Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
Professor. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur
Lecture No. 16
Special Matrices: Rotation and Unitary Matrices. Application: Alamouti Code

Hello, welcome to another module in this massive open online course. So, in this module let us look at another let us look at a special class or rather classes of matrices termed Rotation and Unitary Matrices which arise very frequently in Linear Algebra and Matrix Algebra.

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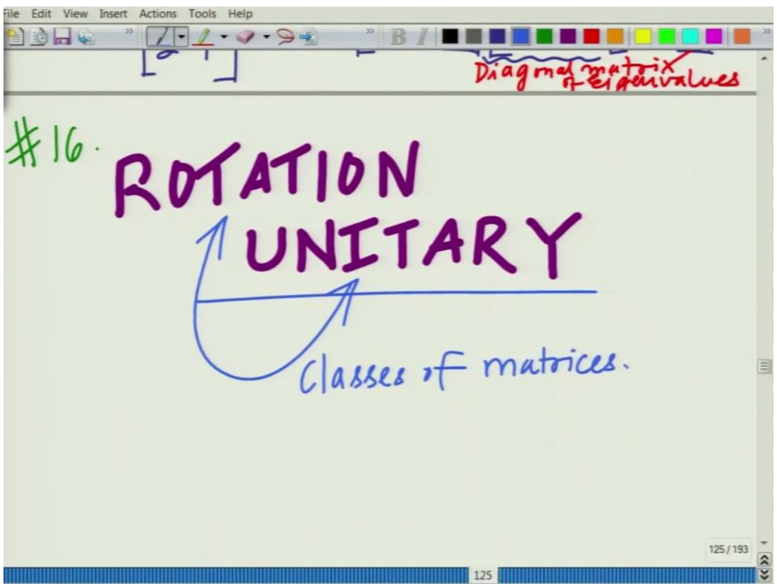

So, let us look at special classes of matrices these are the Rotation and we also have the Unitary. Rotation matrix and Unitary matrices, rotation and unitary and these are special kinds of you can think of as classes of matrices.

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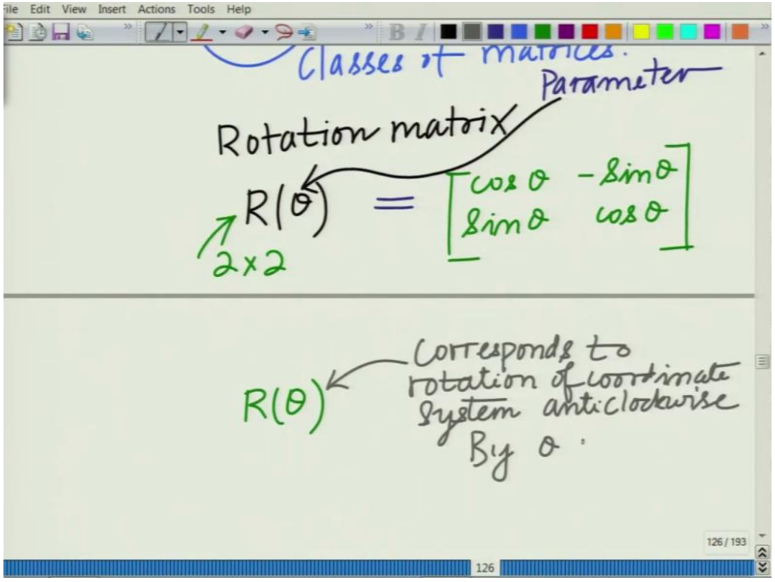

Now, the rotation matrix parameterized by theta what we call as a standard form rotation matrix R of theta this is what we call as a parameter or basically a free variable that is theta can vary independently and depending on that you have the matrix. The different elements of this matrix depend on this parameter theta. So, R of theta this rotation matrix is essentially this is a 2 cross 2 matrix we are looking at the 2 cross 2 rotation matrix, 2 cross 2 rotation matrix this is given us cosine theta minus sine theta and sine theta cosine theta and the rotation matrix satisfies the rotation matrix this R of theta this basically you can show corresponds, corresponds to rotation of the coordinate system anti clockwise by theta.

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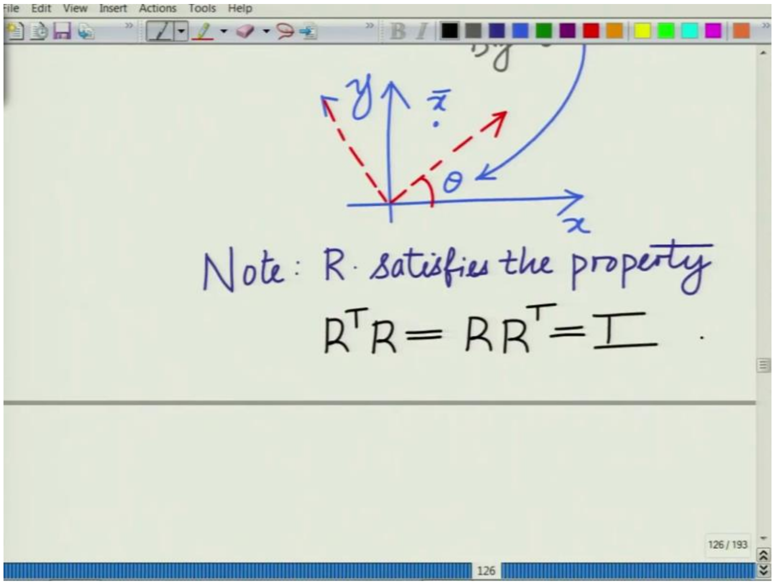

That is if you have a point correct if you have a point x bar and these are your cord this is your current coordinate system and then you rotate this anti clockwise by theta and what are the new coordinates these are essentially given by the rotation matrix. So, this is rotation anti clockwise by theta let me just draw this that is if you have a system that is rotated by theta the new coordinates are given by taking the old coordinates and multiplying them by R theta.

Anyway the rotation matrix satisfies the property note R theta I am going to simplified simply denote this by R satisfies the property that R transpose into R equals R R transpose equals identity this is the interesting property that the rotation matrix satisfies that is R transpose into R R into R transpose is identity which means that R transpose is R inverse that because this is a square matrix.

