Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
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Lecture No. 15
Eigenvalues: Definition, Characteristics equation,
Eigenvalue Decomposition

Hello, welcome to another module in this massive open online course and in this module let us start looking at another very important aspect in the whole of Linear Algebra and that is the concept of an Eigenvalue and the Eigenvalue decomposition, I think this is perhaps one of the most impactful and one of the most important concepts in the entire domain of Linear Algebra and Matrix Algebra.

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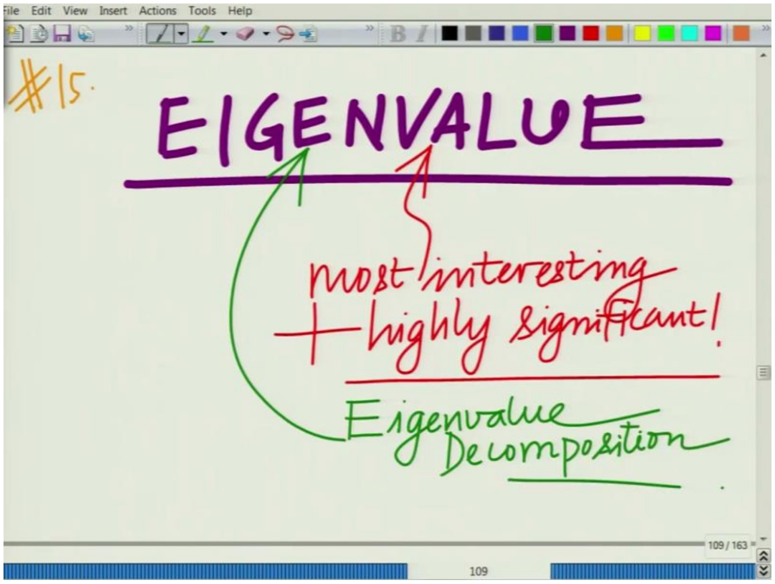

So, this concept of Eigenvalue which is so fascinating and has so many applications, probably one of the most central most I would say this is one of the most interesting plus highly significant because of the large number of applications of this, as one of the most interesting and highly impactful, I would say. And this Eigenvalue is also related to the Eigenvalue Decomposition, these are essentially more or less interlink. Eigenvalue decomposition as in if you find the Eigenvalues and the Eigenvectors that gives the Eigenvalue decomposition.

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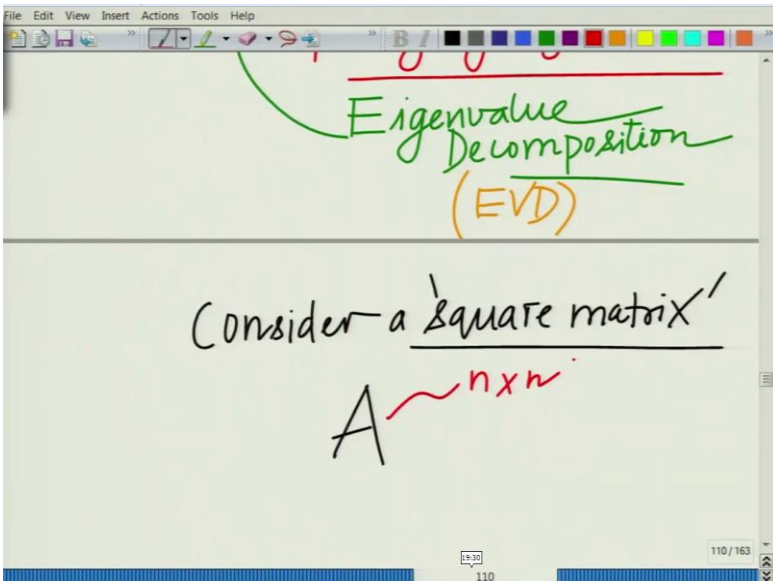

So, which is also often you will see abbreviated as EVD popularly sometimes in literature books and papers people call this as EVD that is the Eigenvalue Decomposition. And this is as I have told you, I would like to explain this very clearly because this is one of the most important concepts that we are probably going to study in this entire course now, what is the concept of an Eigenvalue? This is defined for a square matrix Eigenvalue decomposition, consider a square matrix, consider a square matrix Eigenvalues are defined only for square matrices that is one thing that is let us say we have a matrix A which is an n cross n matrix.

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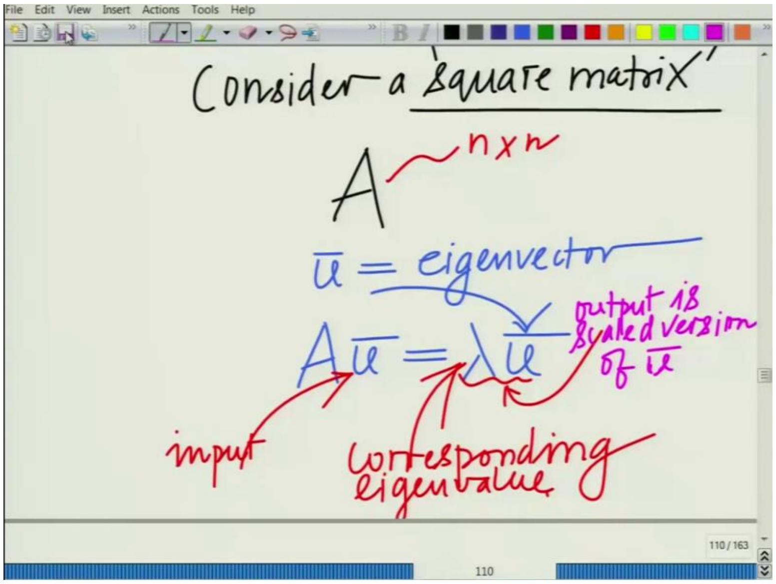

Now, u bar is the Eigenvector which is an n dimensional vector we call this an Eigenvector if it satisfies the property that Au bar equals some lambda times u bar. So, we call u bar as the corresponding Eigenvector and lambda as the corresponding Eigenvalue. So, u bar so, if you have square then you have square matrix A, there is a vector u bar so that is A times u bar is simply a scaled version of u bar that is lambda some scalar quantity lambda times u bar.

