Applied Linear Algebra for Signal Processing, Data Analytics, and Machine Learning
Professor. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur
Lecture No. 17
Positive semi-definite (PSD) matrices: definition, properties, eigenvalue decomposition

Hello, welcome to another module in this massive open online course. So, in this module let us start looking at another important class of matrices termed as PSD that is positive semi-definite or PD positive definite matrices which are again very important and arise very frequently in linear algebra.

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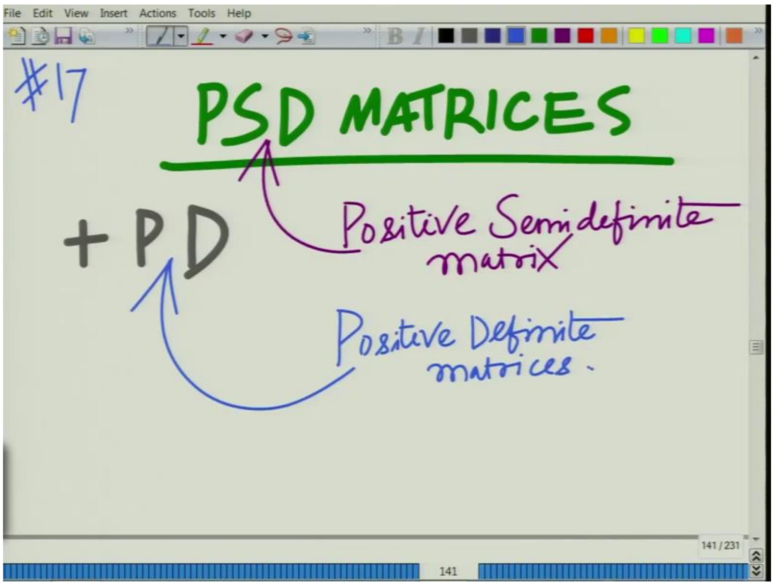

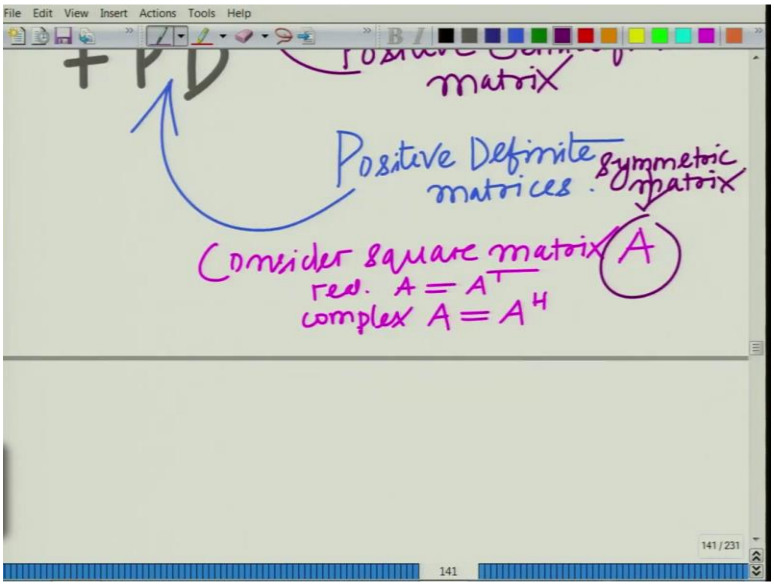

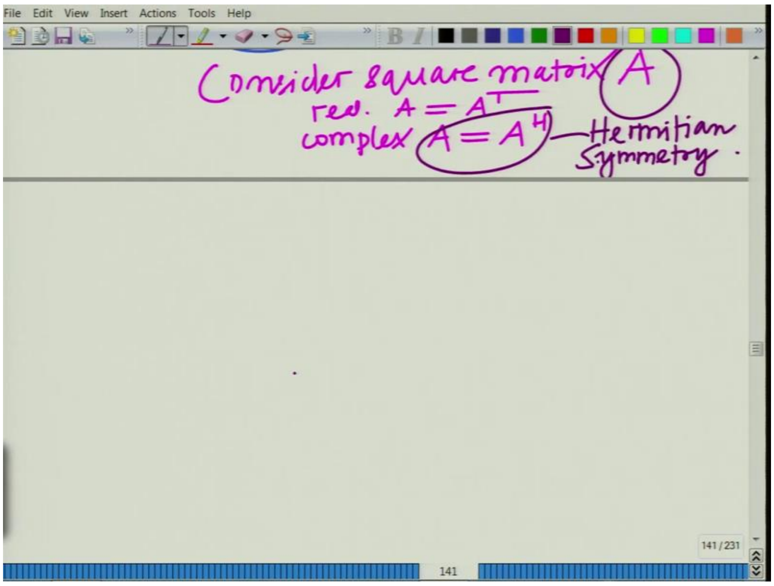

So, we want to start looking at PSD matrices which means, this stands for positive semi-definite matrix (PSD) and we want to also look at positive definite (PD) matrices, what is the meaning of positive and these are related, PD means positive definite matrices. Now, when do we call a matrix as a positive semi-definite matrix? Again, we consider square matrices, consider a square matrix.

In fact, we will consider a symmetric matrix, consider a symmetric square matrix that is if it is a real A equal's A transpose for complex, we will have A equal to A Hermitian that is we are considering a symmetric matrix. What it means is, if it is a real equal to A transpose for complex matrices, we have A equal to A Hermitian that is transpose and complex conjugate, this is also known as Hermitian symmetric. So, you might have also seen this so, this is also known as Hermitian symmetry.