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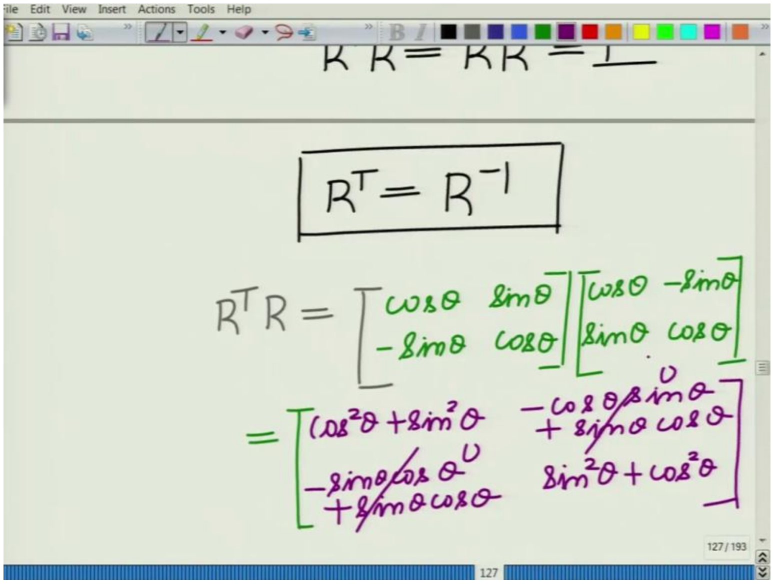

So, this means R transpose is essentially R inverse that is the inverse is obtained simply by taking the transpose of this matrix. So, let us look at this R transpose into R equals you can check this cosine theta sine theta minus sine theta cosine theta times R, which is essentially cosine theta minus sine theta. So, the 1 comma 1 element if you simplify this you can clearly see the 1 comma 1 element this is cosine cos square theta plus sine square theta.

So, this will be cos square theta plus sine square theta this is minus cos minus sine theta cos theta plus sine theta cos theta which is 0 this is again minus cos theta sine theta plus sine theta cos theta plus sine theta and finally, the last element is 2 plus 2 element a sine square theta plus cos square theta which is 1 and this element once again you can see this is also 0.

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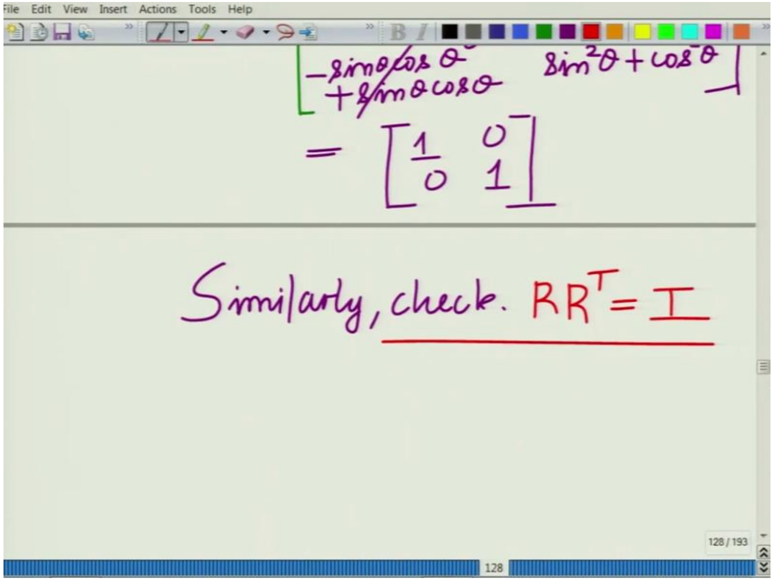

And so, this reduces to it is not difficult to see this reduces to the 2 cross 2 identity matrix 1 0, 0 1. So, this is the interesting property of the Rotation matrix. So, R equals to so, R transpose equals R inverse and similarly, you can also check similarly, check we checked I think R transpose R similarly, you can check R transpose equals identity and you can also see this by the fact that the inverse of a square matrix is unique.

So, if R transpose is the left inverse that is R transpose R equal to identity then it must also be the case that R into R transpose equals identity. So, this is an interesting example. Now, the analog of that for complex matrices is not what is known as a unitary matrix.

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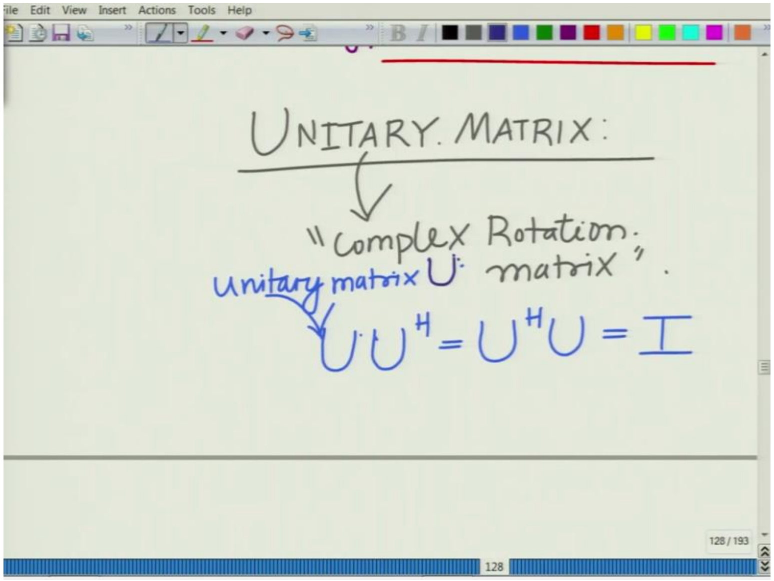

So, unitary matrix is the analog I would say roughly speaking is the analog of a Rotation matrix. So, I can think of this as loosely speaking a complex rotation matrix this is what it is this is a complex rotation matrix which satisfies the property you must have this is R R transpose is identity the unitary matrix U replace the transpose by the Hermitian then this is U U Hermitian equals U Hermitian U equals identity for any Unitary matrix. So, a unitary matrix U satisfies the property that U U Hermitian equals U Hermitian U equals identity.

