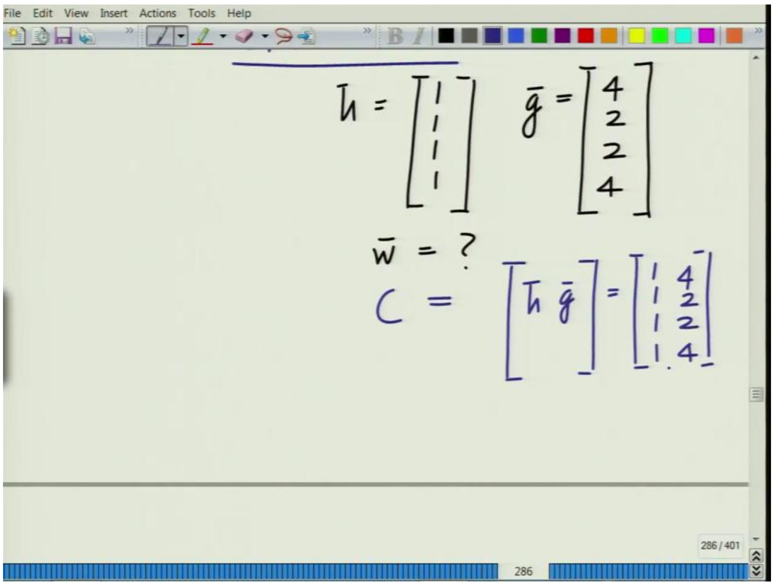


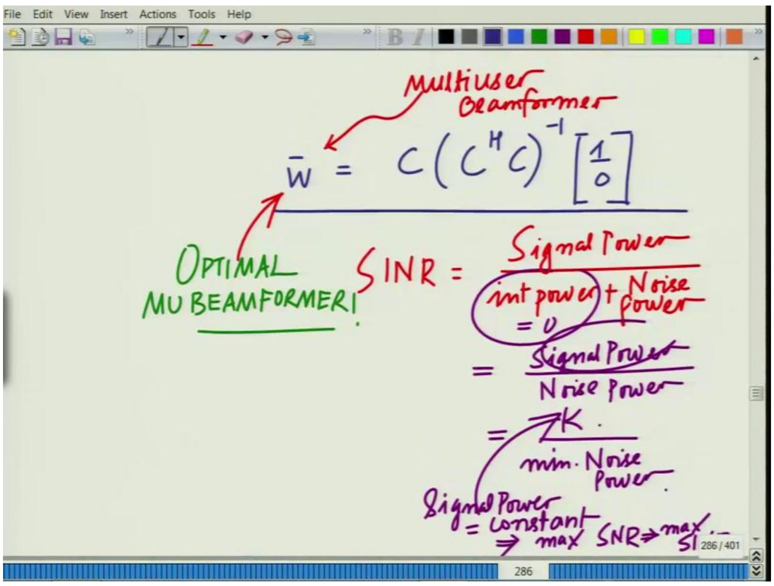


Now, the signal power, the signal gain equal to unity, which means this is a constant divided by what we are doing is we are making constant signal power. So signal power is constant and we are minimizing the noise power. So this maximizes the SNR, which in terms maximizes the SINR, because as I have already told you, the interference power is 0. So, maximizing the signal power, signal to noise ratio is simply is same as maximizing the signal to interference plus noise ratio. So, this is W bar, you can think of this as the optimal multi-user beam former. This is the optimal multi-user beam former, optimal MU that is multi-user beam former that is what your W bar is.

(Refer Slide Time: 27:45)