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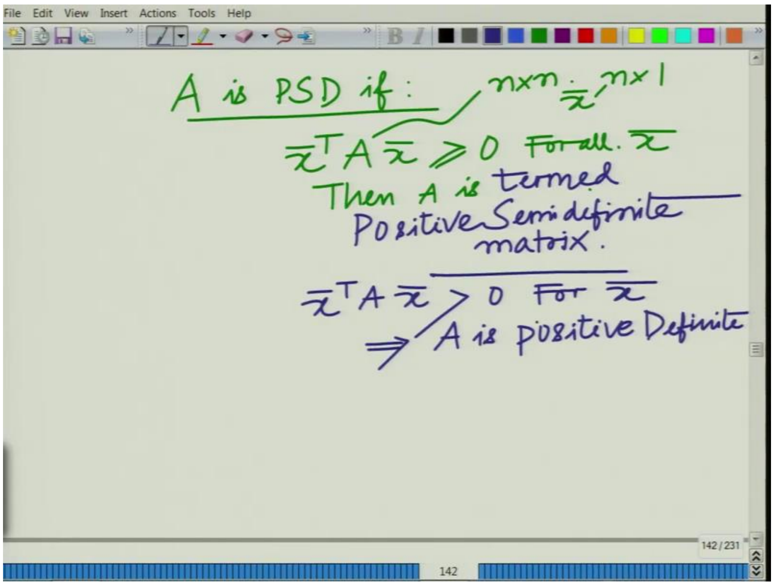

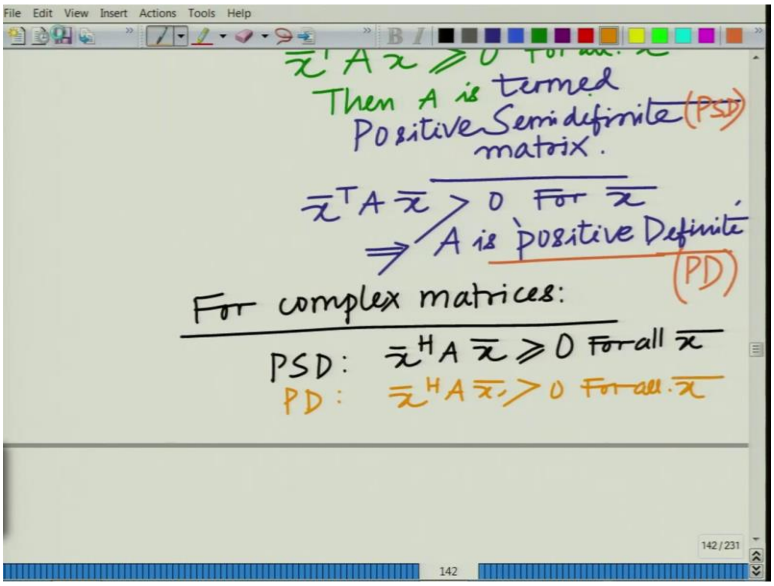

And, we call A is a positive semi-definite matrix, A is positive semi-definite if x bar transpose A x bar greater than or equal to 0 for all x bar, that is if we have an n cross n matrix And x bar is an n cross 1 vector. Then, if x bar transpose, x bar is greater than equal to 0 for all x bar, then A is positive semi-definite, A is positive semi-definite. A is termed a positive semi, let me, let us write it completely, A is termed a positive semi-definite matrix.

On the other hand, if for a complex matrix we have or let us put it again for a positive definite matrix x bar transpose A x bar that is if A is real x bar transpose x bar greater than 0 for all x bar this implies A is a positive definite, this implies A is a positive definite or what we are also calling as a PD matrix. So, positive semi-definite matrix is essentially a PD and this is essentially your P, positive semi-definite is PSD and this is positive definite is a PD matrix.

So, if x bar transfers all vectors x bar, x bar transpose A x bar is always greater than or equal to 0, if that condition holds for all vectors x bar then A is termed a positive semi-definite matrix if it is strictly greater than 0 x bar transpose A x bar strictly greater than 0 for all vectors x bar, then it is termed as a positive definite matrix.

Similarly, you might have already guessed for complex matrices. For PSD, we have x bar Hermitian A x bar greater than or equal to 0 for all x bar, and for PD we have x bar Hermitian A x bar greater than 0 for all x bar for all vectors x bar this is what complex matrices that is we have to consider Hermitian, x bar Hermitian A x bar and if it is greater than or equal to 0 PSD for all x bar if it is simply greater than 0, it is strictly greater than 0 for all x bar then it becomes a positive definite matrix.

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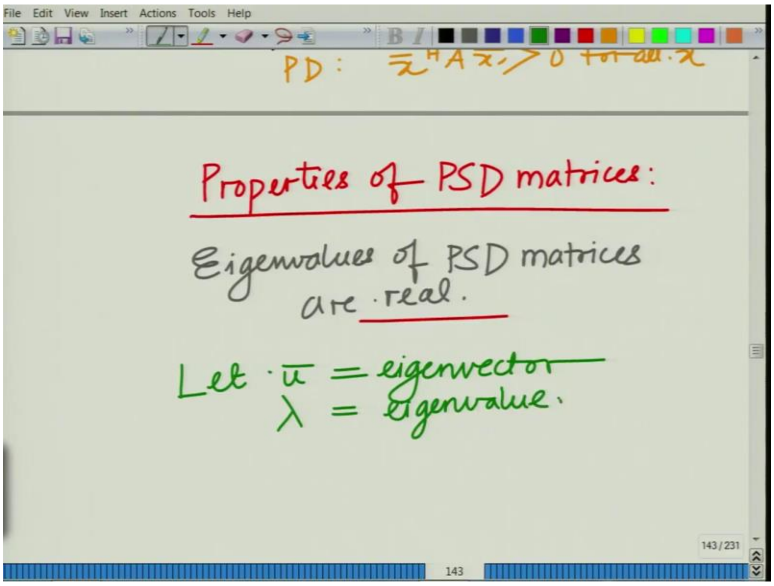

Let us look at the properties of positive semi-definite matrices and of course, properties of positive definite matrices are very similar, the inequalities become strict. So, let us look at the properties of positive semi-definite matrices and what is the property of the positive semi-definite matrices? The first property is that the very important property eigenvalue of PSD matrices are real, the eigenvalue of PSD matrices these are real quantities, these are real numbers, this is a very important property of positive semi-definite matrices and we will prove this for the general case of complex, we will consider general complex positive semi-definite matrices and we will show these properties. And this is a fairly important property, let u bar equals eigenvector, and lambda equal the corresponding eigenvalue.