So, you can think of this as system input u bar output is simply it is not a it is looks it is identical to u bar except that it is simply a scaled version. So, you have the input to this system you can think of this as an input to the system this is the input u bar and output is scaled version of u bar, scaled by lambda. This is one way to think about this. Output is scaled version output is scaled version of u bar, Au bar equals lambda times u bar. So, the vector u bar is known as the Eigenvector and this quantity lambda is known as the Eigenvalue of this square matrix, of the square matrix A.

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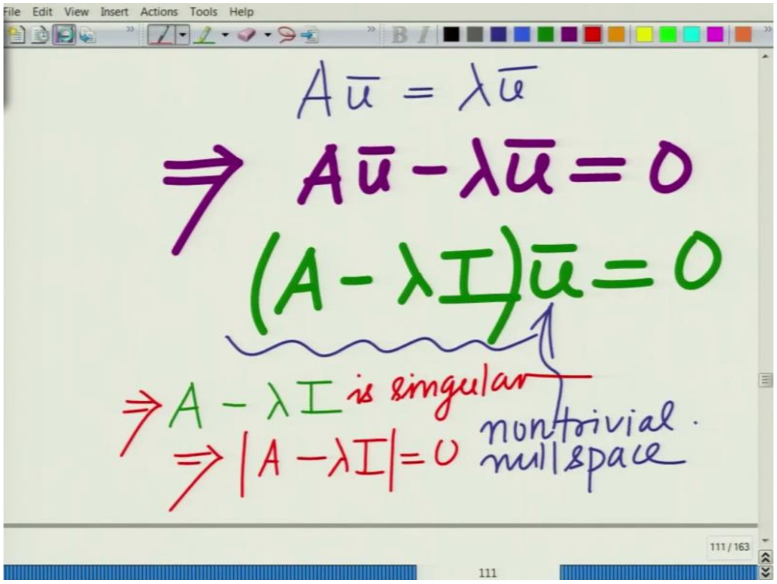

So, this implies note that, this can be simplified as follows Au bar equals lambda u bar. This essentially implies that we write it in big font, this essentially implies that Au bar minus lambda u bar equal to 0 and this essentially implies now, you look at this which means A minus lambda I times u bar equal to 0. And now, if you look at this what does this mean this means that this matrix this is you look at this matrix, this has a non trivial null space, u bar is not 0 this has a non trivial null space.

Remember that is the concept that we have seen this as a non trivial null space, is a singular matrix A minus lambda u bar equal to 0 at singular matrix. This means that A minus lambda I is a singular matrix and this essentially implies that the determinant of A minus lambda I equal to 0 this essentially implies that the determinant of A minus lambda I equal to 0.

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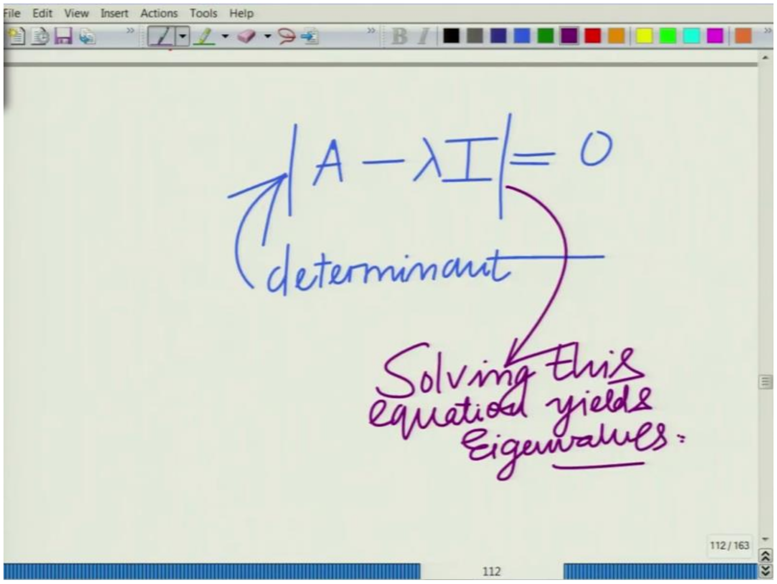

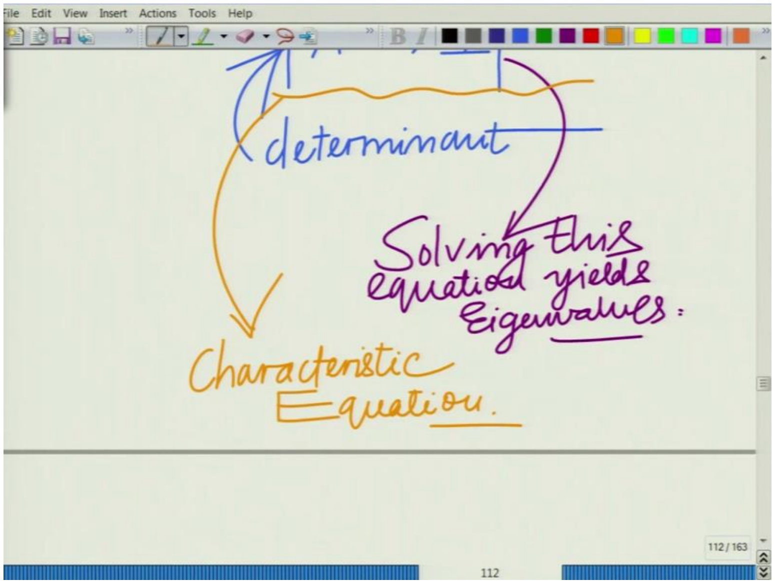

And in fact, now, therefore, the lambdas can be found as the solution to this equation that is the determinant of A minus lambda I, remember this is the determinant and the lambdas can be found as a solution of this equation and this equation is termed as solving this yields the solving this equation yields the Eigenvalues. So, solving this equation yields the Eigenvalues very good. Solving this equation yields the Eigenvalues.

And this equation is known as the Characteristic Equation that is if you look at this determinant of A minus lambda equal to 0, this is known as the Characteristic. This is known as the Characteristic Equation corresponding to the matrix A.

