Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
Professor Aditya K Jagannatham
Department of Electrical Engineering
Indian Institute of Technology Kanpur
Lecture 23
Application: pseudo-inverse and MIMO zero-forcing (ZF) receiver

Hello, welcome to another module in this massive open online course. So, we are looking currently at the least squares technique for finding the vector X in an over determined system of linear equations. Remember the number of linear equations is m the number of unknowns is n and m is greater than n. Let us now continue to look at applications of the least squares technique and one of the most important applications, at least one of the very important modern applications in wireless communications is in MIMO systems, that is multiple input multiple output wireless systems.

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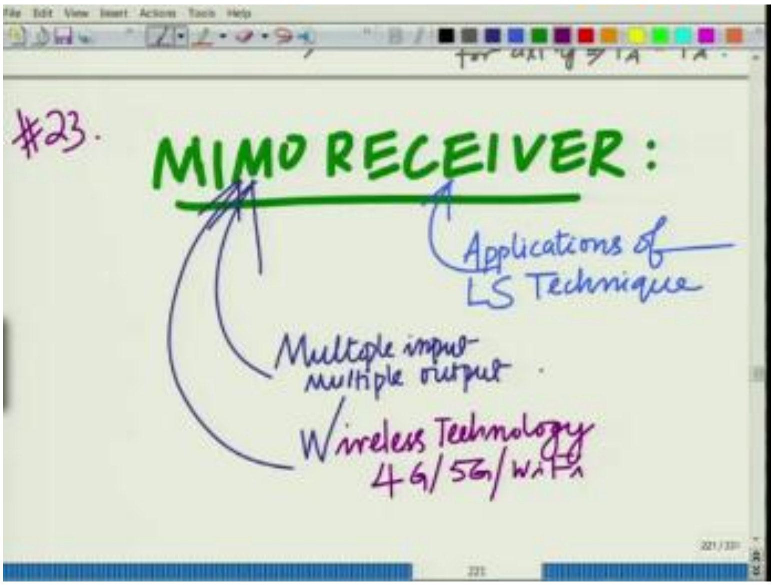

So, let us look specifically in a MIMO system, how to implement the receiver and remember what we are looking at? We are looking at applications of the LS technique and MIMO is basically, you might remember, this is multiple input multiple output and this is important wireless technology, a very important wireless technology. In fact, many of you, wireless technology, this is used in 4G, this is used in 5G and this is used in Wi-Fi so on and so forth.

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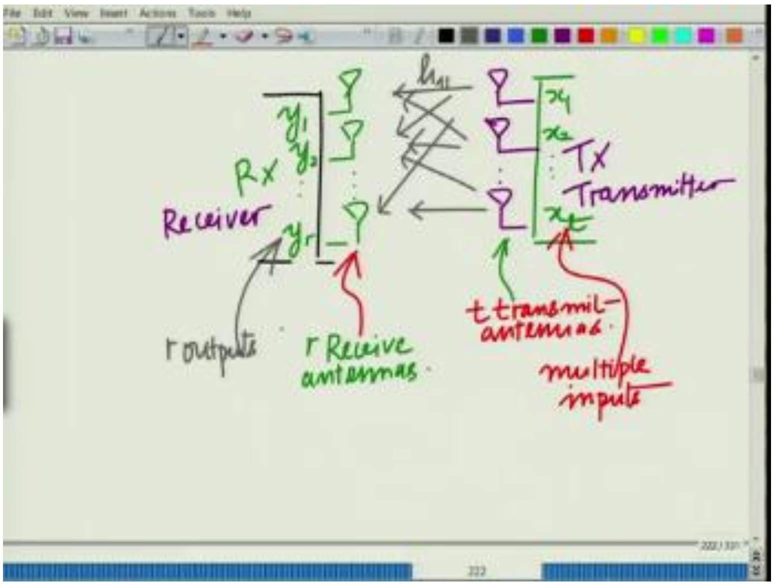

So, now what happens in the MIMO system is as we might have already talked about couple of times in this, you have a transmitter, you have a receiver, you have a receiver with multiple receive antennas that is the multiple outputs and then you have a transmitter with multiple transmit antennas that is the multiple inputs. So, this is the transmitter, this is the receiver and you are transmitting the symbols x, so you can think of this saying notation x1, x2, xn, y1, y2, ym, or let me denote it by more appropriate notation for wireless system that is y1, y2, yr, where r is the number of receive antennas and xt where t is the number of transmit antennas.

And these are the various channels and then you will have the corresponding channel coefficients. So, for instance between the transmit antenna 1 and the receive antenna 1, you have the channel coefficient which is h11. So we have a MIMO wireless system and these are the t antennas, these are the multiple t inputs and these are the r receive antennas, these are the t transmit antennas.

In fact, let me write this more, these are the t transmit antennas and these are the r receive antennas and these are the r outputs. So, you have a multiple input multiple output system. You have r receive antennas on which you are receiving the outputs y1, y2, yr, here the t transmit antennas on which you are transmitting the transmit symbols x1, x2, xt which form the input, so you have multiple outputs multiple inputs. This is the multiple input multiple output system.