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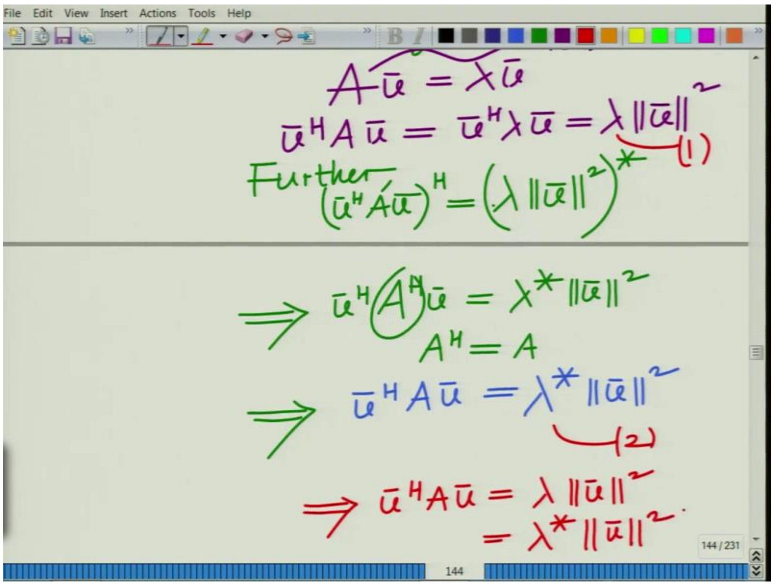

Such that we have A u bar equals lambda u bar, where A is we have said a PSD matrix. Then, u Hermitian if we perform u Hermitian, u bar Hermitian A u bar this would be u bar Hermitian lambda u bar which is lambda norm u bar square. Now, further, if you take the conjugate Hermitian on the left, u bar Hermitian A u bar if you take the Hermitian this is equal to lambda. So, after that, if you take the Hermitian then you have your Hermitian u bar equal to lambda norm u bar square you can take the conjugate on the right.

Since it is a scalar quantity, this implies u bar Hermitian A u bar or A Hermitian or rather A Hermitian u bar Hermitian, A Hermitian u bar equals lambda conjugate norm u bar square, since, norm bar square is earlier quantity. Again, we have a symmetric matrix that is A Hermitian equal to A. So, this implies, you can easily see, u bar Hermitian A u bar equal to lambda conjugate norm u bar square.

So, if you look at this what we can call as properties 1 and 2. This implies u bar Hermitian A u bar equals lambda norm u bar square equal to lambda conjugate norm u bar square from 1 and 2, from results 1 comma 2.

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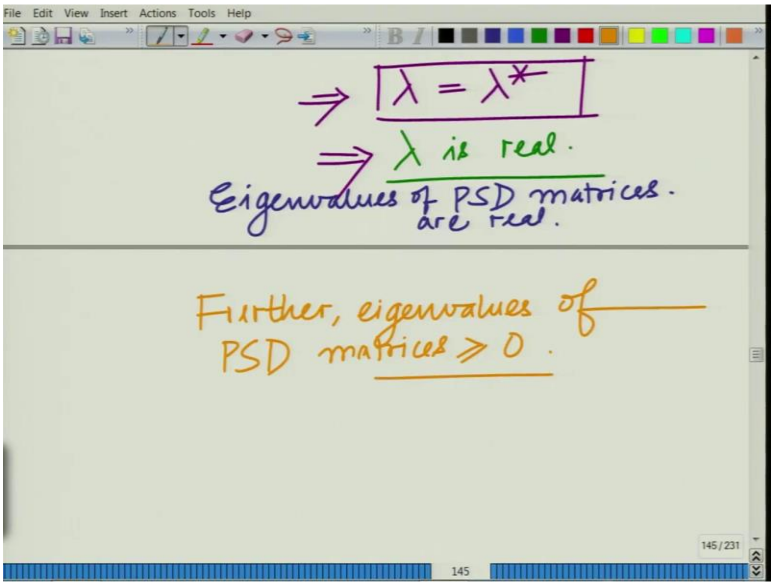

And finally, this implies you can see that lambda equals lambda conjugate and therefore, this implies that the eigenvalues, this implies that lambda is real. Therefore, eigenvalues of PSD matrices are real. And further, these eigenvalues are greater than or equal to 0, further, it is not very difficult to see, further, all eigenvalues of PSD matrices are greater than or that is they are non-negative, the eigenvalues of PSD matrices are greater than or equal to 0 and that is also not very difficult to see.

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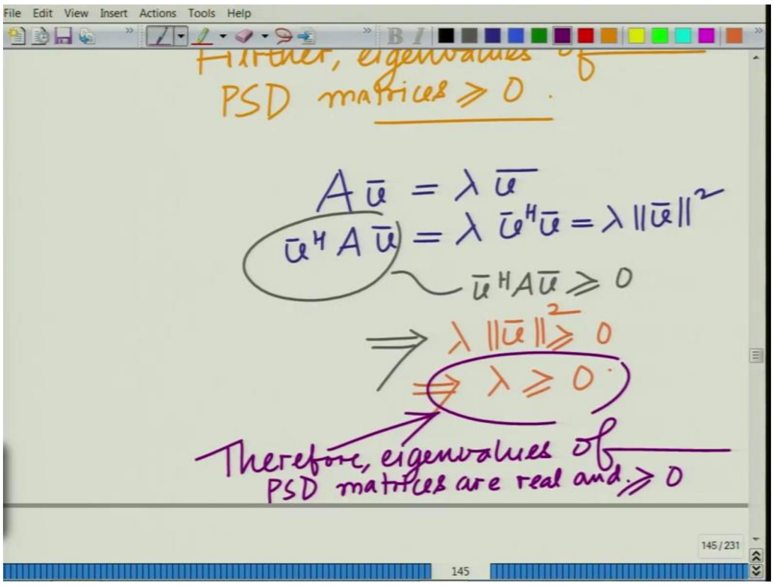