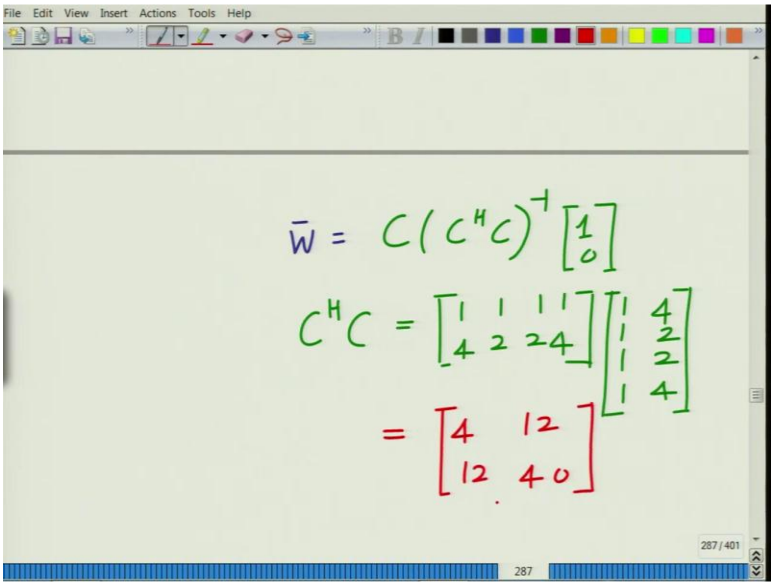


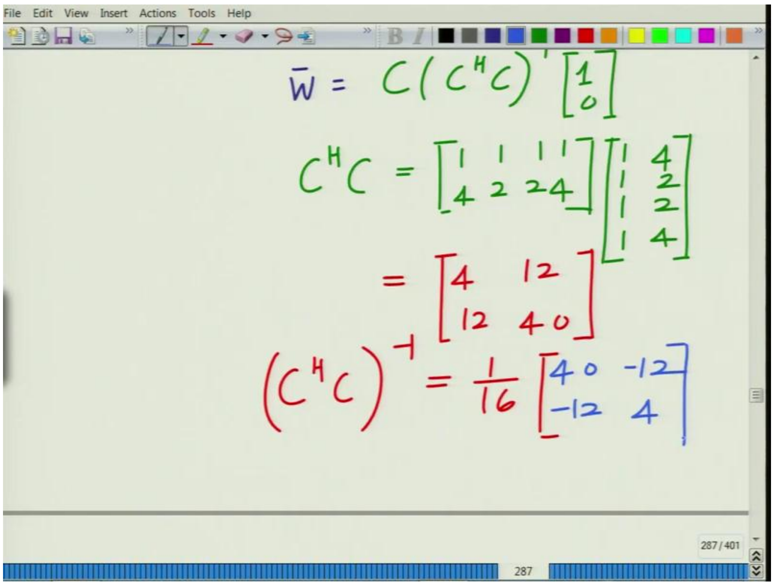




As usual let us do a simple example to understand this. Simple example, let us say you have given a problem in which your h bar is 1, 1, 1, 1 and your g bar that is a interference channel is 4, 2, 2, 4. And one can ask the question, what is the optimal multiuser beam former W bar? For that you first form the matrix C, which is h bar, g bar, which is essentially your matrix 1, 1, 1, 1; 4, 2, 2, 4.

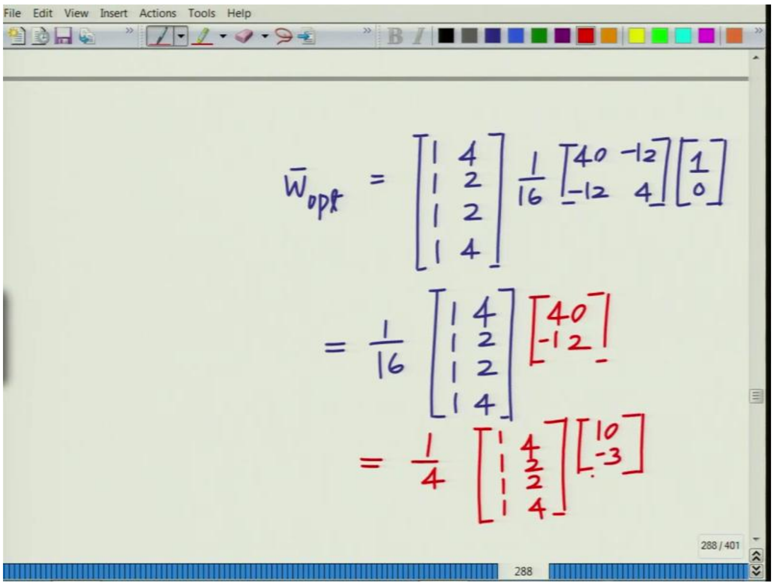
(Refer Slide Time: 28:54)