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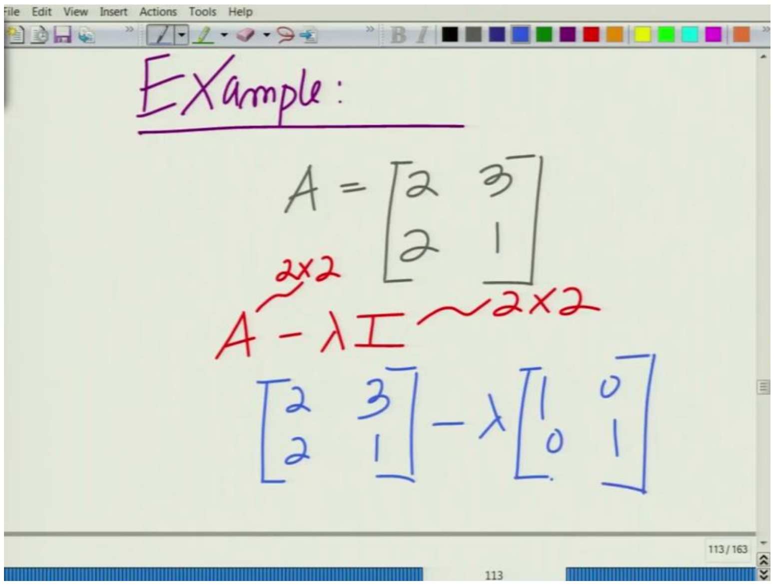

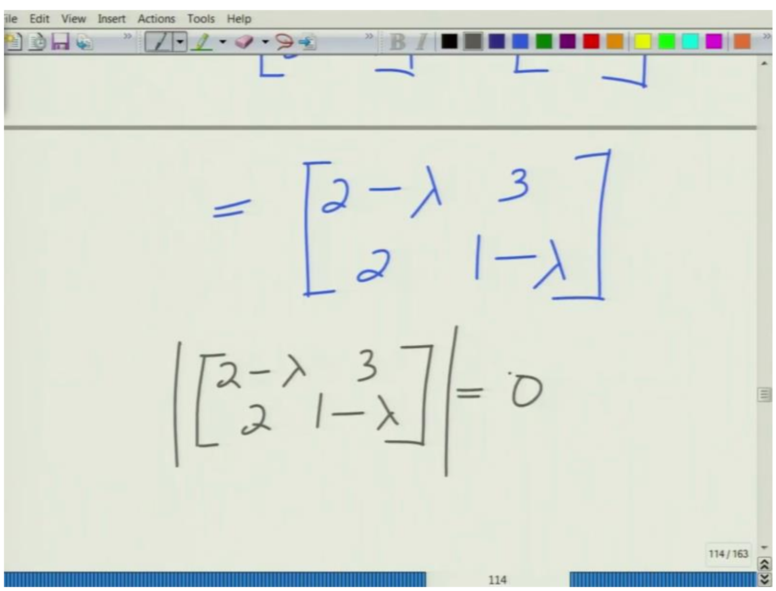

Let us look at a simple example, example. So, hope you have been able to follow what we are saying so, far you have the square matrix A, Au bar equal to lambda u bar if there is a vector u bar that satisfies this that is the output is a scaled version of the input, u bar is the Eigenvector lambda is the corresponding Eigenvalue and the Eigenvalues can be found as the solution of the characteristic equation which is given by determinant A minus lambda I equal to 0.

Let us look at a simple equation, let us look at a simple example to understand this for instance, let us look at the example A equal to 2 comma 2, 3 comma 1 and now, what is A minus lambda I A minus lambda times identity of course, I have to take the identity of the same size as A so, A is 2 cross 2 which means, I will also be 2 cross 2 so, we have A minus lambda I, so, you have 2 2, 3 1 minus lambda 1 0, 0 1 which is essentially now you take this, this is essentially going to be 2 minus lambda 1 minus lambda 3 2 that is A minus lambda I now the determinant of A minus lambda I that is essentially determinant of 2 minus lambda 1 minus lambda 3 2, this is essentially equal to 2 minus lambda into 1 minus lambda minus 6. Now, we have to set this equal to 0 remember this set this equal to 0.

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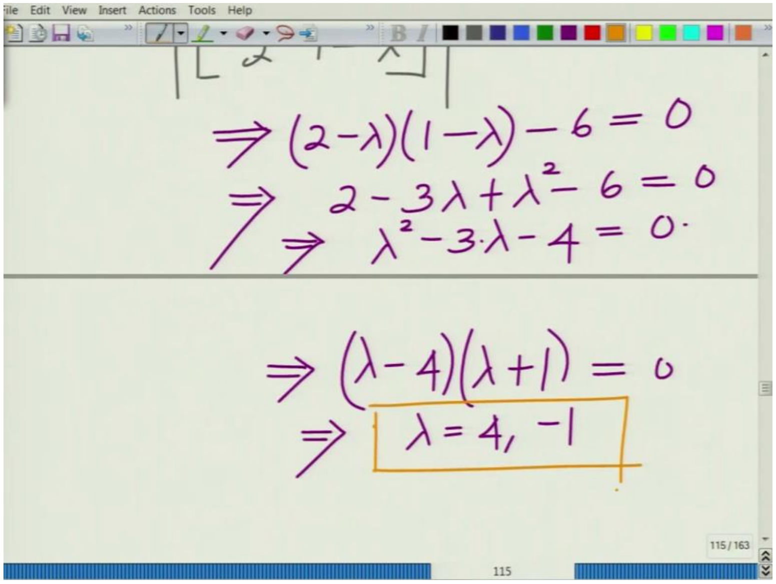

This implies essentially that 2 minus lambda into 1 minus lambda minus 6 equal to 0, this implies that 2 minus 3 lambda plus lambda square minus 6 is equal to 0. This implies that lambda square minus 3 lambda minus 4 equal to 0, this implies lambda minus 4, this essentially implies lambda minus 4 into a lambda plus 1 equal to 0, this implies that lambda equal to 4 comma minus 1. So, this is essentially what this implies. So, this is essentially what this implies.