Follow the same logic, we have matrix A, eigenvector u bar, A u bar equals lambda times u bar, u bar Hermitian A u bar equals lambda u bar Hermitian u bar equal to lambda norm u bar square. Now, you know from the property of the PSD matrices u bar Hermitian A u bar is greater than or equal to 0 that is the left-hand side is greater than or equal to 0 which means the right-hand side lambda norm u bar square is also greater than or equal to 0, which implies that norm u bar square we can see is the square norm which is always greater equal to 0 which implies that lambda is greater than or equal to 0.

Therefore, this is a very interesting property put together, therefore, eigenvalues of the PSD matrices are real and greater than or equal to 0, these are real and eigenvalues are greater than or equal to 0 that is these eigenvalues are non-negative. The eigenvalues of PSD matrices are essentially non-negative, is a very important and interesting property which has a lot of applications as I already told you because positive semi-definite and positive definite matrices are as frequently and of course, the counterpart for the analogs property for a positive definite matrix will be that eigenvalues are strictly greater than 0 that is eigenvalues are positive because u bar Hermitian A u bar is strictly greater than 0.

Now, let us you can simply replace the week inequalities by the strong inequalities. Now, the next important property of eigenvalues are that I use the following let me first describe the property.

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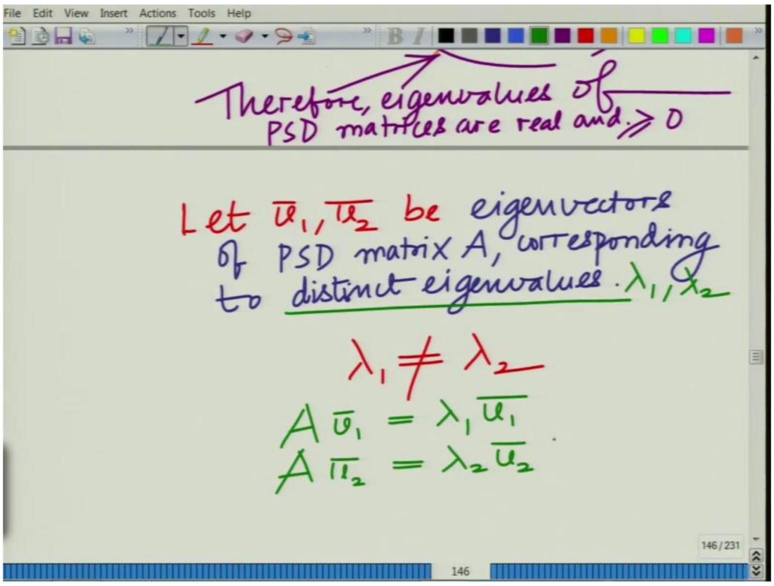

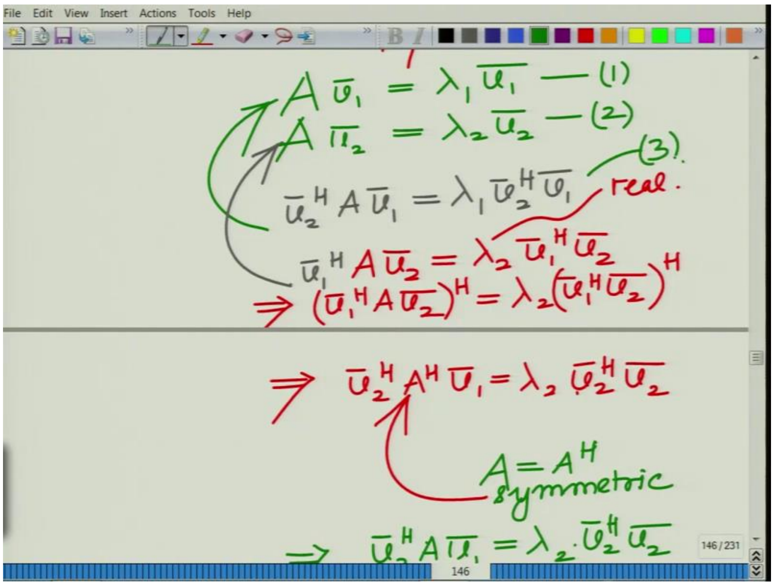

Let u bar, let u1 bar comma u2 bar be distinct, be eigenvectors corresponding to distinct eigenvalues, this is important or eigenvectors of PSD matrix A, corresponding to distinct eigenvalues lambda 1 comma lambda 2, that is lambda 1 is not equal to lambda 2, these are eigenvectors implies A u1 bar equals lambda 1 u1 bar, A u2 bar equals lambda 2 u2 bar. Then, what is the property that these eigenvectors u1 bar and u2 bar satisfy? We are going to see that these satisfy a very interesting property.

Let us start with equation 1, let us start with this perform u2 bar Hermitian A u1 bar this is lambda 1 u2 bar Hermitian u1 bar and now, let us perform on this u1 bar Hermitian A u2 bar this is lambda 2 u1 bar Hermitian u2 bar. Now, take the complex Hermitian or complex conjugate because these are scalar quantities both of these things are (sca) same. So, u1 bar Hermitian A u2 bar equals, now, we already seen this is a real quantity eigenvalues are real, so this remains lambda 2. So, u1 Hermitian u2 bar Hermitian, so this implies u2 bar Hermitian A Hermitian u1 bar equals lambda 2 u2 bar Hermitian u2 bar.