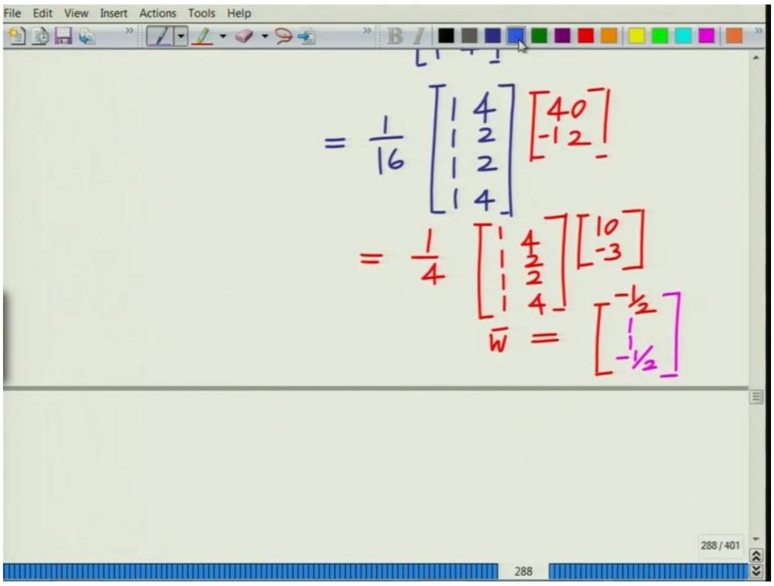




And, remember, using the expression W bar equals C C Hermitian, C inverse 1, 0. So, let us first form C Hermitian C, that is 1, 1, 1, 1; 4, 2, 2, 4 C Hermitian times C. That is 1, 1, 1, 1; 4, 2, 2, 4, which is equal to 4, 40, 12, 12. And therefore, C Hermitian C inverse, this is equal to 40 into 4, 160 minus 12 into 12, 144 that is 1 over the determinant which is 16. Once again, now we are very familiar with this computing the inverse of 2 cross 2 matrix switch the off, switch the diagonal elements negative of the off diagonal elements, this is C Hermitian C inverse.

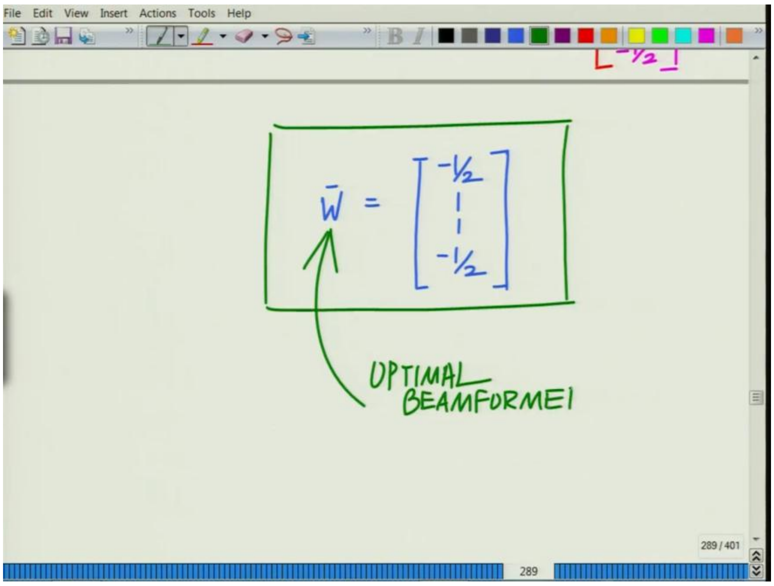
(Refer Slide Time: 30:11)





And therefore, W bar optimal which is equal to C, which is essentially your, what is this, this is your matrix 1, 1, 1, 1; 4, 2, 2, 4 times 1 over 16, 40, minus 12, minus 12 , 4 times the vector 1 comma 0. Which is essentially, now if you look at it, this is 1 over 16, 1, 1, 1, 1; 4, 2, 2, 4, times 40, minus 12, which essentially if you divided by 4, 1 over 4, this will be 1, 1, 1, 1; 4, 2, 2, 4, this will be 10, minus 3. And you can evaluate this, it is not very difficult to see your W bar reduces to 10 minus 12 divided that is minus 2 divided by 4 that is minus half, 10 minus 6, 4 divided by 4, 1, 1. And once again 10 minus 12 that is minus 2 divided by 4 that is minus half.

(Refer Slide Time: 32:02)



So, the optimal beam former W bar for this system that maximizes the SINR is essentially half, 1, I am sorry, minus half, 1, 1, minus half. And this essentially is the optimal beam former. What is it doing? Signal gain is constant, interference is 0, minimizing the noise power, therefore, maximizing the signal to noise power ratio, at the same time maximizing also the signal to interference plus noise ratio.

So, this is an interesting, very interesting application, in fact, one of the most profound applications of least norm, you can see it is a very practical application in the multi-user system. If you want to maximize the gain of the desired signal, but at the same time suppress the interference, so this can therefore have very interesting and very important applications in wireless communication.

In fact, you can easily extend it to a scenario with more than two users. It is not very difficult to see how you can suppress, maximize the signal gain of a desired user while minimizing the interference caused by more than one users, two, three, and so on. And I urge you to think about this, how would you extend this to a scenario with more than one interfering user. So with that, let us stop this module here and let us continue in the subsequent modules. Thank you very much.