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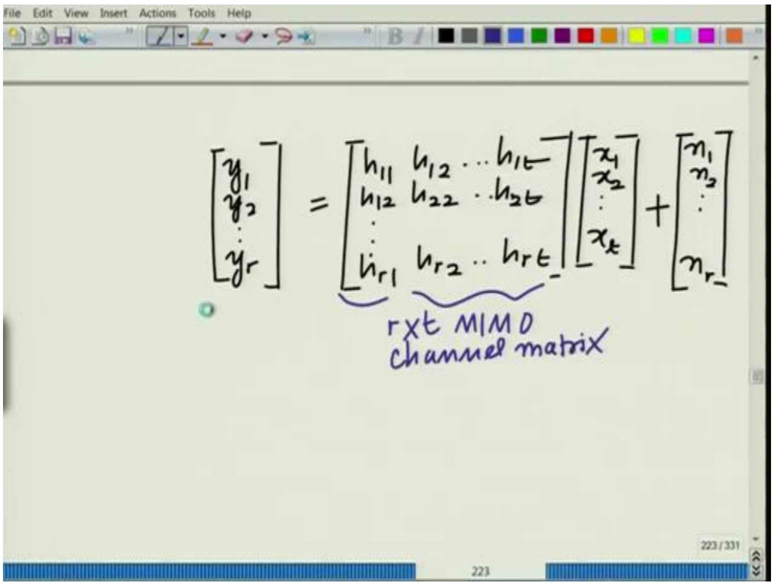

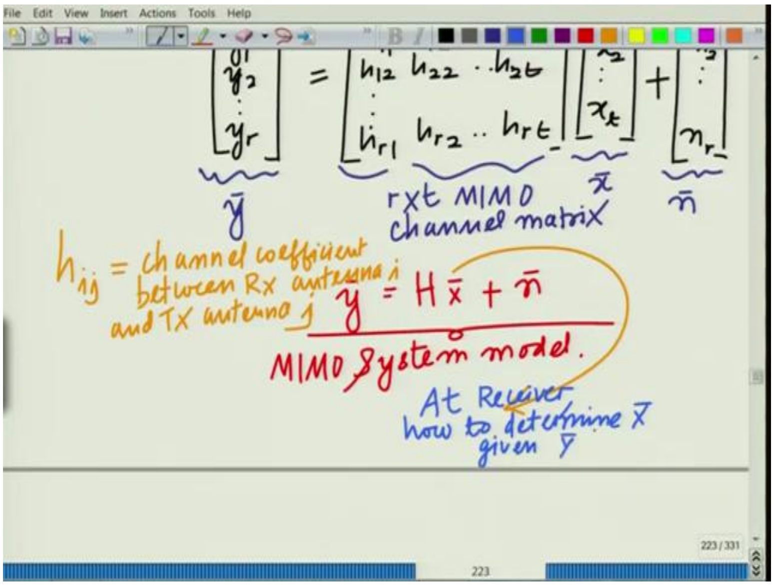

Therefore, you have the channel vector, I can write the model, and this is in fact, often known as system model y1, y2, yr. This is a neat demonstration of MIMO technology you have the channel matrix which is r cross t, then you have. So, this is your r cross t channel matrix, this is your transmit vector comprising of the t symbols and then you have the noise vector.

So, this is basically your r cross t MIMO, this is your r cross t MIMO channel matrix, this is your vector y bar, this is your vector x bar, this is your vector n bar. So, you have y bar equal to H x bar plus n bar that is your system model. This is often known as the system model. In fact, this is your MIMO system model and H is your channel matrix. Now the point is and of course, what is h, i, j, it is worth repeating this h, i, j is the channel coefficient between receive antenna j and transmit antenna.

This is the channel coefficient between receive antenna, I am sorry, receive antenna i and the transmit antenna j, for instance h12 that is the channel coefficient between receive antenna 1 transmit antenna 2, h21 is the channel coefficient between receive antenna 2 transmit antenna 1. Now the question is, how to determine x bar, this is the receiver, at receiver, you are transmitting the symbols x, in x bar x1, x2, xt at the receiver, how to decode, how to determine x bar given y bar?

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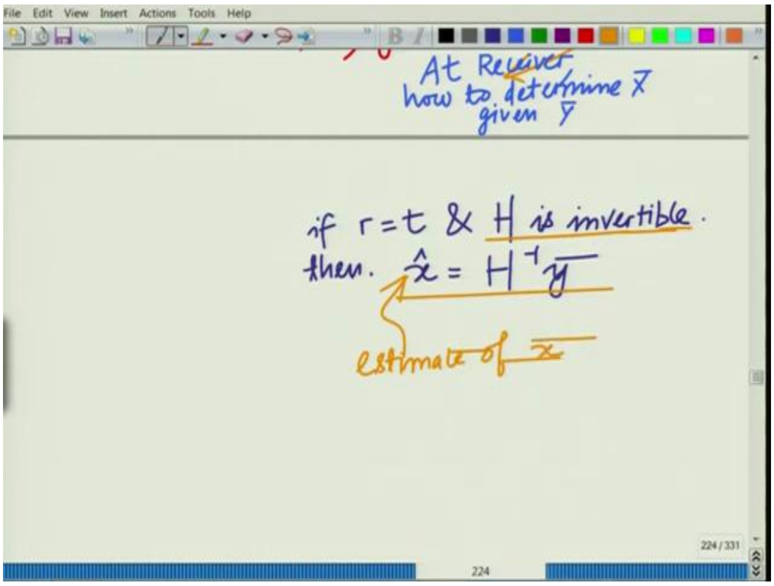