Now, once again, A is symmetric which implies A equals A Hermitian and this implies u2 bar Hermitian A u1 bar equal to lambda 2 u2 bar Hermitian u2 bar. And therefore, we have lambda 1 u2 bar Hermitian u1 bar, and therefore, from now, if you look at these equations 3 and 4, so now let us call this 3.

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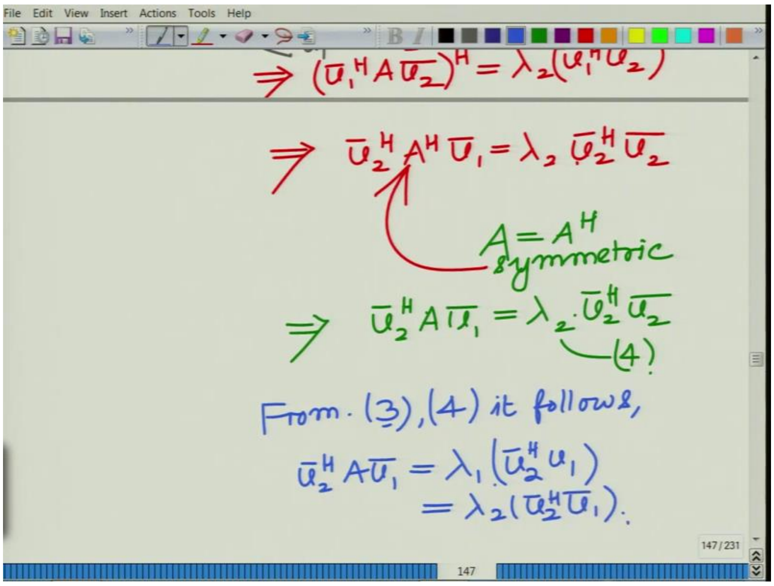

And let us call this equation as equation number 4, from 3 and 4 it follows that u2 bar Hermitian A u1 bar equals lambda 1 u2 bar. From 3, we have this is equal to lambda 1 u2 bar Hermitian u1 bar and from 4 we have the same quantity equal to lambda 2 u2 bar Hermitian u1 bar.

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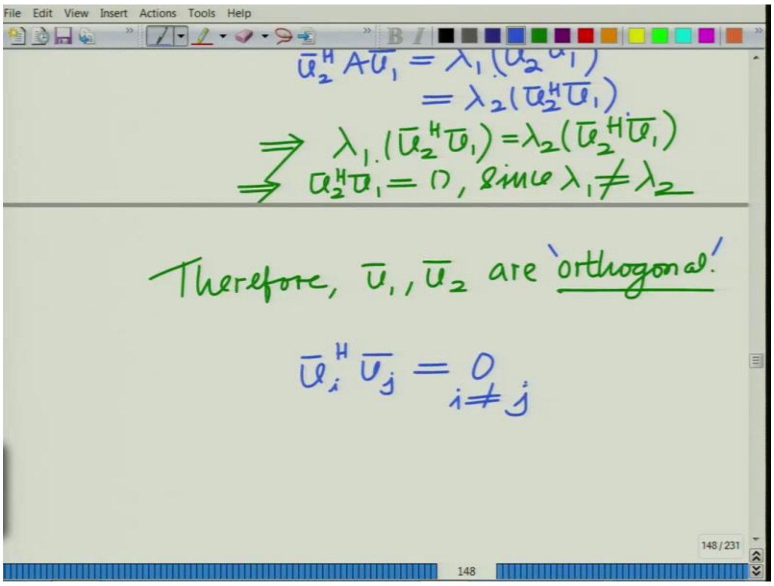

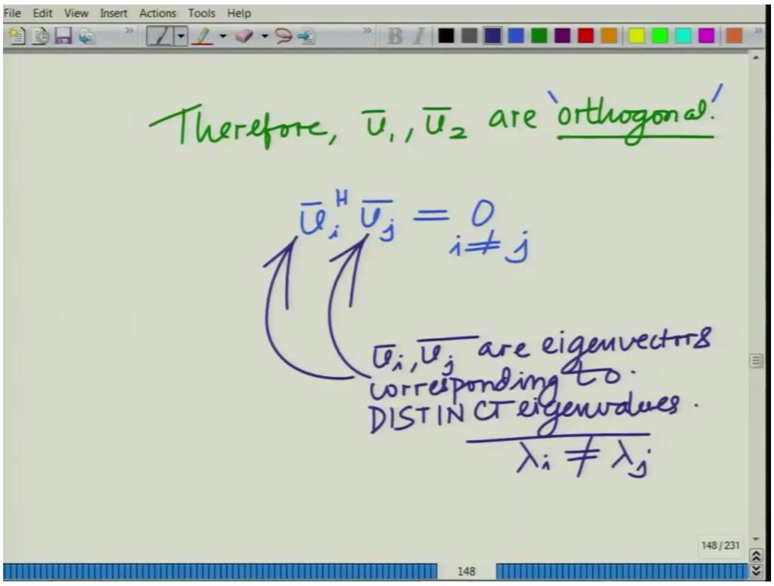

Which implies now, if you look at it something very interesting lambda 1 u2 bar Hermitian u1 bar equal to lambda 2 u2 bar Hermitian u1 bar. Now, note these are distinct eigenvalues lambda 1 is not equal to lambda 2 that is the assumption to begin with. Therefore, this is only possible when u2 bar Hermitian u1 bar is equal to 0. So, this implies u2 bar Hermitian equal to 0, since lambda 1 is not equal to lambda 2.

Therefore, what this implies, u1 bar comma u2 bar orthogonal and this is a very interesting property, these are orthogonal that is if you look at any ui bar Hermitian uj bar is going to be 0 for i not equal to j and these are eigenvectors corresponding to the distinct eigenvalues of a positive semi-definite matrix. Where ui bar comma uj bar are eigenvectors corresponding to DISTINCT eigenvalues.