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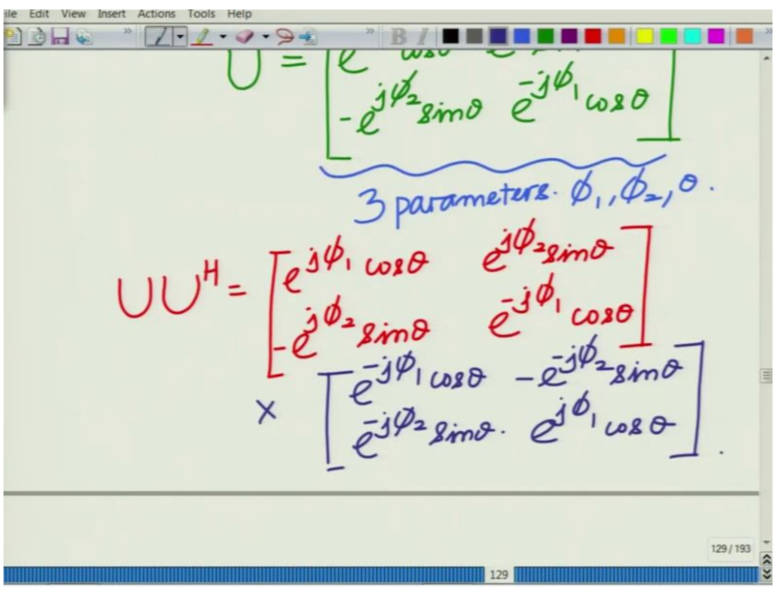

For example, you have let us say a canonical example is you can write a complex unitary matrix this will be a product of this will depend on all more parameters because it is complex. So, this will be e raised to j phi 1 cosine theta e raise to j phi 2 sine theta minus e raise to j phi 2 sine theta and e raise to minus j phi 1 cosine theta and you can see this depends on 3 parameters phi 1 phi 2 theta this is the matrix that depends on 3 parameters phi 1, phi 2, phi theta.

And you can once again check U U Hermitian if you have U U Hermitian let us write to U, U will be e raise to j phi 1 cosine theta e raise to j phi 2 sine theta minus e raise to j phi 2 sine theta e raise to minus j phi 1 cosine theta times the product U Hermitian so, U Hermitian will be e raise to minus j1 phi 1 cosine theta, so, you take the transpose and Hermitian. So, minus e raise to minus j phi 2 sine theta e raise to minus j phi 2 sine theta e raise to g phi 1 cosine theta.

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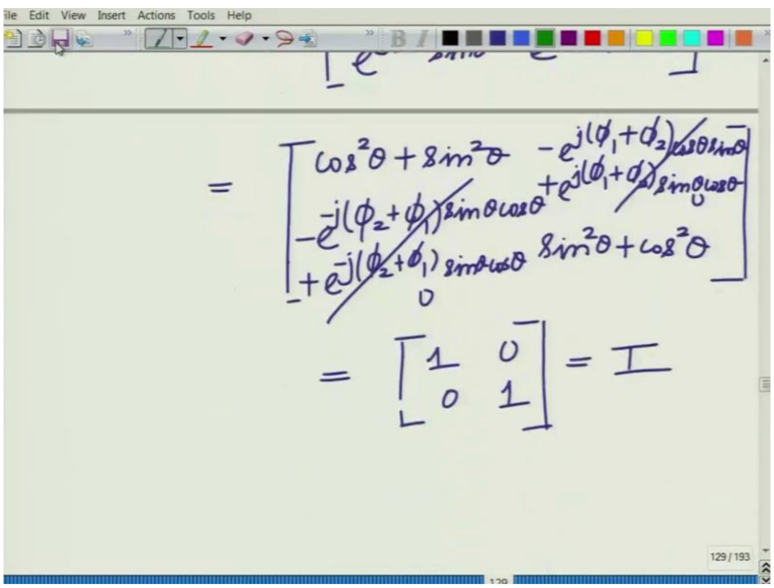

And if you multiply this, this will be you can once again see the diagonal terms will be cosine square theta plus sine square theta and this will also be sine square theta plus cosine square theta off diagonal terms will be that will be if you look at this minus e (rai) that will be minus e raise to minus j phi minus e raise to minus j phi 2, I think this should be there is a problem here there is there has to be a minus sign over here there has to be a minus sign over here.

So, this will be minus e raise to minus j minus e raise to and therefore, once you do this, this will be e raise to so, there is a negative sign up so, there is no sign over negative sign over here and this will be minus e raise to minus j phi 2 plus phi 1, phi 2 plus phi 1 sine theta cosine theta plus e raise to minus j phi 2 plus phi 1 sine theta cosine theta and therefore, this is obviously going to go to 0 and similarly here you will have minus e that is the 1 comma 2 element minus e raise to j phi 1 plus phi 2 cosine theta sine theta plus e raise to j phi 1 plus phi 2 sine theta cosine theta.

And you can clearly see that this will also be 0 and therefore, this will be the identity matrix again this will be 1 0, 0 1 which is essentially the identity matrix. So, this will be doing one final check e raise to minus j phi 1 cos e raise to minus, e raise to minus e minus e raise to j phi 2 sine theta e raise to minus j phi 2 sine theta e raise to j phi 1 cosine theta. So, this is going to be you can check this and if there is there are certainly typos you can correct this so, as to ensure that this is essentially correct.

So, just ensure just make sure that what we have written over here it should be by and large correct, but if there are any residual typographical errors, then please check this.

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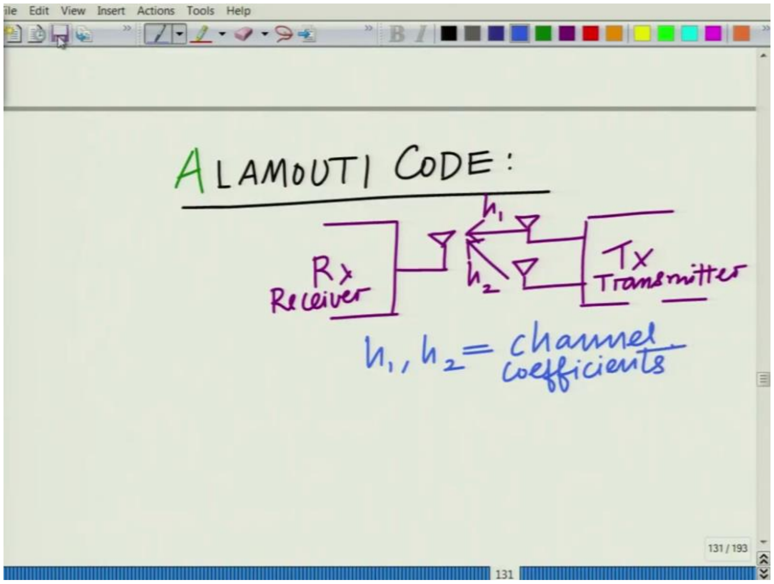