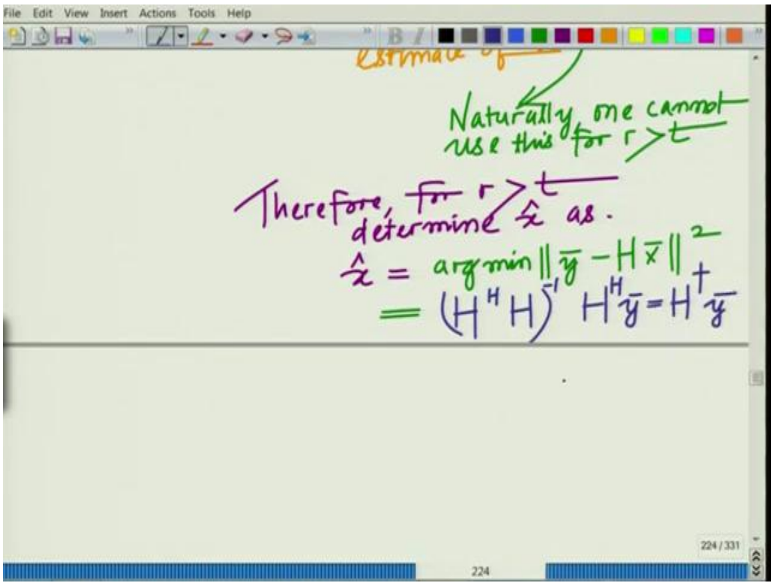

So, we have already seen if r equal to t and H is not singular, that is H is invertible, then x hat can simply be determined. This is what we have already seen in a previous module that if H r equal to t, that is H square matrix and H is invertible then we can form x hat that is the estimate of x, x bar which is an estimate of x bar. The transmit vector x bar is exact equal to simply H inverse y because H is invertible and at that point we have realized that we cannot use this if r is greater than.

Then how do we do it? And now we are at the right point to answer that question, address that question. So, naturally since the inverse does not exist for r greater than t, naturally one cannot use this for r greater than t and for r greater than t now we know therefore determine, therefore r greater than t determine or evaluate x hat as x hat is equal to, remember we solve the approximation problem.

y bar minus H x bar norm square which is basically given by the least square estimate which is H Hermitian H inverse, H Hermitian h, inverse H Hermitian y bar or which is also basically you can write this as H dagger y bar we already seen. H dagger is thing but the pseudo inverse of H bar, the matrix H that is H Hermitian H inverse into H and we are using the Hermitian because the channel matrix in general can be a complex channel matrix. So, we are using the Hermitian.

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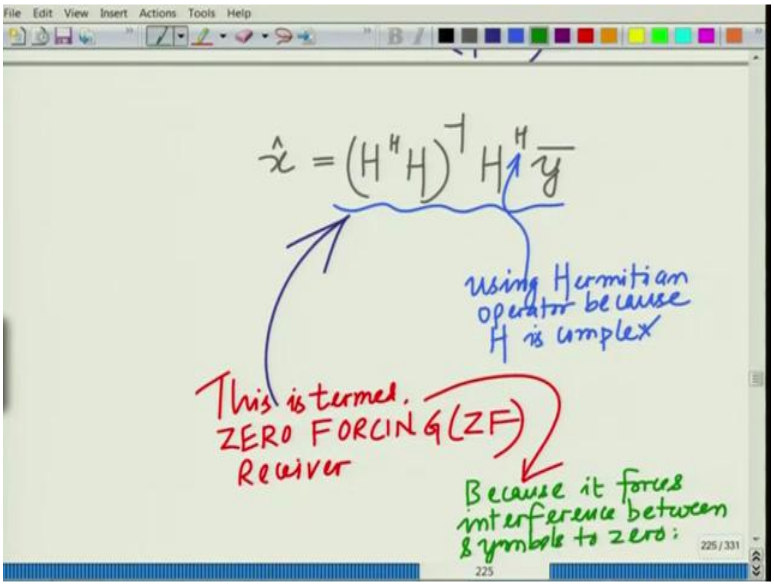

So, let us right this again. This is H Hermitian H, inverse H Hermitian y bar using the Hermitian instead of the transpose using the Hermitian operator, H is typically a complex. Because H is typically we look at the communication system in the base band and the base band, the channel can be typically represented as a complex channel. A complex channel coefficient. Hence, the matrix each channel coefficient h, i, j is complex and the channel matrix H is therefore naturally complex.

So, we typically we cannot use the Hermitian. Hermitian covers the general case of a complex channel matrix. And, this basically, this receiver which is basically, this is a name, this is known as the zero forcing receiver. This is termed, the zero forcing or the ZF receiver because the name arises, because it forces interference between the symbols to zero. Because it forces the interference between symbols to zero.

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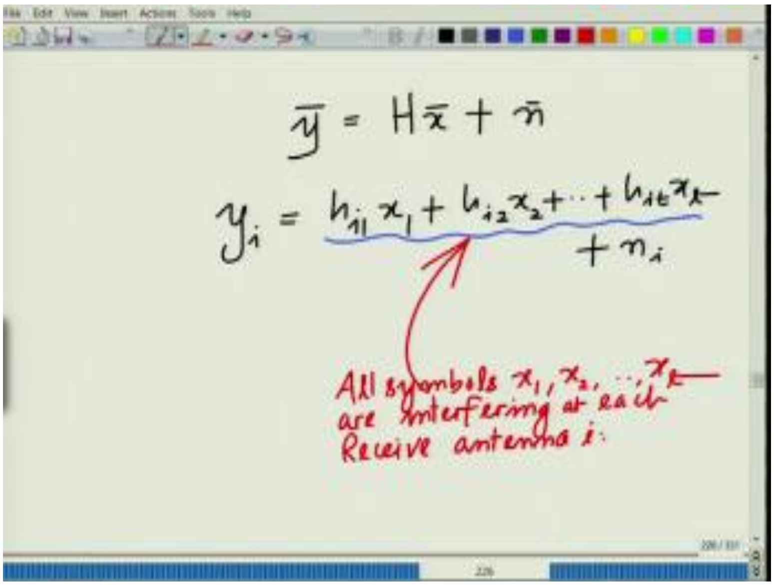

Let me explain this, you have y bar equals H times x bar plus n bar and therefore if you look at any y, i that is received on ith antenna that will be given as h1i, I am sorry hi1 times x1 plus hi 2 times x2 plus hi t times xt plus, of course ni and therefore if you look at any receive antenna i, all the symbols x1, x2, up to x 3 are interfering. There is inter symbol interference. We can say inter transmit antenna interference at any receive antenna. So, if you look at this, all the symbols, these are interfering at each antenna i.