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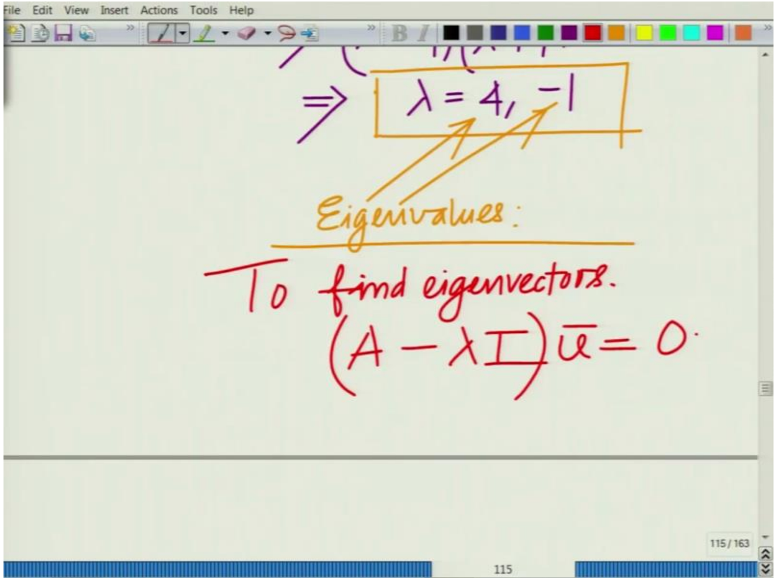

So, these are the Eigenvalues lambda equal to 4 lambda equal to minus 1. So, you have a quadratic equation solving that yields the 2 Eigenvalues, very good. Now, to find the Eigenvectors remember we have to solve remember Eigenvectors satisfy the equation A minus lambda I times u bar equal to 0 that is if you look at the Eigenvector u bar that lies in the null space of the matrix A minus lambda I, where lambda is the corresponding Eigenvalue.

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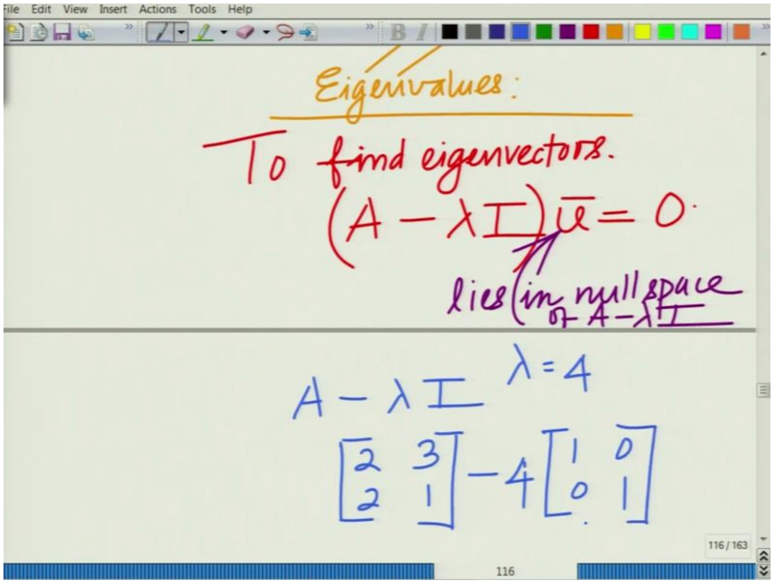

So, the u bar lies in the null space of A minus lambda I. So, u bar lies in the null space of A minus lambda I now, let us take this A minus lambda I set a set lambda equal to 4. So, that will be your, your 2 comma 2 2 and 3 1 minus lambda times I or minus 4 times 1 0, 0 1 which if you look at this now,

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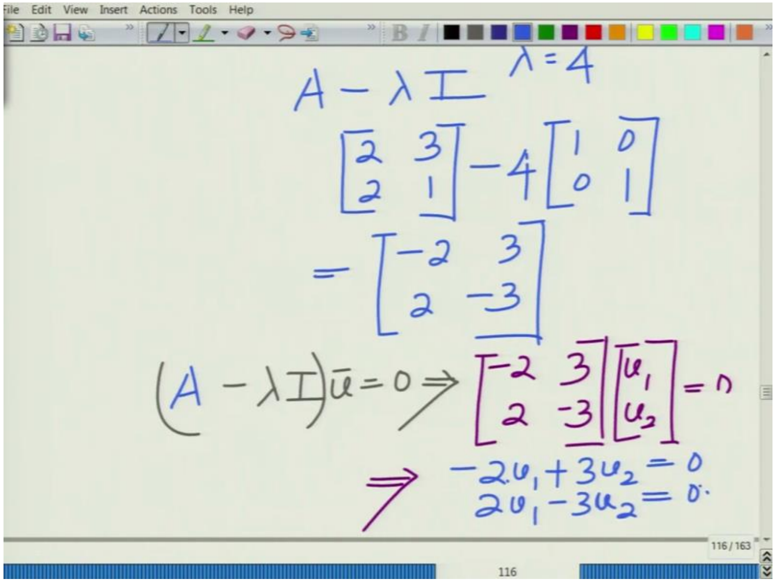

This is essentially you will have minus 2 3 2 minus 2 3 to minus 3 now, we said A minus lambda u bar equal to 0. So, A minus lambda I times u bar equal to 0 this implies that if I substitute this matrix minus 2 3 2 minus 3 u bar, u1, u2 equal to 0 this implies you will see minus 2 u1 plus 3 u2 equal to 0 2 u1 minus 3 u2 equal to 0 remember look, look at this, this is basically the first equation scaled by minus 1 gives you the second equation and this is also always a characteristic of when you try to solve the null space there will always be a free variable.