That is you have the eigenvalue lambda i, as eigenvalue lambda corresponding to uj lambda i not equal to lambda j. So, lambda i not equal to lambda j, and the corresponding eigenvectors ui bar Hermitian ui bar uj bar will be orthogonal that is ui bar Hermitian uj bar equal to 0, eigenvectors of a positive semi-definite matrix corresponding to distinct eigenvalues are orthogonal.

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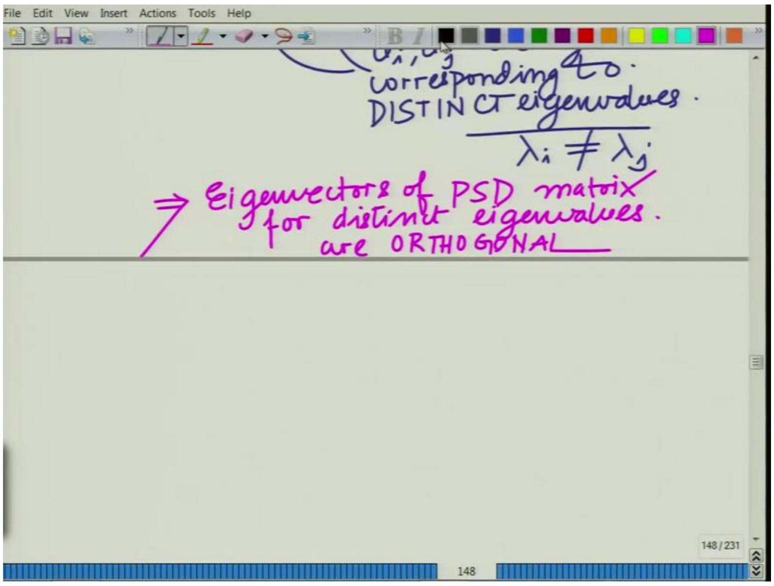

So, the result is that the eigenvectors of PSD matrix, so this implies eigenvectors of PSD matrix for distinct eigenvalues are orthogonal that is ui bar Hermitian uj bar equal to 0 if lambda i is not equal to lambda j. This gives a very interesting decomposition.

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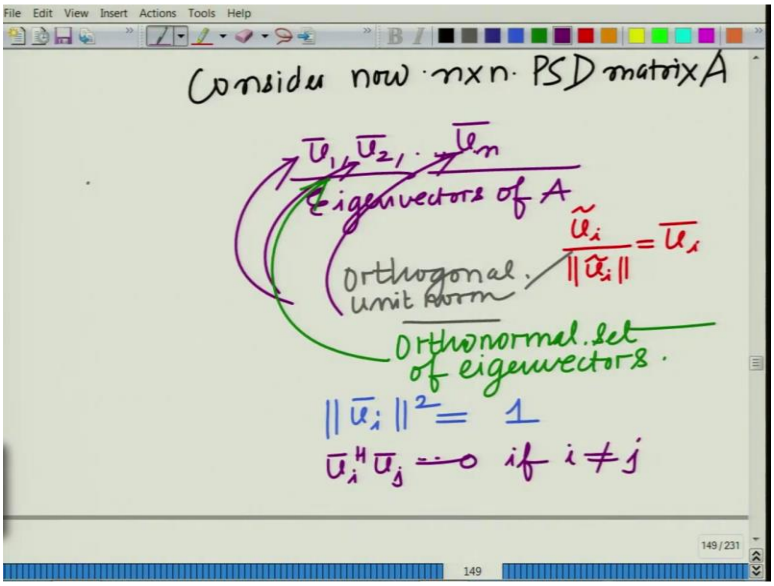

Consider now, n cross n PSD matrix A with eigenvectors are u1 bar, u2 bar, un bar, these are the eigenvectors of A, we already seen these are going to be orthogonal. Now, let us also make them unit norm and that is not difficult to see because if ui tilda an eigenvector which is not unit norm I can always divide it by norm ui tilda and call that as ui bar which is essentially a unit norm vector. Remember, if you idealize an eigenvector, scaled version of ui tilda is also an eigenvector we have already seen this therefore, ui tilda divided by norm of ui tilda is also an eigenvector which we are calling as ui bar.

So, he can always construct a unit norm eigenvector from. And we have already seen that these different eigenvectors corresponding to the different eigenvalues these are orthogonal so you have orthogonal unit norm that makes it an orthonormal set of eigenvectors. So, we have in a sense, we have an orthonormal set of eigenvectors that exists for a PSD matrix, orthonormal set of eigenvectors because of the positive properties of the PSD matrix such a set of eigenvectors exists.

Which essentially means that again, you might, you will know by this point orthonormal means norm ui bar square equal to 1, ui bar Hermitian uj bar equal to 0, if i not equal to j. This we have already seen eigenvectors corresponding to distinct eigenvalues are orthogonal.

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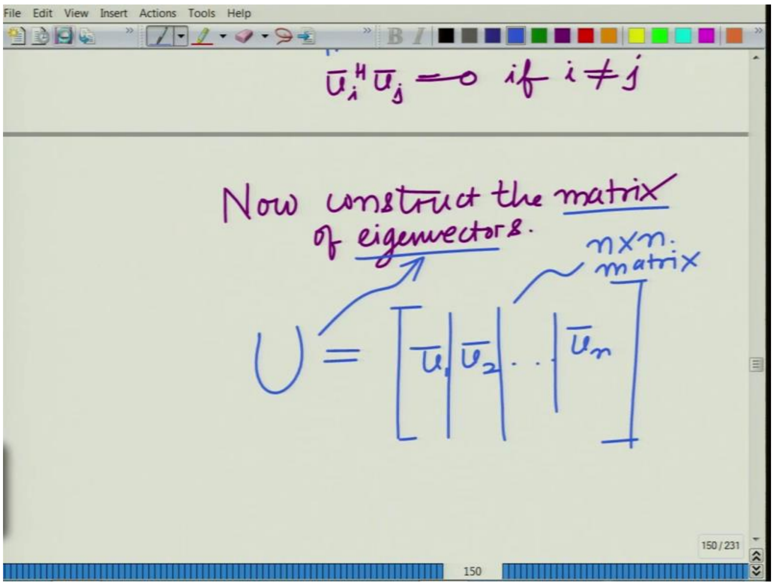