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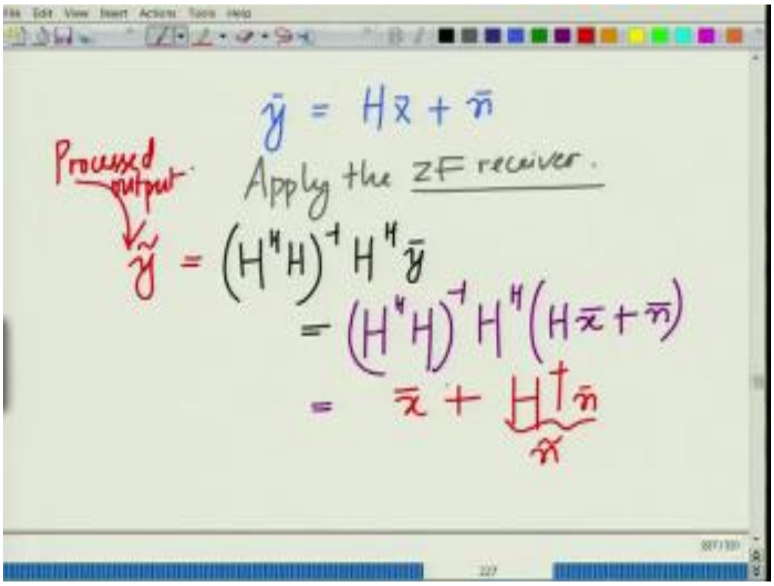

And now when you apply the zero forcing receiver, so you have y bar equal to Hx bar plus n bar. So, apply, so now what we do is, we apply the ZF receiver so we perform H Hermitian H inverse H Hermitian into y bar, which if you apply the zero forcing receiver you can see this reduces to H Hermitian H inverse H Hermitian substitute y bar, y bar equal to Hx bar plus n bar. Now if you multiply it out you will see this H Hermitian H inverse H Hermitian into H, that is nothing but identity so you have x bar plus you have this matrix pseudo inverse of H, let us call that H dagger, H dagger times n bar. This I can call as n tilde, so I can call this output, this is the processed output y tilde.

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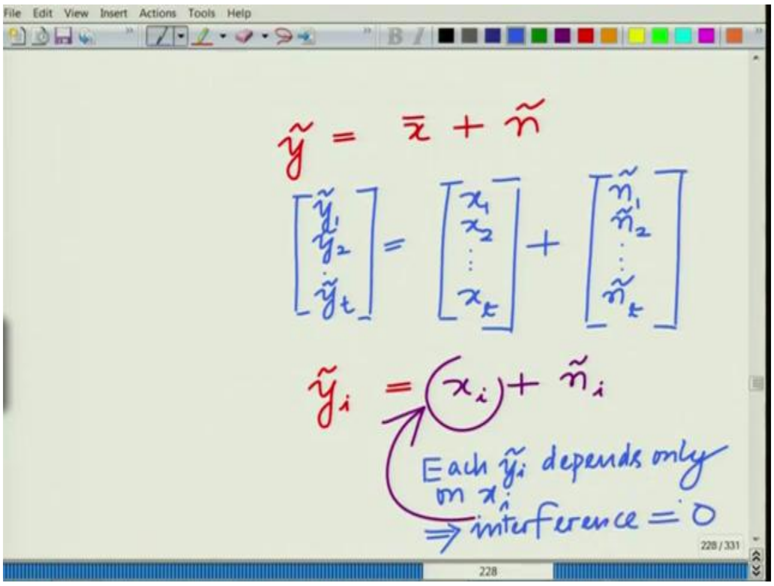

So, the processed output y tilde equal to x bar plus y tilde equal to x bar plus n tilde and therefore, now if you look at this processed output you can verify that this processed output is a t cross 1 vector because H Hermitian because the pseudo inverse is a t cross matrix. Therefore, now once you have this thing, you will have y1 tilde, y2 tilde, this is equal to n1 tilde.

And, now you can see each yi tilde is simply equal to xi plus ni tilde, so there is no interference among the different xi's at each yi tilde, although yi has the interference from the different xi's, yi tilde depends only on xi each, y1 tilde depends only on x1, y2 tilde depends only on x2 so on and so forth. t tilde depends only on xt, so therefore, using this receiver we have suppressed this interference. The interference is reduced to zero therefore this is known as the zero forcing receiver. So, each yi tilde depends only on xi implies interference between the symbols of the different transmit antennas, interference equal to zero.

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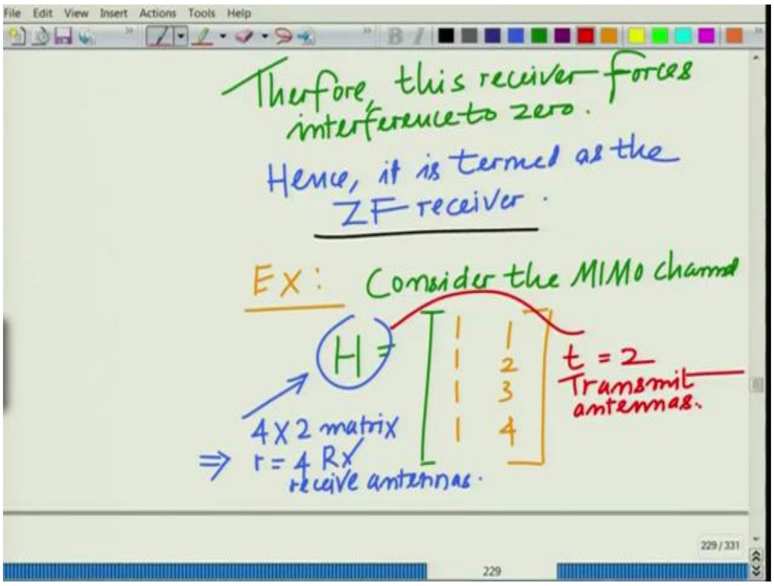

Therefore, this receiver forces the interference to zero hence it is termed as the zero forcing receiver, hence this is termed as the zero forcing receiver. This forces the interference between the symbols belonging to the different transmit antennas to zero, therefore, naturally this is termed as the zero forcing receiver.