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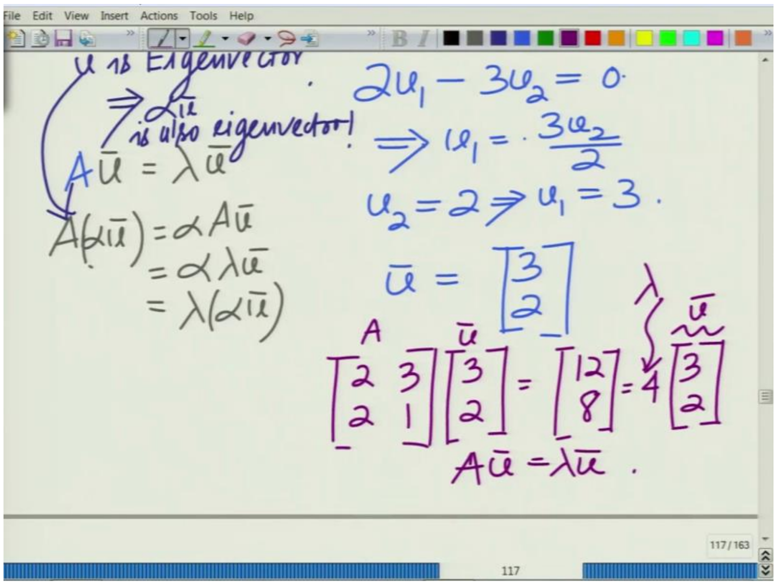

So, in this case I can only I only have till 1 equation 2 u1 minus 3 u2 equal to 0 which means u1 equals 3 u2 by 2 set u2 equal to 2 this implies u1 equals 3. So, I get one of the vector is u bar equals u1 comma u2 that is 3 comma 2. And in fact if we say this is not unique if I said u2 equal 2 in fact if I say u2 equal to 1, I will get u1 equal to 3 by 2. So, the Eigenvector you can see this interesting property of the Eigenvector, Eigenvector is independent of a scaling parameter if you scale it by a constant alpha it will still be an Eigenvector.

Because if you look at the property of the Eigenvector Ax bar or Au bar equal to lambda u bar Au bar equal to lambda u bar, if I scale it by alpha A into alpha times u bar equal to alpha times Au bar which is alpha times lambda u bar which is essentially lambda alpha times u bar. So, if you scale a vector, so, this is scaling does not impact Eigenvector so, alpha u bar, u bar is an Eigenvector implies alpha u bar is also an Eigenvector corresponding to the same Eigenvalue is also is also u bar is an Eigenvector implies alpha u bar is also an Eigenvector.

So, in fact 3 by 3 comma 2 will be an Eigenvector if you scalar multiplied both by 2. So, 6 comma 4 will also be an Eigenvector so on and so forth. Scaling does not have scaling by any constant alpha does not impact at all, it will still be an Eigenvector corresponding to the Eigenvalue lambda and therefore, you will see that there is a free variable that I can choose u2 and u1 can be determined appropriately or in this case I can also choose u1 and u2 will be determined appropriately.

And now, we can do a quick check for instance, let us multiply this Eigenvector by this matrix 2 2 31, this is our matrix A multiplied by this 3 2 which is u bar that gives you essentially to enter the rest 6, 6 plus 6 12 and 2 into 3 6 6 plus 2 8 which is nothing but 4 times your vector 3 comma 2, 4 is nothing but recognize this is your lambda and this is your u bar. So, you have very much satisfies the equation Au bar equals lambda u bar. So, that is very interesting.

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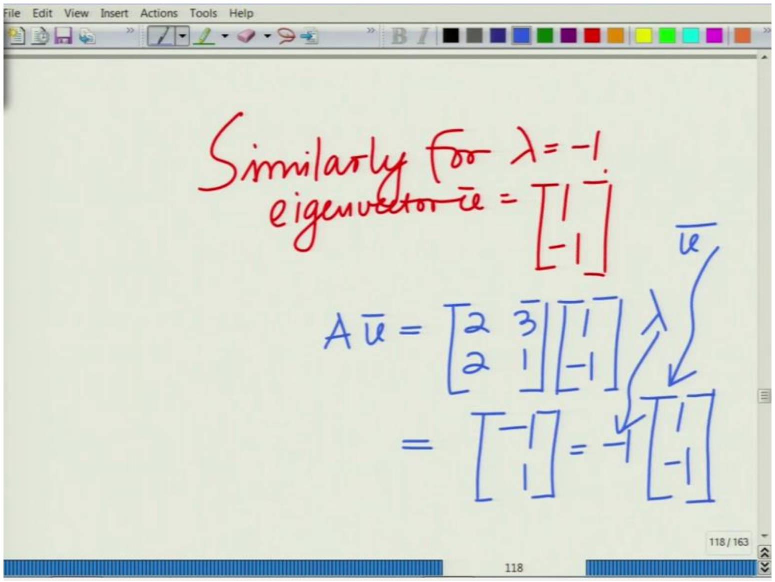

Now, the other Eigenvalue similarly, you can solve this we already solved it even the Eigenvalue is equal to minus 1, 1 similarly, lambda equal to minus 1, Eigenvector u bar this is given as 1 comma Eigenvector u bar this is given as 1 comma minus 1 and you have Au bar equals 2 2, 3 1, 1 comma minus 1 which you can see is essentially just to check this, this is essentially equal to 2 into 1 minus 3. So, this will be minus 1 and 2 minus 1. So, this is 1 which is essentially minus 1 comma 1 minus 1 and this once again you can see this is basically your lambda and this is your vector u bar, Au bar equal to lambda u bar.

So, again corresponding to the Eigenvector corresponding to the Eigenvector 1 column corresponding to the Eigenvector 1 comma minus 1, we have the Eigenvalue minus 1. And of course, if 1 comma minus 1 is an Eigenvector 2 comma minus 2 is also an Eigenvector 3 comma minus 3 is also an Eigenvector in fact, for that matter multiplied by minus 3 minus 3 comma 3 is also Eigenvector which is independent of scaling. Now, let us look at the Eigenvalue Decomposition.