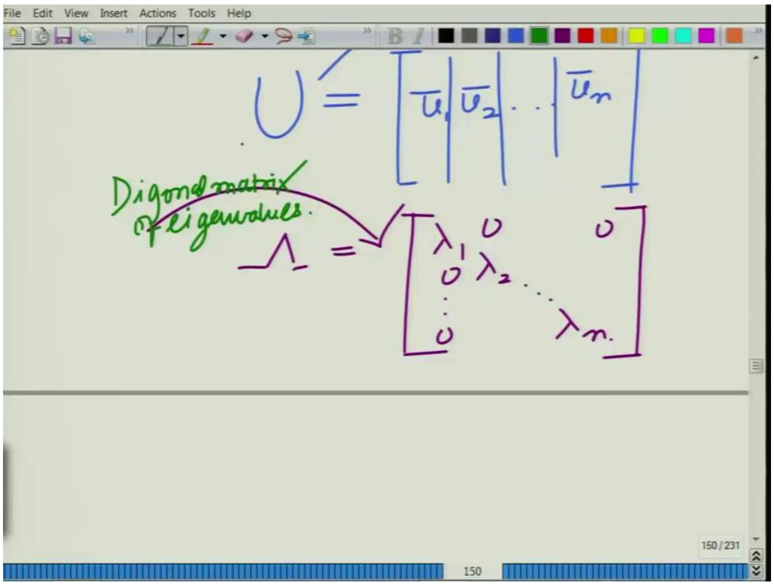

Now, construct the matrix of eigenvectors. What is the matrix of eigenvectors? That is we know this is u1 bar u2 bar, the columns are given by the eigenvectors, u1 bar, u2 bar, un bar this will be an n cross n matrix this is your matrix of eigenvectors, where each column is an eigenvector and the corresponding eigenvalues are lambda 1, lambda 2, lambda n, and we can write that as a diagonal matrix.

So, we have the diagonal matrix lambda where you have lambda 1, lambda 2, lambda n this is the diagonal matrix of eigenvalues and notice since this u1 bar, u2 bar, un bar these are orthonormal, it follows that remember from the property of unitary matrices and that is why unitary matrices are important remember, that is what we said because these arise frequently and now, you can see the matrix of eigenvectors of a PSD matrix is a unitary matrix because the columns are orthogonal unit norm it means it satisfies.

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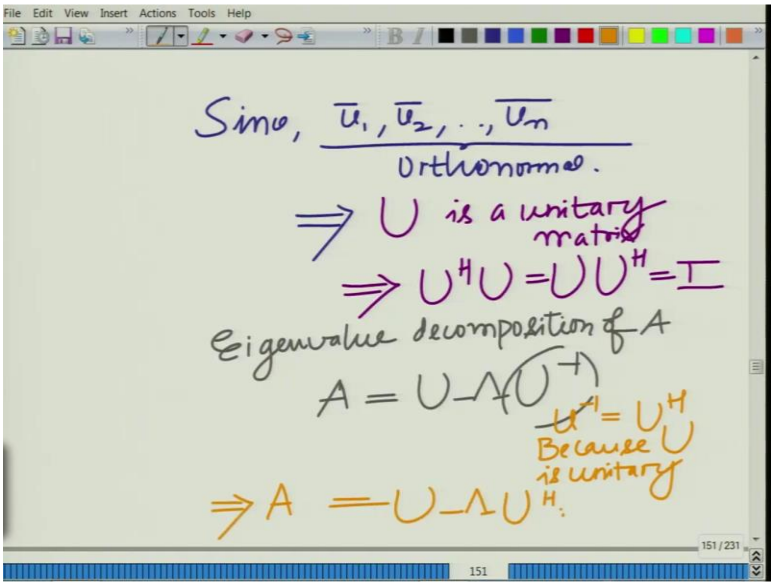

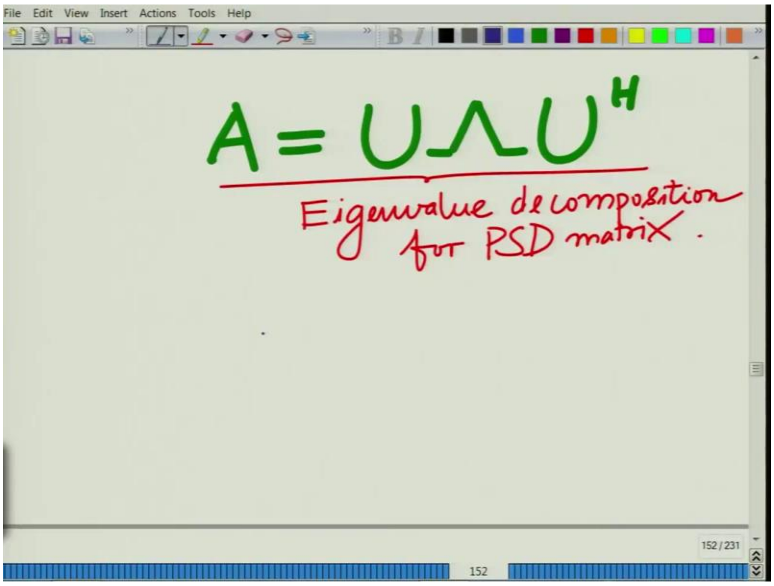

Since, ui bar, since, u1 bar, u2 bar, un bar these are orthonormal, this implies U is a unitary matrix which essentially means that U satisfy the property U Hermitian U equals U U Hermitian is identity and therefore, the eigenvalue decomposition of the PSD matrix A, remember any matrix can be decomposed as U lambda U inverse this is eigenvalue decomposition, U matrix which is a matrix of eigenvectors times lambda which is a diagonal matrix of eigenvalues times U inverse.