Let us look at a simple example to understand this, example, so this is a very important concept and a very interesting as well as important as you are going to see there are several applications, in fact, this is just one application in the context of wireless communication that we are looking at. So, consider the MIMO channel H equals, let us write the channel matrix

1, 1, 1, 1; 1, 2, 3, 4 so just to make the example simple, I am considering a real channel matrix, of course you can see this is a simple example purely for this purpose of illustration.

Practice the actual channel matrix will be complex, in fact, these will be sort of the real and imaginary parts of each complex number is going to be some kind of a fixed point decimal number. So, this is your channel matrix, this is what we are terming as the channel matrix and if you observe this is a 4 cross matrix implies r equal to 4, this is a system that has r equal to 4 receive antennas and we have t equal to 2, we have t equal to 2 transmit antennas.

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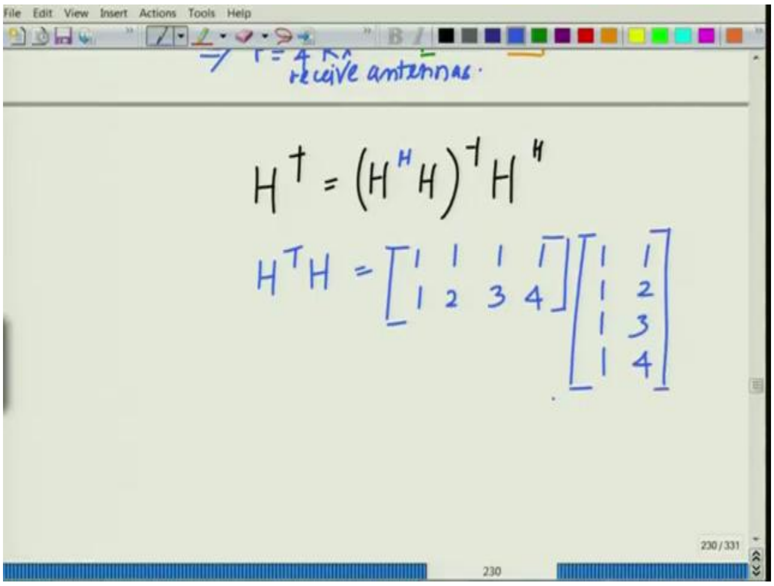

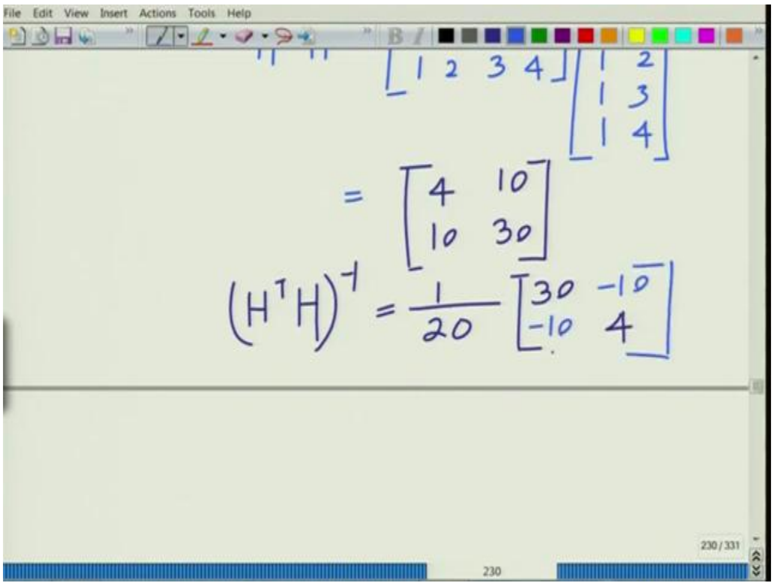

Now let us start evaluate the zero forcing receiver that is H dagger, remember which is H Hermitian H inverse H Hermitian, so first let us evaluate this matrix that is pseudo inverse matrix, now H of course here I can replace this by Hermitian by transpose so H transpose, because this is a real matrix. This will be 1, 1, 1, 1; 1, 2, 3, 4; 1, 1, 1, 1; 1, 2 and if you perform this you will get this as 4 10 10 then you have 1 plus 4 5, 5 plus 9 14, 14 plus 16 so this is going to be, the 2 cross 2 entry, this is going to be 30.

Let us evaluate now H transpose H inverse so that will be 1 over the determinant 30 into 4 minus that is 120 minus 10 into 10 that is 100 so this will be 120 minus 100 this is equal to 20 times interchange the diagonal elements, negative of the off diagonal elements, done. That is the inverse of your 2 cross 2 matrix.

I mean these kind of tricks I mean one has to become more and more familiar with the techniques of linear algebra and I hope with enough practice you are also going to be an expert at these kind of manipulations because as I told you linear algebra, matrix algebra, matrix manipulation, now is the standard in many, many fields of science and engineering be it wireless communication, be it signal processing, machine learning, data analysis, one has to be comfortable with matrices and linear algebra to have a better understanding and in fact, a much better grip on all these techniques.