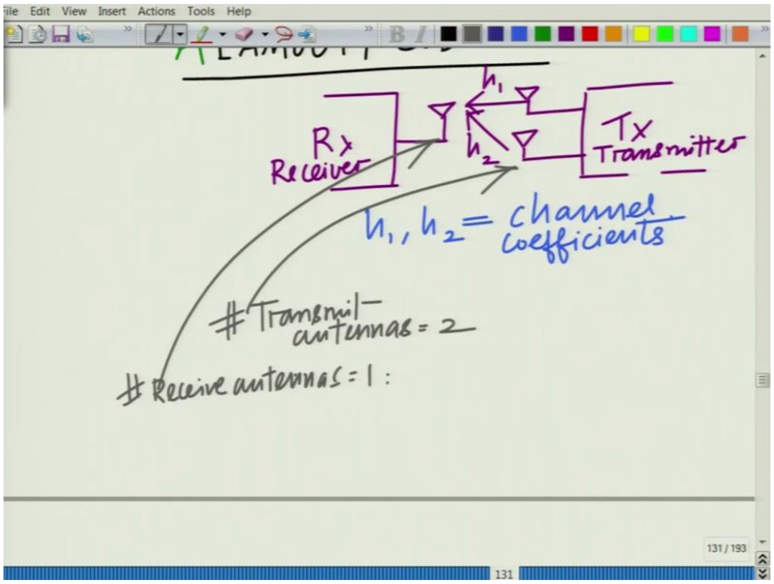

Now, therefore, and as I told you these are very practical for instance, one place where they arise for instance, some of you might have heard in the context of modern wireless communication we have codes. Especially when you have wireless systems with multiple antennas 1 such code is what is known as an Alamouti code. I will just briefly describe this example because this arises in practice. So, you have the Alamouti code and in this Alamouti code what you have is you have the receiver with a single antenna and you have the transmitter with 2 antennas and these are the channel coefficients h1, h2.

So, this is the receiver, this is the transmitter and these are the channel coefficients h1, h2, h1, h2 these are the channel coefficients and what you can see is that that is basically if you see this, this is the 2 transmit antenna system number of transmit antennas equal to 2 and number of receive antennas equal to 1.

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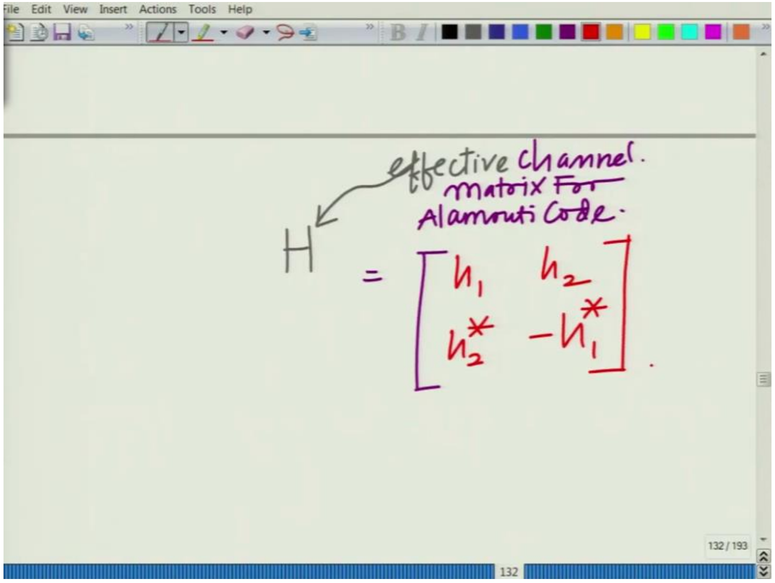

And the channel matrix for this system if you look at this is what we call as, the effective channel matrix for the Alamouti code, effective matrix channel matrix we can call this effective channel matrix, effective channel matrix for the Alamouti code this is given by the expression h1, h2, h2 conjugate minus h1 conjugate now, this by itself is not unitary.

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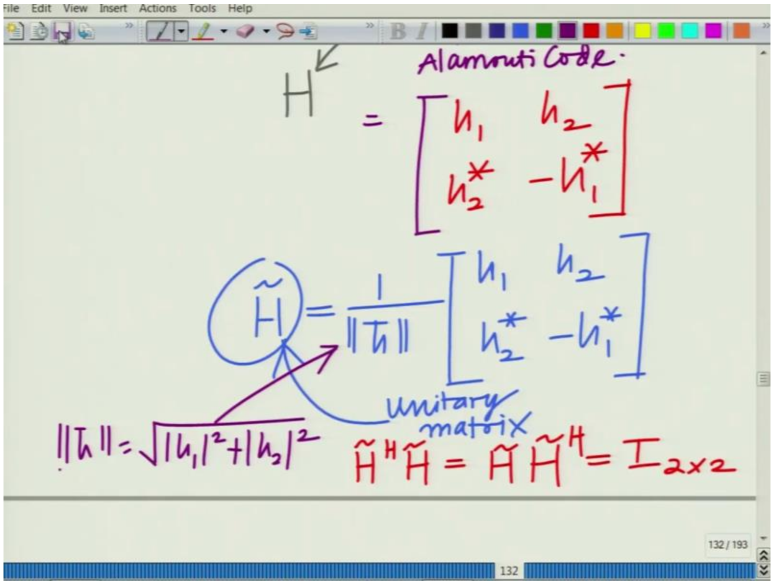

But if you look at H tilde which is obtained by dividing the matrix by the norm of the channel vector that is norm of h bar that is if you look at h1, h2, h2 conjugate minus h1 conjugate now, this matrix H tilde you can show easily is in fact you can check this, this matrix you can easily show this as a unitary matrix that is H tilde Hermitian H tilde equals H tilde Hermitian equals to cross to identity matrix and in fact this quantity norm of h this norm of h bar is also obvious this is square root of should be fairly easy for you to understand.

This is the norm of each column that is norm of h bar equals square root of magnitude h1 square plus magnitude h2 square. In fact, this is a practical example of this concept and in fact rotation and unitary matrices arise very, very frequently especially, we will see a concept later on in this MOOC, we will be we will talk about for instance the singular value decomposition and in the singular value decomposition we have unitary matrices.

We have the singular vectors and essentially we will see that the unitary matrices are a key component of the singular value decomposition. Let us look at some properties of these Unitary Matrices, a Unitary Matrices are very, very interesting properties and as I said they arise very frequently and they have very interesting properties.