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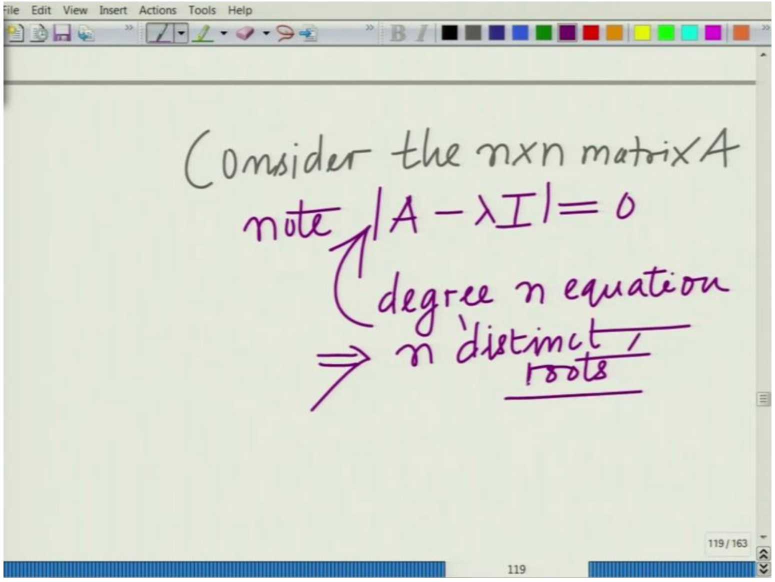

Consider the n cross n matrix A, consider the n cross n matrix A. Now, note that A minus lambda I is equal to 0 note that this has this is an equation of degree n, this is an equation of degree n this implies generally this has n roots, this has n roots, let us assume this has n distinct roots, I will just make it a little subject, the general theory is a little complicated. So, let n this because we are not interested in going into remember this is not a course on pure Linear Algebra, people, people can talk I mean Eigenvalue when Eigenvalue decomposition is such, such a such an important and such a vast area that people can spend lectures and lectures talking about it.

So, we want to simplify our discussion. So, let us consider a simple scenario where this characteristic equation which is an nth degree polynomial has n distinct roots and you have the Eigenvectors corresponding to n distinct roots that is me that makes our discussion simple. So, in that case, once you have these Eigenvectors

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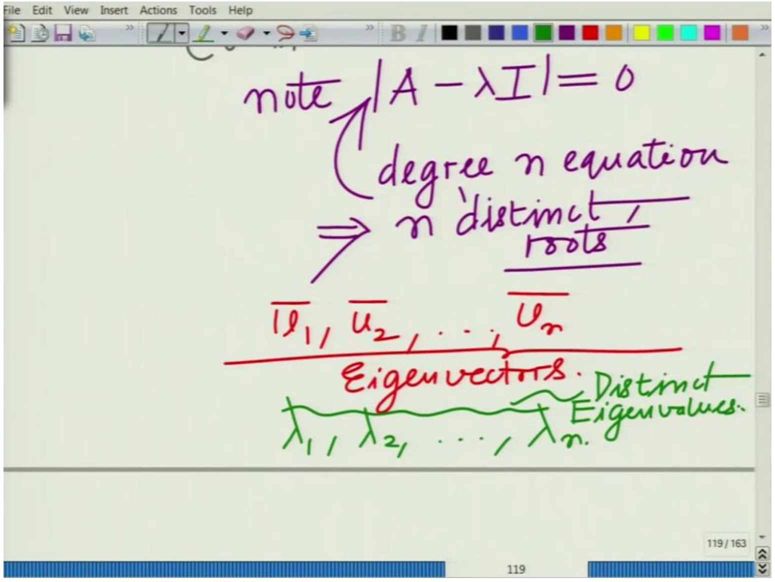

Now, let us let say u1 bar u2 bar so on u1 bar these denote the Eigenvectors and lambda 1 lambda 2 lambda n denote the n distinct Eigenvalues because remember the characteristic equation has distinct roots. So, these are the distinct these are the distinct Eigenvalues.

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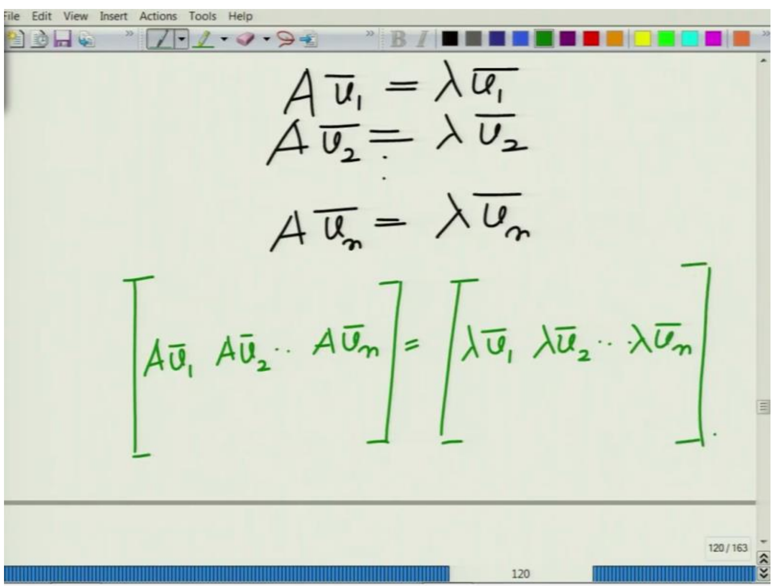

Then you have you are going to have remember, the Eigenvalue satisfy A u1 bar Eigenvector satisfy A u1 bar equals lambda u n bar A u2 bar equals lambda u2 bar and A un bar equals lambda un bar. Now, if I put these stack these together put these together concatenate this as one big matrix remember A u1 bar A u2 bar A un bar I can put these things together as let me just write this a little bit more clearly I can write this as A u1 bar, Au2 bar these are the n columns this is equal to lambda u1 bar lambda u2 bar lambda 2n bar.