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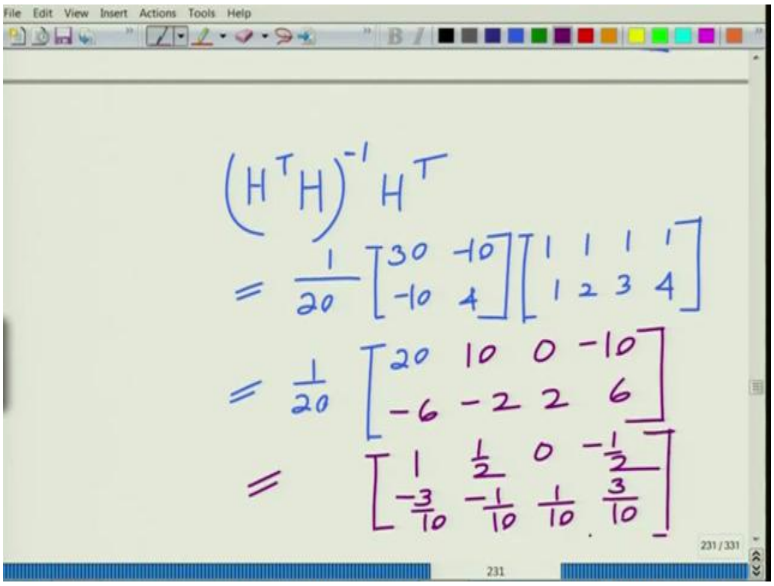

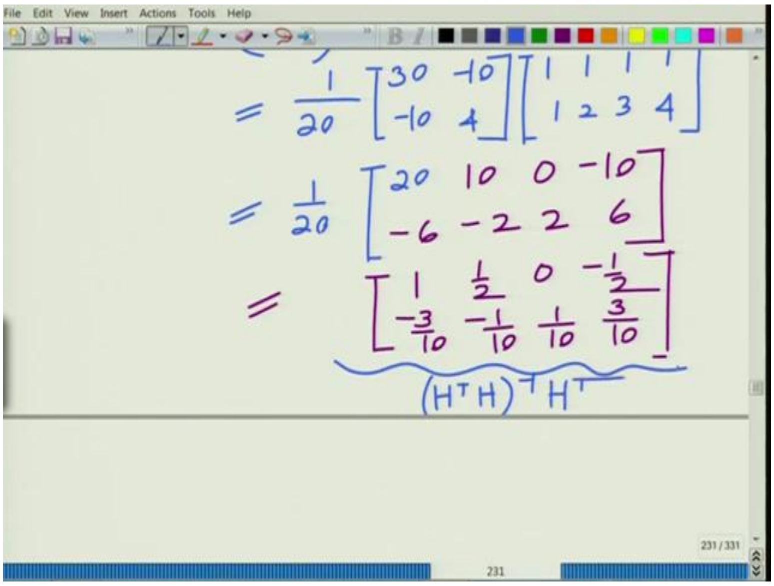

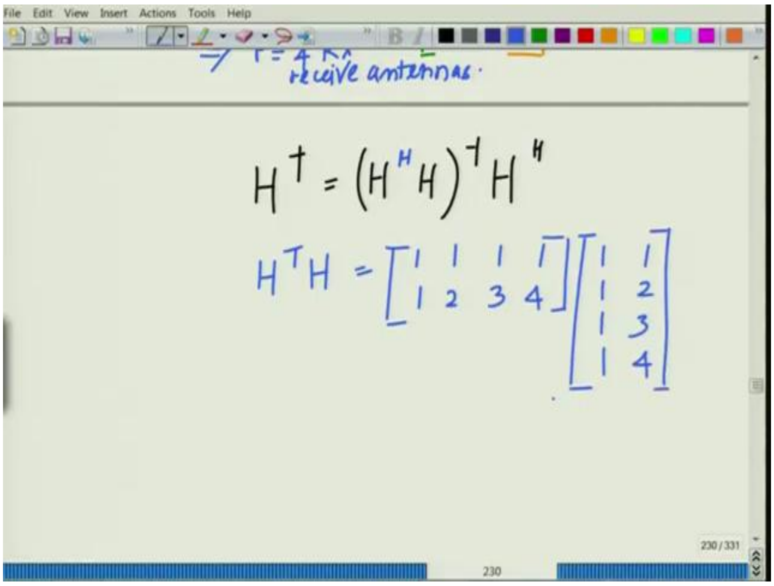

So, therefore, now we evaluate the matrix H transpose H inverse H transpose, this will be given as 1 over 20, 34 minus 10 minus 10 times 1, 1, 1, 1; 1, 2, 3, 4 hopefully you got that. 1 2 3 4 absolutely, so this is going to be 1 over 20. Let us write down the elements, 30 minus 10 that is 20, 30 minus 20 that is 10, 30 minus 30 that is 0, there is minus 10 and the entries in the second row minus 10 plus 4 that is minus 6, minus 10 plus 8 that is minus 2, minus 10 plus 12 that is 2, minus 10 plus 16 that is 6, hopefully got those right.

And now divide them by 20 so you will get, I guess you will get 1 half 0 minus half minus 6 over 20 that is minus 3 over 10, minus 1 over 10 1 over 10 and 3 over 10, I guess, that complete our this thing. So, this is your H transpose H inverse H transpose.