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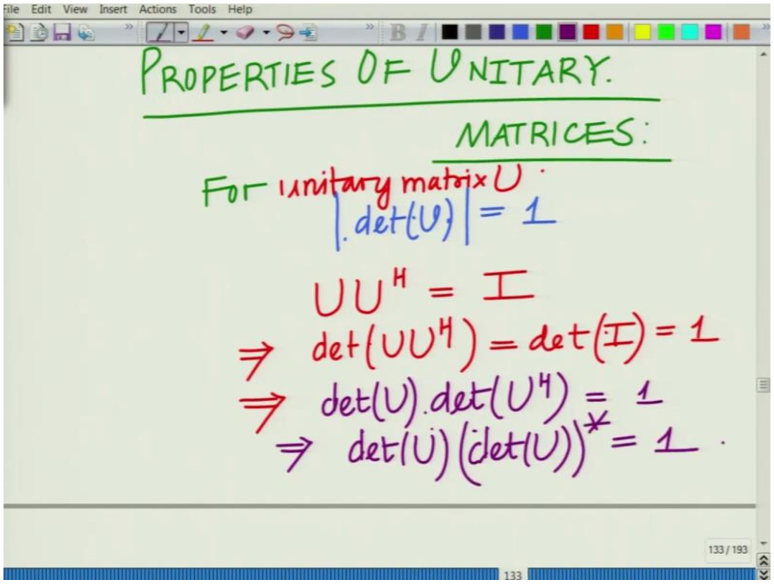

So, properties, Properties of Unitary Matrices for any unitary matrix U, it is easy to show that the determinant of U or the magnitude of the determinant of U equals 1 this is not very difficult to show because remember for Unitary matrix and this is not easy to show because remember we have the property U U Hermitian equals identity this implies that determinant of U U Hermitian equals determinant of identity which is equal to 1, but the determinant of the product of 2 matrices is nothing but the product of the determinant. So, determinant of U into determinant of

U Hermitian equal to 1. This implies but determinant of U Hermitian is the conjugate of the determinant of U that is a product of property of the determinants. So, this is essentially you can write this as the determinant of U times determinant of U conjugate equal to 1.

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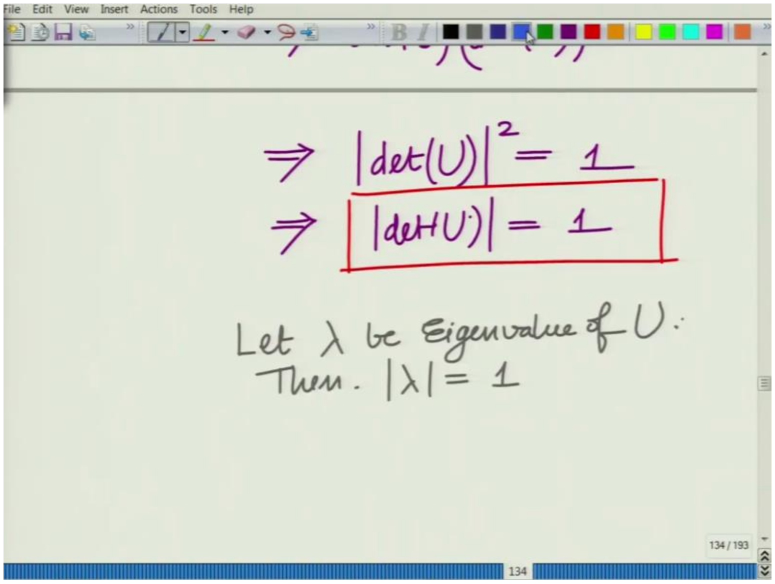

This essentially implies that magnitude determinant of U square equal to 1 this implies magnitude of determinant of U is equal to 1. So, magnitude of this or any unitary matrix the magnitude of determinant U is equal to 1 in fact, we have an even more interesting property that is if you look at any Eigenvalue of the unitary matrix that has to have the magnitude equal to 1. All Eigenvalues of unitary matrices have unit magnitude that is a very interesting property. So, let lambda be Eigenvalue of U, then magnitude lambda is equal to 1. This is an interesting property for instance, let us consider v bar to be considered since we are considering U to be matrix.

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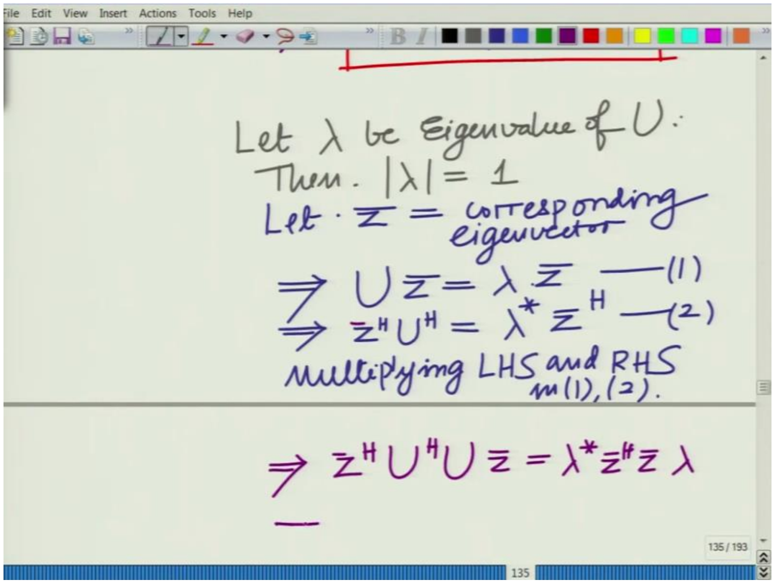

Let, let us say let us call z bar be the corresponding Eigenvector this implies U times z bar equals lambda times z bar and this also implies that z bar Hermitian U Hermitian. Taking the Hermitian on both sides this is a lambda conjugate z bar Hermitian. Now multiplying LHS and RHS in both 1 and 2 multiplying left hand side, right hand side in 1 comma 2, this implies we have z bar Hermitian, U Hermitian U z bar equal to lambda conjugate z bar Hermitian z bar into lambda.