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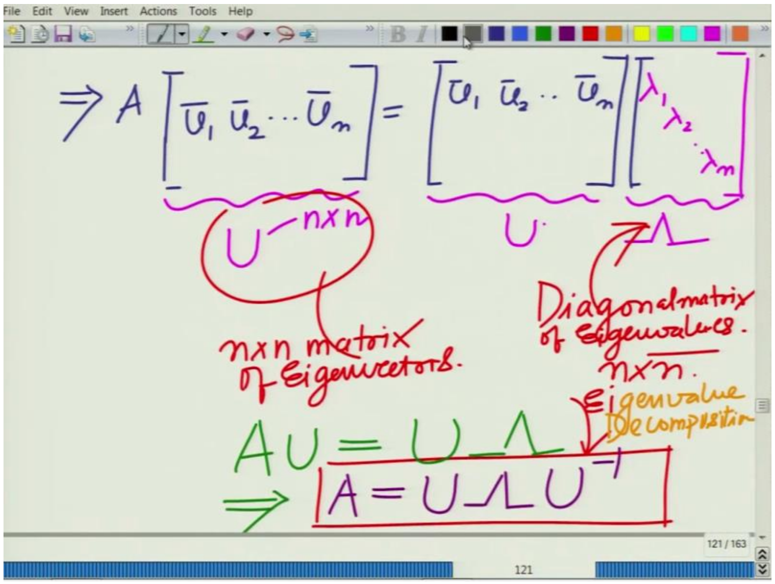

And now, if I pull A out on the left I can write this as A times the matrix containing the vectors u1 bar, u2 bar until un bar this is equal to the diagonal matrix the diagonal matrix so, this is equal to u1 bar u2 bar u1 bar times what is here is the diagonal matrix lambda 1 lambda 2, lambda n and you can see this holds that is this is nothing but lambda 1 u1 bar, lambda u2 bar, lambda n. So, I can this is A times if I now call this as the matrix U, which is essentially an n cross n matrix this is a n cross n matrix of Eigenvectors, this is also u.

Now this is an interesting matrix this is a lambda which is essentially you can see this is the diagonal matrix containing the Eigenvalues this is important, this is the diagonal matrix of Eigenvalues this is the diagonal matrix of Eigenvalues then you have at this u is the n cross n matrix this is also n cross n this is a diagonal matrix and this u is the n cross n matrix of Eigenvectors this is the n cross n matrix of Eigenvectors and what we have shown is that you have Au equals u times lambda where lambda is a diagonal matrix of Eigenvalues and this essentially now take u to the right.

That is, you can show that u is invertible and as I have said we are not let us assume that u is invertible. So, as I said we are not going to go into the finer details of this that is when is u going to be invertible and so on. And then, you can write that equal to u lambda u inverse and this is known as the Eigenvalue Decomposition. This is essentially known as the Eigenvalue Decomposition. This interesting thing is known as the Eigenvalue Decomposition and this is something that is very interesting, which is essentially you have the, you have we are writing this as let me just write this with a very big font.

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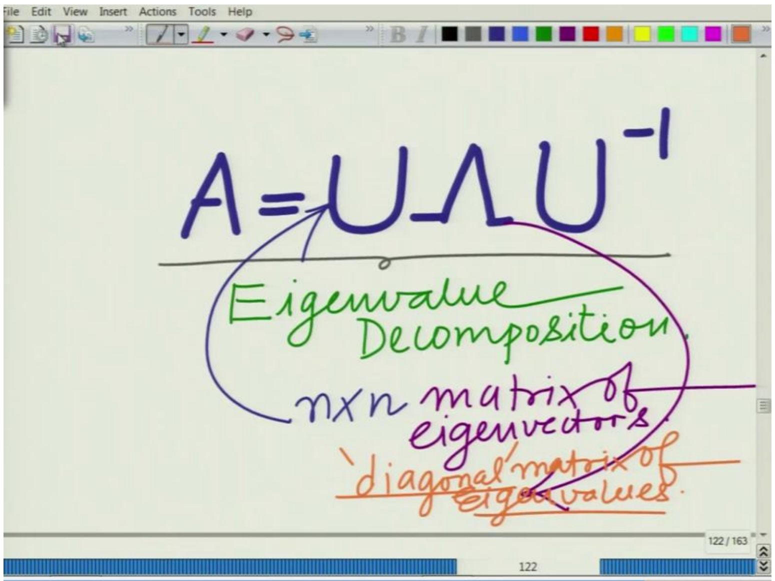

Because this is something that going to be A equal to u lambda u inverse, this is something that is very, very interesting and this is what we are calling as the this is what we are calling as the Eigenvalue Decomposition. This is essentially something that is very interesting and so, this is your u, which is the n cross n matrix of Eigenvectors and the most interesting matrix is this lambda, which is the diagonal matrix of this is the diagonal, this is what is interesting that diagonal matrix of Eigenvalues of this lambda is essentially a diagonal and this is something that is very, very fundamental and you will see this everywhere.

Wherever you have applications of matrices such as Signal Processing, Machine Learning, Data Analysis and so on and so forth. These applications are huge and applications this wide variety of applications that is the property of Eigen the principle of Eigenvalues. And this Eigenvalue decomposition has applications, it is ubiquitous it has applications everywhere from big data to signal processing to machine learning applications are everywhere.

And this is one of the most important components, one of the most important I would say decompositions and one of the most important concepts, let them square matrix A can be written as A u lambda u inverse, where u is the matrix of Eigenvectors and lambda is no diagonal matrix of Eigenvalues. Excellent.

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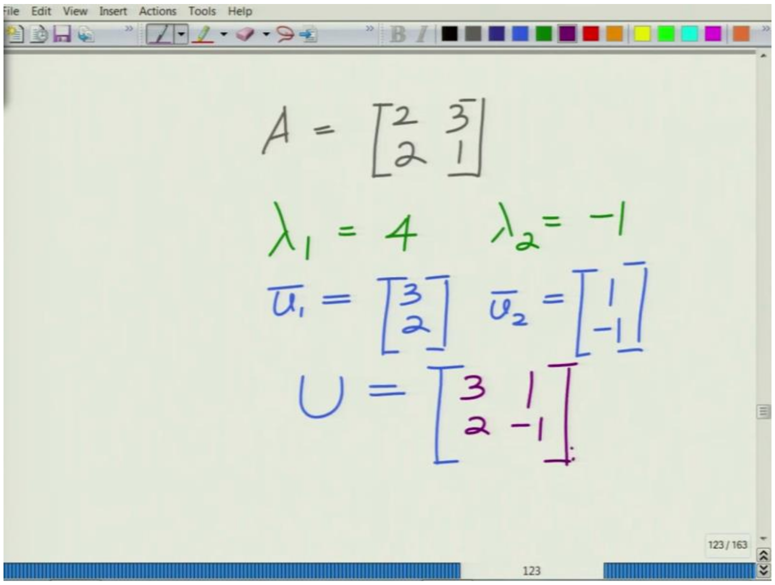