But now, you see because U is unitary U inverse equal to u Hermitian because U is unitary. So, this implies A equal to U lambda U Hermitian for a positive semi-definite matrix. This is the interesting property for a PSD matrix A equal to U lambda U Hermitian. So, u inverse becomes u Hermitian, this is essentially the EVD or what we call as the eigenvalue decomposition for PSD matrix, this is the eigenvalue decomposition for a positive semi-definite matrix that is for the general square matrix it can be written as U lambda U inverse, but for positive semi-definite and of course, for also a positive definite matrix this becomes U lambda U Hermitian.

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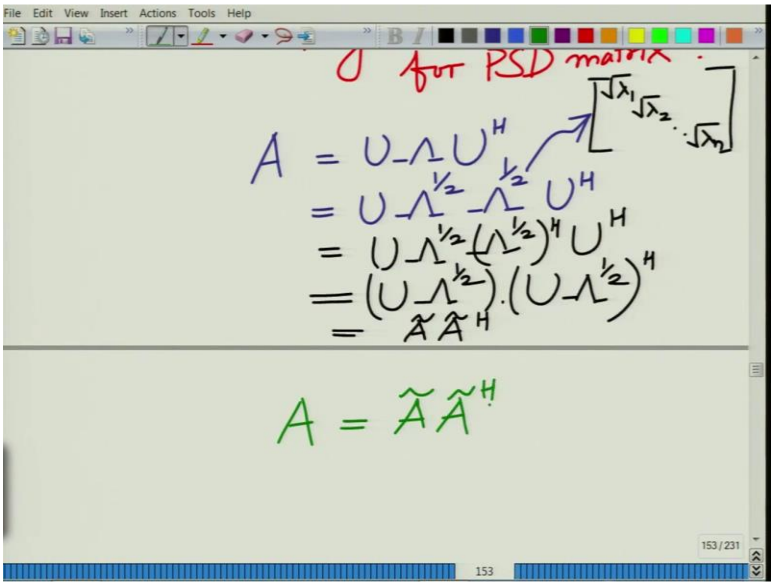

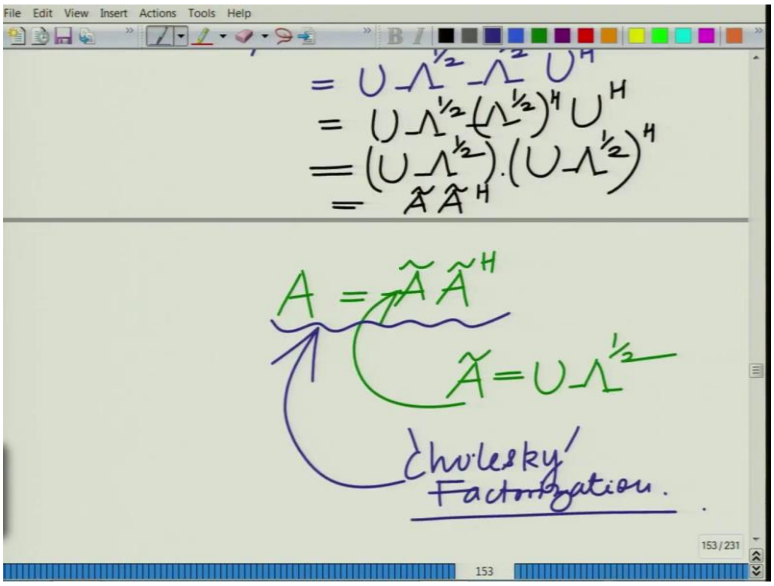

Let us look at another small property, so you have A equals U lambda U Hermitian which I can write as U, I can write the square root of lambda times itself lambda raise to the power of half times itself, What is this lambda raise to the power of half? This is the diagonal matrix of eigenvalues, eigenvalues are real. So, I can take the square root, so I can write lambda 1 square root lambda 1, square root lambda n this is the diagonal matrix containing the square root of eigenvalues and naturally these are real.

So, I can also write this as a lambda half, lambda half Hermitian U Hermitian which is essentially U lambda half times U lambda half Hermitian. Therefore, this implies this is A tilda, A tilda Hermitian, so I can write any positive semi-definite matrix A can be expressed as it can be decomposed as A tilda, A tilda Hermitian, where one possible value of A equals U lambda raised to the power of half.

So, any positive semi-definite matrix A can be decomposed as the product of two matrices that is A tilda into the Hermitian of its, A tilda into A tilda Hermitian. And this is termed as the Cholesky factorization of a positive symmetry and this is very helpful. The Cholesky factorization has a very important role to play in Data Analytics, Machine Learning, Signal Processing this that is you have a positive semi-definite matrix.

And that can be that has positive eigenvalues, whether it is non-negative eigenvalues and therefore, from the properties, of course, orthogonal eigenvectors and so on and so forth. And you can write it as A U lambda U Hermitian. And essentially what has followed is you can write essentially any positive semi-definite matrix as A tilda times A tilda Hermitian. So, it is very similar to a kind of a square root decomposition of a positive real number.

So, if I have a positive real number, x, I can write it as square root of x into square root of x, this is roughly something similar. It is not exactly that, of course, because this generalizes this concept of matrices, and of course, we have the Hermitian it is not exactly A tilda into A tilda, but it is rather A tilda into A tilda A Hermitian. So, this is a very, very interesting. So, let us get decomposition positive definite matrices, the decompositions their eigenvalues their eigenvectors arise very, very frequently in all applications in Linear Algebra, Machine Learning, Data Analytics and so on and so forth. So, let us pause here and we will continue in the subsequent modules. Thank you very much.