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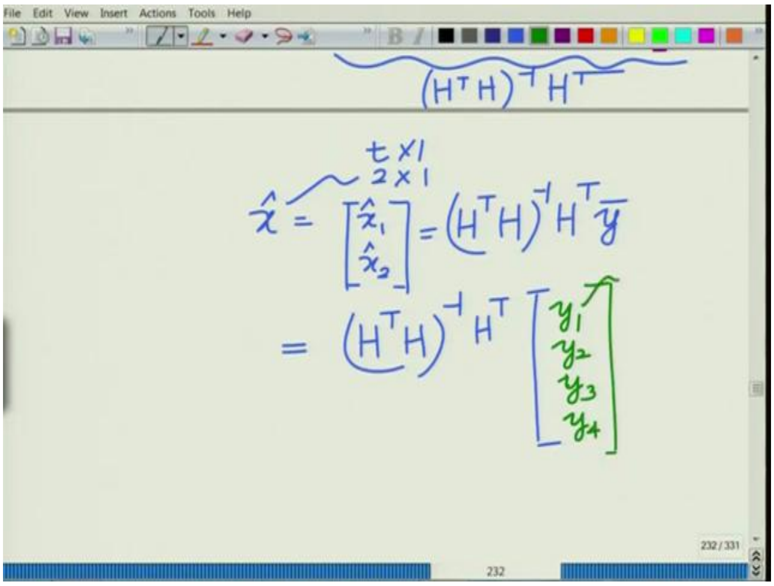

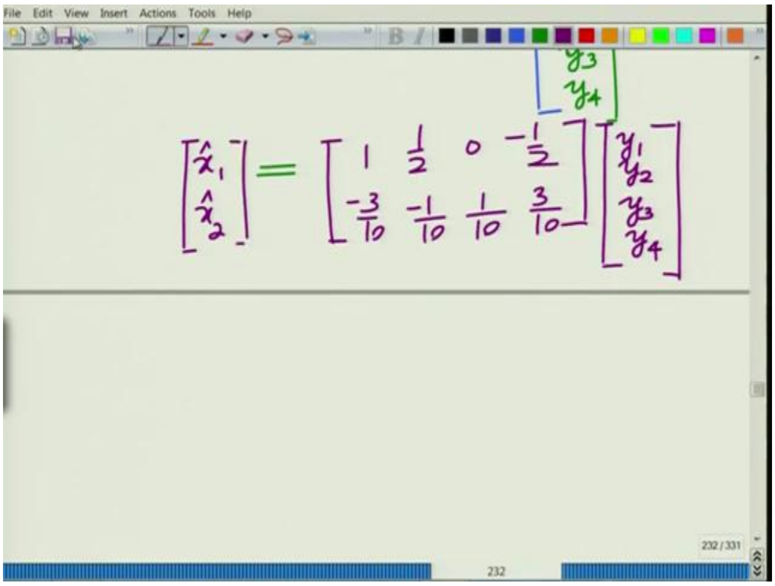

And now we apply this to the received vector so we have x hat, remember this is going to be t cross 1 or basically a 2 cross 1 vector, this is going to be x1 hat, x2 hat which is essentially your H transpose H inverse H transpose into y bar which is going to be your H transpose H inverse H transpose and it is not difficult to see that y bar for this scenario will be contain 4 elements because you have 4 receive antennas, this is r cross 1 or this is a 4 cross 1 vector. Now substitute this matrix that we just obtained which is 1, half, zero minus half, minus 3 over 10, minus 1 over 10, 1 over 10, 3 over 10, y1, y2, y3, y4 and this is your x1 hat, x2 hat.

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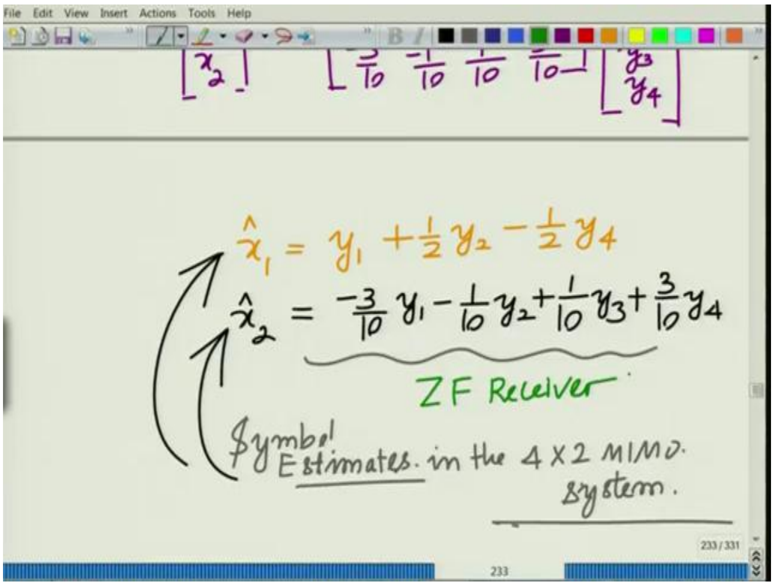