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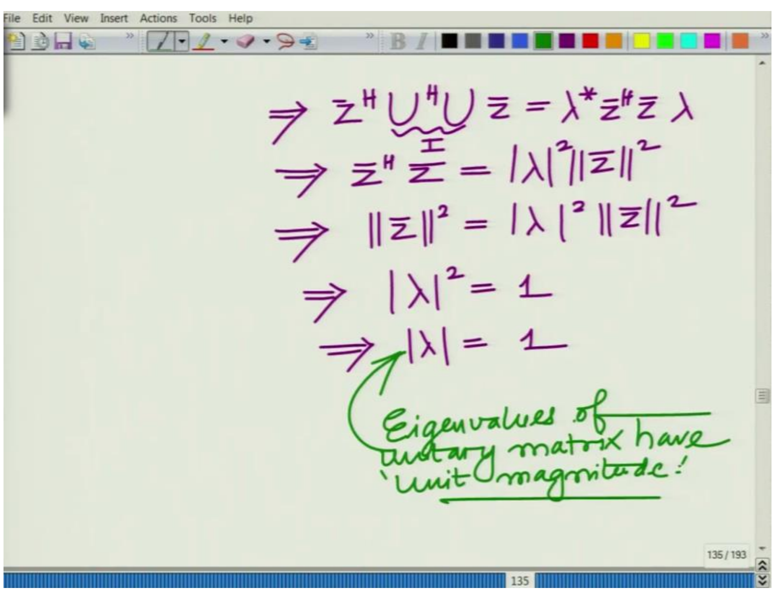

There is now we have U Hermitian equal to identity. So, this implies z bar Hermitian times z bar equals lambda conjugate into lambda that is magnitude lambda square into norm z bar square and on the left also you have norm z bar square z bar Hermitian z bar a norm z bar square equal to magnitude lambda square norm z bar square this implies magnitude lambda squared equal to 1 This implies that magnitude lambda equal to 1. So, all Eigen values of the unitary matrix have unit magnitude this shows Eigenvalues of the unitary matrix these have unit magnitude. And let us look at other another interesting property.

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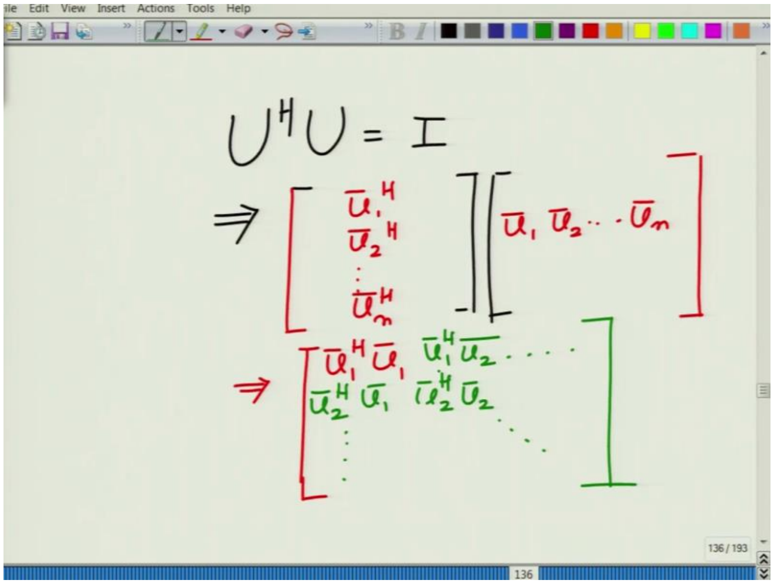

We have u Hermitian U equal to identity which means we write it in terms of the columns that is of the matrix U has the columns u1 bar u2 bar if we see the unitary matrix as the columns u1 bar u2 bar un bar let us say this is an n cross n matrix and then the U Hermitian will automatically have rows u1 bar Hermitian u2 bar Hermitian un bar Hermitian and this implies that if you look at the product this will be of the form.

Well, the diagonal entries will be u1 bar Hermitian u1 bar u2 bar Hermitian u2 bar and so on and the off diagonal elements will be un bar Hermitian u2 bar, u2 bar Hermitian u1 bar and so on. And these are the diagonal entries are 1.

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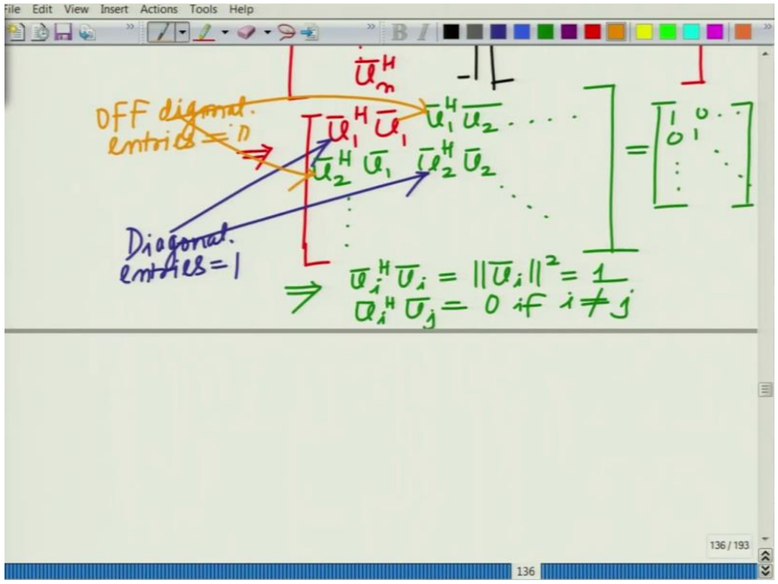

So, and this is equal to identity and what is identity and identity is nothing but a matrix which has all the diagonal entries which are 1 and the off diagonal entries are 0. So, this implies essentially that ui bar Hermitian ui bar which is equal to norm ui bar square which is equal to 1 and ui bar Hermitian uj bar equal to 0 if I naught equal to j that is off diagonal entries are 0. So, diagonal entries are 1.

So, if you look at the diagonal entries of this matrix diagonal entries are 1 and if you look at the off diagonal entries what are that is the 1 2 there is ij elements where i is not equal to j. So, if we look at the off diagonal entries so, off diagonal entries are 0 that is ij elements ij is equal to 1 which means that if you look at any columns we look at any columns u1 bar u2 bar un bar this satisfies the property that norm ui bar square equal to 1 that is their unit norm.

And any two different columns ui bar herm ui bar and uj bar satisfy the property ui bar Hermitian uj bar equal to 0 that is the any two different columns are orthogonal. So, we say the columns are orthonormal, they are unit norm and orthonormal. So, is this so we basically have these 2 properties.