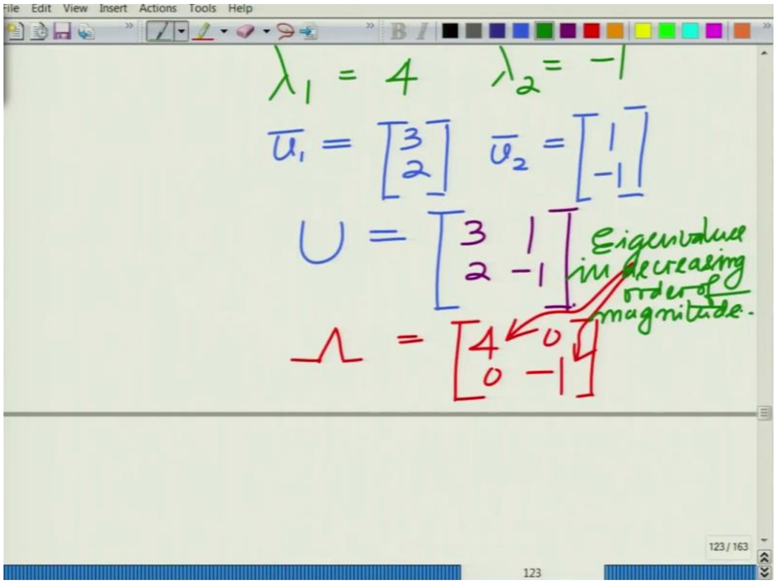

Let us look at a simple example we have A, let us go back to our matrix before we have A equal to 2 2, 3 1, this is 2 2, 3 1. This is our matrix we have the Eigenvalues lambda 1 equal to 4 lambda 2 equal to minus 1 and corresponding to these we have the Eigenvectors u1 bar u1 bar equal to 3 comma 2, u2 bar equal to 1 comma minus 1 therefore, you can form the matrix u which is the matrix of Eigenvectors which will be 3 comma 2 and you will have 1 comma 1 comma minus 1. And the diagonal matrix of Eigenvalues lambda that we are talking about you have to write the Eigen all you have to do is you have to write the diagonal matrix there are Eigenvalues on the diagonal so 4 comma minus 1.

Now, the other interesting thing about this is if you look at this we have arranged the Eigenvalues in decreasing order of magnitude, we have the Eigenvalues 4 and minus 1, so, we have arranged the Eigenvalues in decreasing order of magnitude. Although it is not very important. In fact, Eigenvalue decomposition we can write the Eigenvalues of in any particular in any order, but it is usually very useful you will see realize that in many applications, if you write the Eigenvalues in the decreasing order of magnitude that has some significance as we are going to see later.

So, we are writing the Eigenvalues in decreasing order of magnitude we can also say that is a convention Eigenvalues in decreasing order of magnitude, Eigenvalues in decreasing order of magnitude. And now therefore, we have the property.

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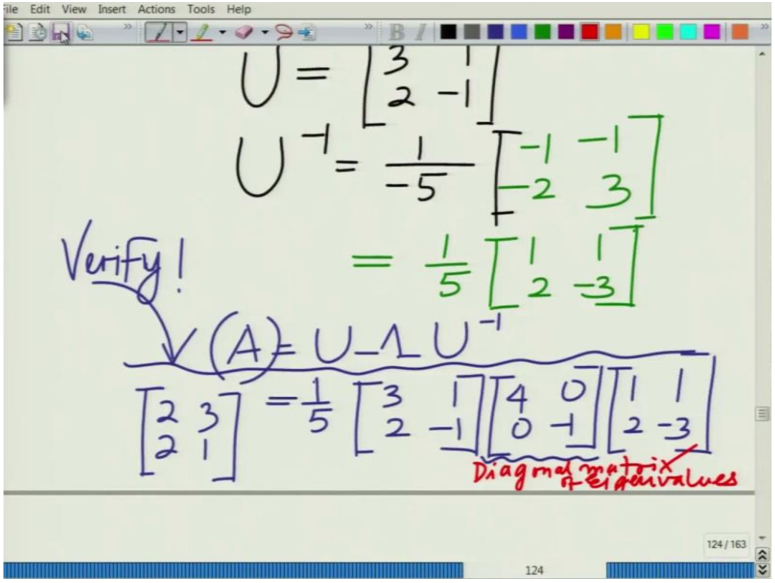

Now, let us calculate u we already know u inverse. So, we have seen u equals well what is u, u equals 3 2, 1 minus 1 so, u inverse remember this is 2 cross 2 matrix so 1 over the determinant minus 3 minus 2, that is minus 5, swap the diagonal elements to cross 2 matrix inverse we have swapped the diagonal elements minus 1 comma 3 and negative of the off diagonal elements negative of the off diagonal elements, so minus 1 minus 2. And now if you take the negative in front of the minus y inside, so, this becomes 1 over 5, 1 1, 2 minus 3.

And therefore, now you have finally A equal to u lambda u inverse which essentially take the constant 1 over 5 outside you can take the constant 1 over 5 outside you can write this as first is u, that is essentially your 3, 2 1 minus 1 the diagonal matrix of Eigenvalues that is your 4 0 0 minus 1 times u inverse, we remember we have already taken this 1 by 5 outside. So, that is 1 1, 2 minus 3 and you can multiply it out and you can ensure that you get the original matrix which is A which is essentially your remember your 2 2, 3 1 this is your original matrix A you can multiply this, so verify this.

So, my suggestion to you is verify this, multiply it out verify this and you can see so, this is basically u1 lambda of course, u inverse will have the 1 1 over 5 over here and this is essentially the diagonal matrix of Eigenvalues. This is your Diagonal Matrix. This is essentially your diagonal matrix of Eigenvalues and this is the Eigenvalue Decomposition which has several, several applications and each I have already said alluded to at the beginning of this module, that this is easily probably one of the most important concepts in entire Linear Algebra.

So please, I hope you paid attention please go over this if you have not understand anything clearly and understand this completely and assimilate this concept because this has several applications, tremendous applications. So with that, let us stop here and let us continue this discussion in the subsequent modules. Thank you.