Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
Professor. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur
Lecture No. 13
Linear Transformation of Gaussian Random Vectors

Hello, welcome to another module in this massive open online course. So, we are looking at Gaussian random variables and Gaussian random vectors that is the multivariate Gaussian random this Gaussian probability density function. Let us continue our discussion, let us look at linear transformation of Gaussian random vectors because this is what arises frequently in the analysis of linear systems.

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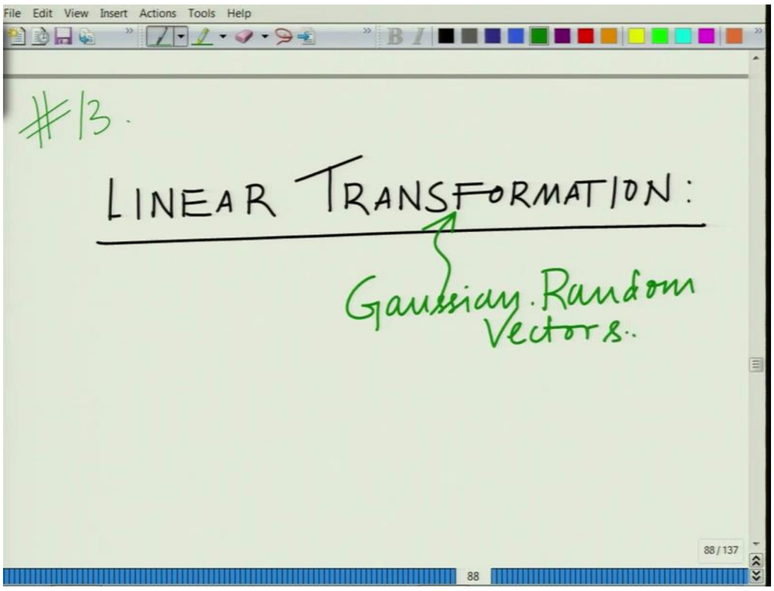

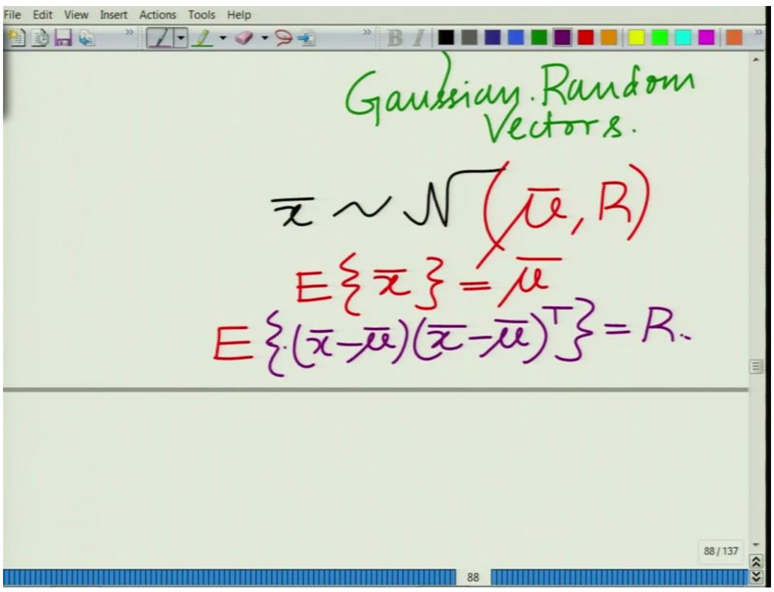

So, let us look at a linear transformation; linear transformation of Gaussian random vectors. Linear transformation, linear transformation of Gaussian random vectors. What do we mean by this? What do we mean by this is the following if x bar is a Gaussian. Remember this is normal; this is the notation we are using for Gaussian with mean mu bar, covariance R. That is what we are saying is expected value of x bar equals mu bar and expected value of the covariance matrix, expected value of x bar minus mu bar into x bar minus mu bar transpose equal to R. Now, what happens when I consider a linear transformation of this?

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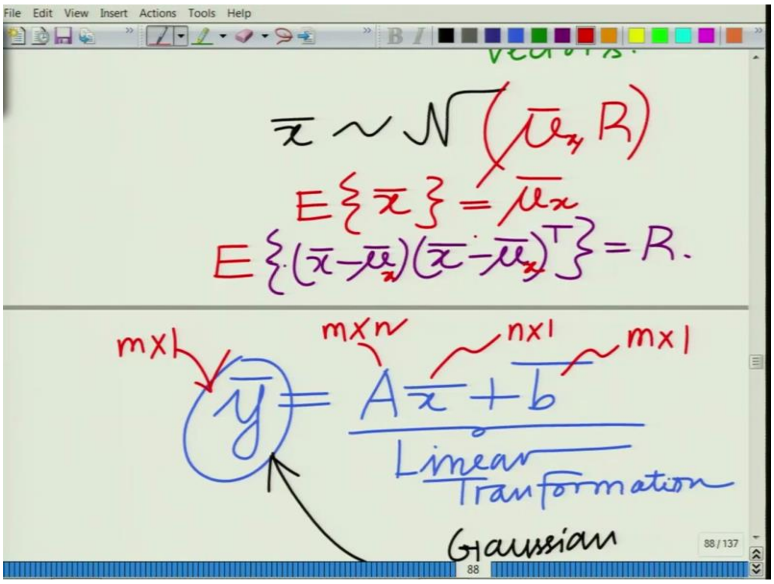

That is, I consider y bar equals A x bar plus b bar. This is a linear transformation. In fact, to be more precise, it is affine transformation. This is simply known as linear transformation. This is a linear transformation of the Gaussian random vector. So, y bar you can see is linear related to x bar. Now, what is interesting is that a linear transformation of Gaussian random vector leads to another Gaussian random vector.