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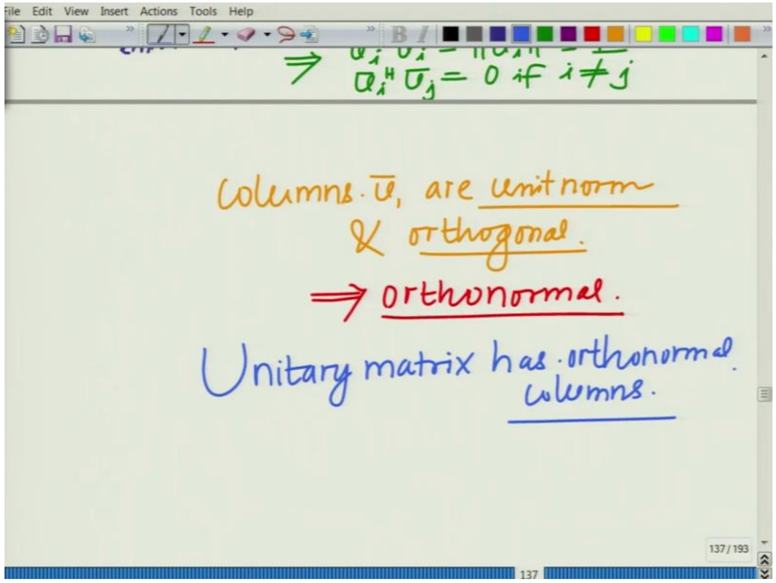

So ui bar the columns ui bar, Unit norm they are unit norm and orthonormal and this implies they are essentially these are this is what is termed as orthonormal columns. So, unitary matrix has orthonormal columns. So, there is an interesting property. So Unitary matrix, Unitary matrix has orthonormal columns.

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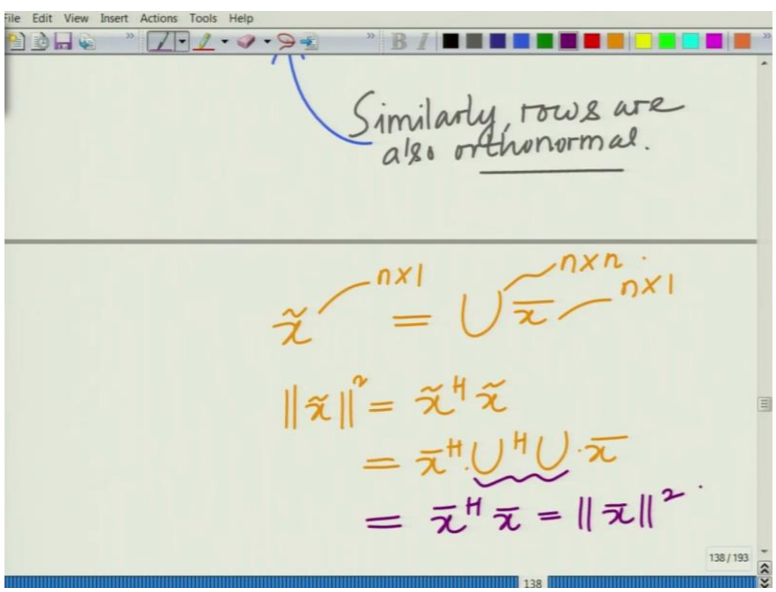

Similarly, you can show similarly, it is not very difficult to see again row rows are also rows are also orthonormal that is they have unit nor that is the rows of the unitary matrix also have unit norm and they are orthogonal to each other.

Now, let us look at another interesting property and this and all these properties, these arise very frequently that is we take any vector x tilde n dimensional vector and cross 1 which is given as u times x bar where use n cross n unitary matrix and x is x bar is m cross 1 unit vector, then if you look at norm of x tilde square that is equal to x tilde a Hermitian x tilde which is x bar Hermitian u bar Hermitian u times x bar which is now you can see U by U Hermitian use identity So, this is simply x bar Hermitian x bar which is equal to norm of x bar square.

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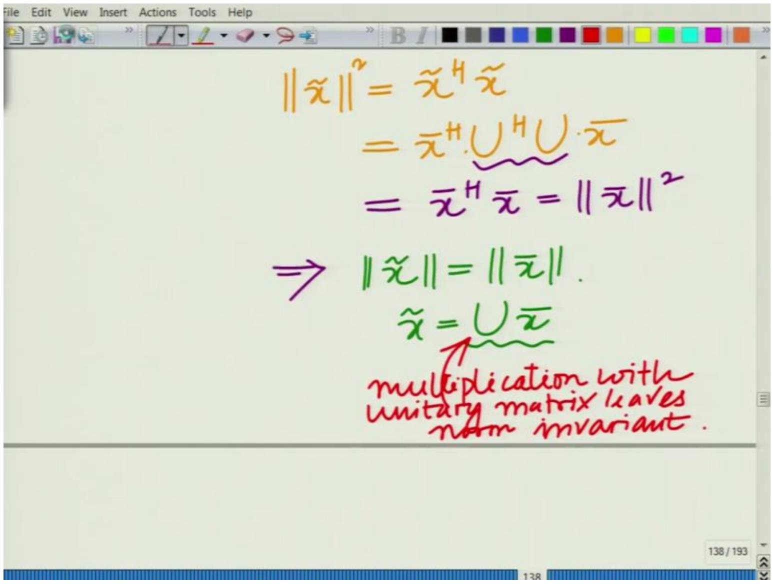

So, it implies that norm of x tilde this interestingly implies the norm of x tilde equal to norm of x bar where x tilde are equal to u times x bar that is you multiply any vector by a unitary matrix it does not change the norm that is a norm of the output vector will be the same as the norm of the input vector and this is expected because U is simply a complex rotation matrix this can be thought of as complex rotation. So, multiplying by unitary matrix, matrix leaves does not change leaves the norm invariant multiplication the use the norm invariant.