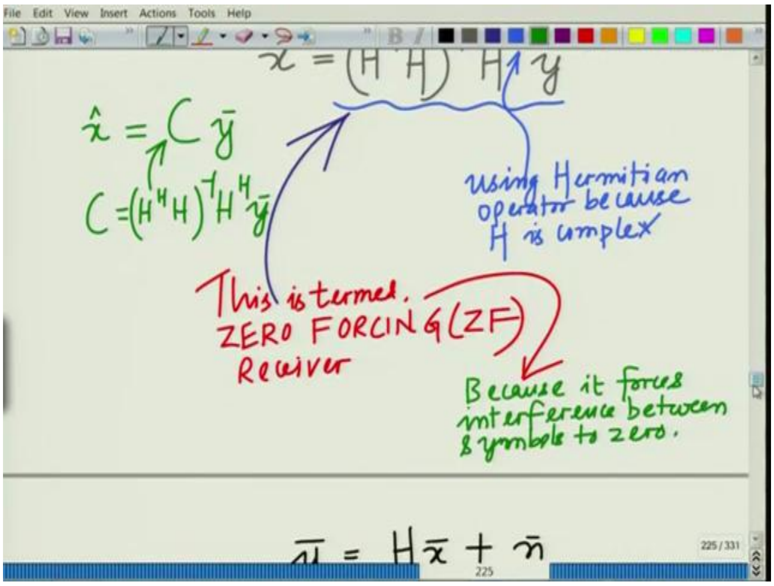

And therefore, finally if you look at this, you can simplify this as x1 hat equal to y1 plus half y2 of course zero times y3 minus half y4, that is the estimate of symbol x1 and your x2 hat equals minus 3 over 10 y1 plus or minus 1 over 10 y2 plus 1 over 10 y3 plus 3 over 10 y4. So, these are the estimates of the symbols. These are your symbol estimates in the MIMO system. And this is basically nothing but essentially your ZF receiver which forces the interference to zero, so this is your ZF receiver which forces the symbols with 2 interference between the symbols from the different transmit antennas x1, x2, xt to zero.

The only other point that I wanted to make here is that, basically you are obtaining the estimate x via linear transformation of the receive vector y. So, x hat equal to some matrix C times where C is basically H dagger or H Hermitian H inverse H Hermitian. So, x hat is obtained by a linear transformation of y bar, whenever the receiver has such a structure this is known as a linear receiver.

So, the ZF receiver in the MIMO wireless system belongs to a class of receivers known as linear receivers. So, the only other point that I want to make over here is that, if you look at this thing, you have x hat equal to H Hermitian, which is essentially you have x hat is equals to C times y bar, where C equals H Hermitian H inverse times H Hermitian y bar.

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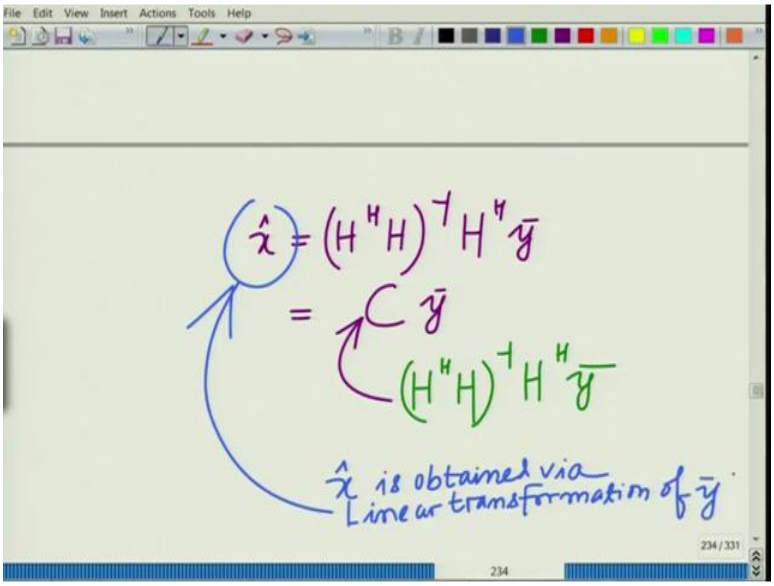

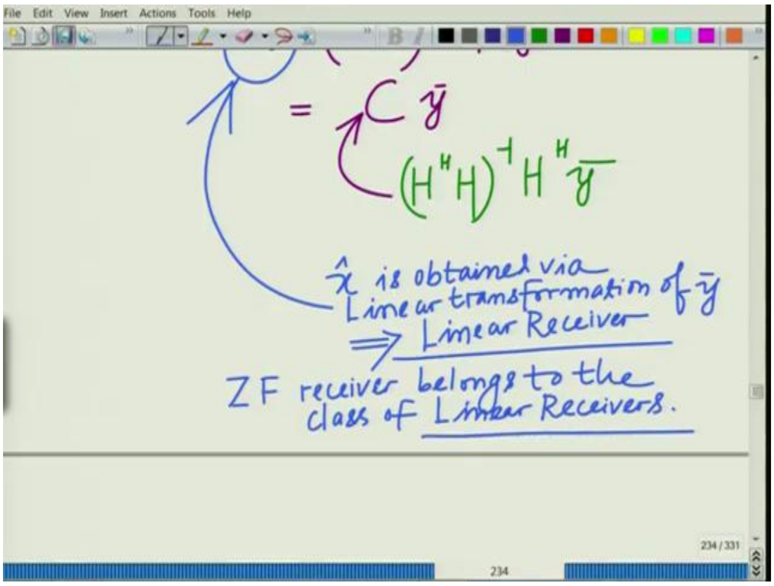

Let me just write it a little bit more clearly so what I am trying to say over here is that you have, what do you have? You have x hat equals H Hermitian H inverse times H Hermitian y bar which is equal to C times y bar, where the C is the matrix. H Hermitian H inverse H Hermitian y bar. So you can see in this case, for the zero forcing receiver x hat is obtained via linear transformation. Y bar implies, this is a linear receiver belongs to the, therefore ZF receiver belongs to the class linear receiver.

There is one more linear receiver which is known as the LMMS receiver, Linear Minimum Mean Square Error Receiver, and we are going to look at that also in the due course. So, it is important at this point to remember that we have the zero forcing receiver which is in fact, nothing but a very practical application of the concept of least squares and least squares estimation in the context of MIMO wireless communication which is of course MIMO technology used in 4G, 5G, Wi-Fi, airdots 11n, airdot 11ac, pretty much most of the modern wireless communication systems that you know are based on multiple input multiple output technologies.

And naturally you can see the extent, the kind of wide applicability of this technique, of this concept of least squares that we have learnt. And in fact, that is just one area of applicability, you have immense number of applications of least squares and linear algebra in general. And we have also done a simple example to illustrate the specific application. So, let us stop here and we will continue with other such applications and concepts in the subsequent modules. Thank you very much.