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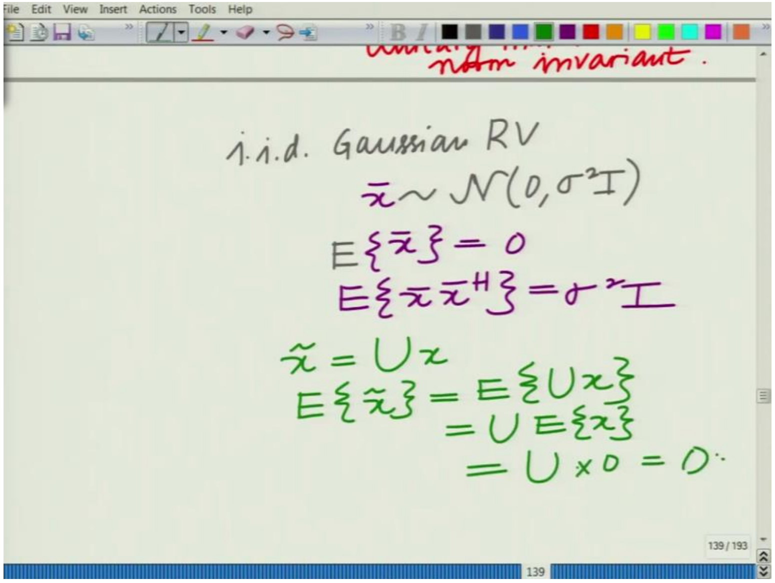

And further you can also have another interesting property that is, let us go back to consider or ID Gaussian random vector that is x bar which is distributed as Gaussian mean equal to mu and variance is sigma square i that is variances proportional to identity, sigma square times identity arises when the Gaussian components are 0 mean. I mean, are independent and these are identically distributed each as variance sigma square.

So, this essentially implies that expected value of let us call this as x bar. So, expected value of x bar is equal to 0, expected value or x bar x bar Hermitian equal to sigma square times identity, then now, if you have x tilde are equal to again u times x, it is not difficult to see that expected value of x tilde is equal to expected value of u times x which is equal to u times expected value of x which is equal to u times 0 which is equal to 0. So, expected value of x tilde is 0.

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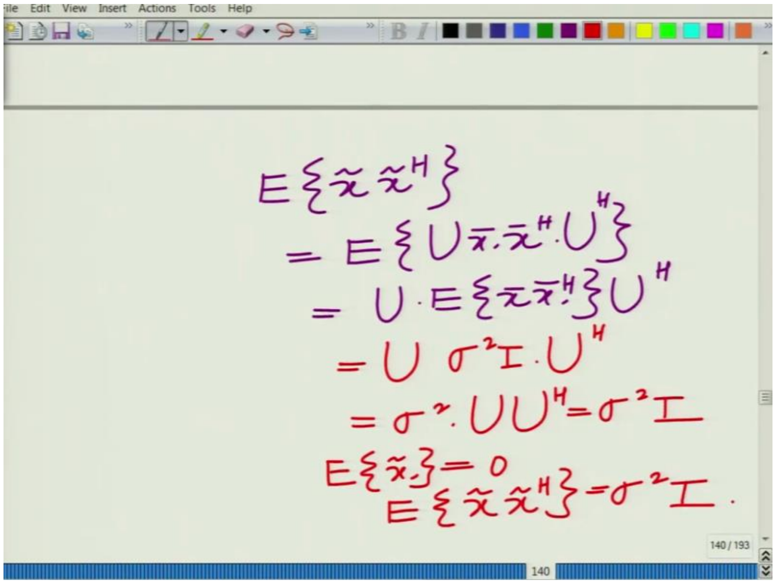

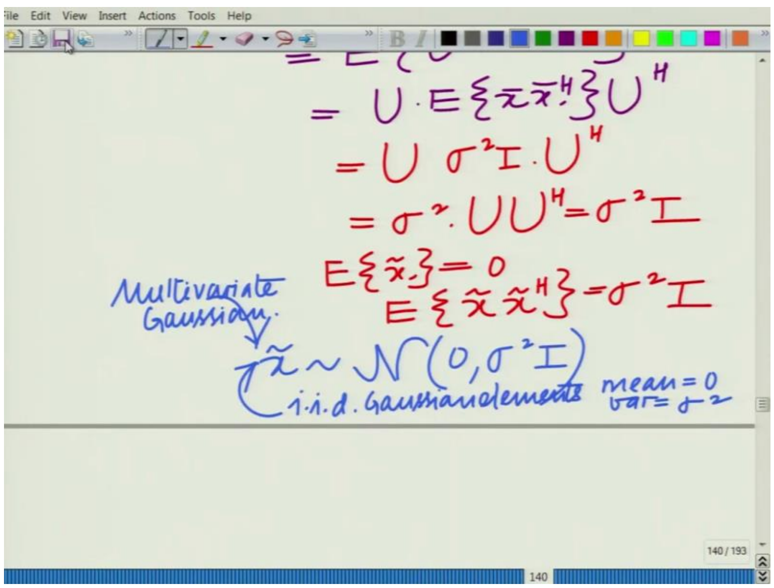

Further, if you look at the covariance matrix of x tilde that is expected value of x tilde x tilde a Hermitian that is equal to expected value of U times x bar, x bar Hermitian times U Hermitian which is U now, expected value of x bar, x bar Hermitian into U Hermitian which you can essentially say is U expected value of x bar x bar Hermitian is sigma square times identity times U Hermitian which is now take sigma square outside U, U Hermitian which is sigma square times identity. So, you have now expected value of x tilde that is mean is 0 and covariance is again exactly the same expected value of x tilde, x tilde Hermitian equals sigma square times identity.

And therefore, if you look at this x tilde is proportional to is our has the distribution multivariate was in not since this is a linear transmission linear transformation externalize also you can see multivariate Gaussian because we said this we have seen earlier that is a linear transformation linear transformation of a multivariate Gaussian gives another multivariate Gaussian what is striking is that you have independent identical Gaussian you have an Gaussian multivariate Gaussian with iid Gaussian component Gaussian elements.

And when you transform it with a unitary matrix what you will see is again in the resulting multivariate Gaussian the components are independent identically distributed Gaussian random variables with mean 0 and variance sigma square. So, the multivariate Gaussian with independent identical identically distributed Gaussian random variables under unitary transformation results in another multivariate Gaussian with iid Gaussian distributed component so, this is an interesting property. So, x tilde law also has is also as iid Gaussian elements mean equal to 0 and variance equal to sigma square.

So, that is the interesting property that you have all. So, I think that is an exhaustive introduction that completes our module on unitary matrices that is rotation matrices and their analog in the complex world that is unitary matrices which satisfy the property U Hermitian equal to U U Hermitian equaled identity and these arise very, very frequently as I already alluded to, for instance, unitary matrices arise and the singular value decomposition. They also arise in the Eigenvalue Decomposition of what are known as Positive Semi Definite and Positive Definite Matrices which we are going to see next.

So, and these arise very frequently and these have a very important role to play often in Linear Algebra. So, with that, let us stop here and let us continue in the subsequent modules. Thank you very much.