So, Gaussian that is multivariate Gaussian with the components x1, x2, xn jointly Gaussian, linearly transformed leads to another multivariate that is leads to another multivariate Gaussian probability density function that is leads to a Gaussian random vector. So, y bar is also Gaussian in nature. This is a very interesting thing; y bar is Gaussian in nature. That is Gaussian remains Gaussian. It is a very interesting property: Gaussian remains Gaussian under linear; Gaussian remains a Gaussian under linear transformation.

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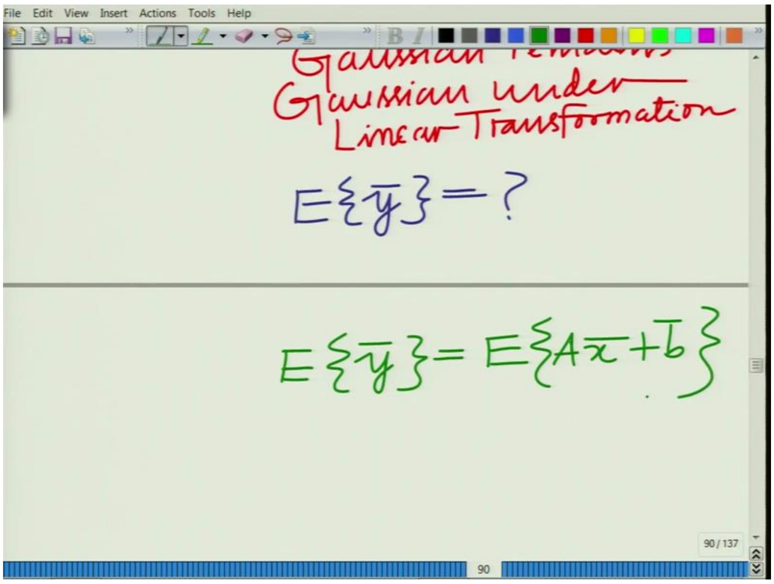

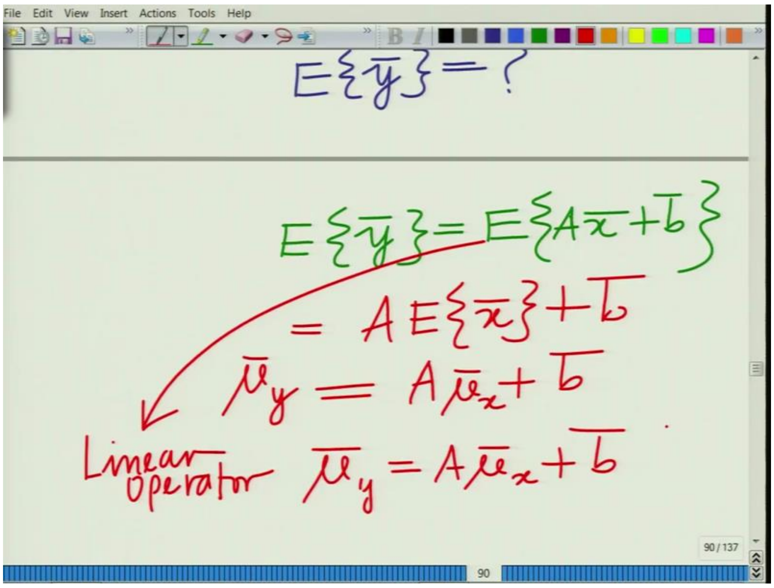

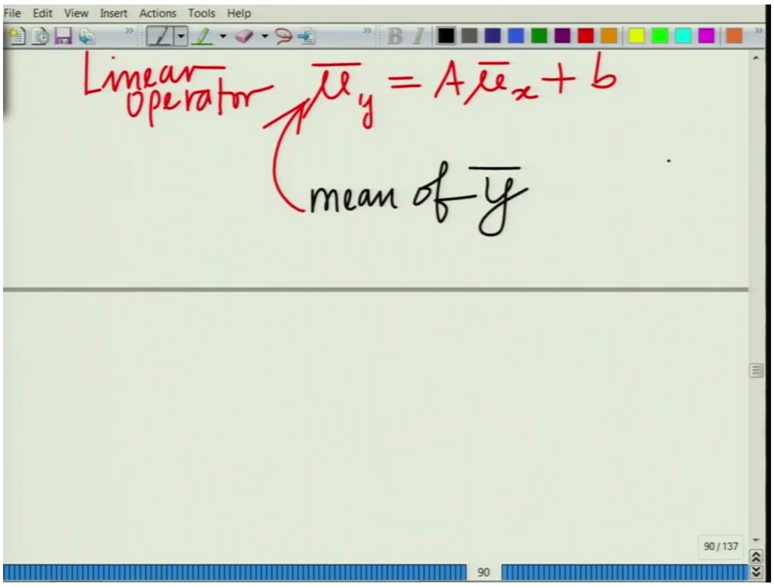

Further now, let us find let us ask the question, what are the mean and variance? That is what is expected value of y bar? This is very simple to find. Expected value of y bar equals expected value or A times x bar, A is a matrix. So, let us write some dimension so that it becomes additionally clear. So, let us say this is an n cross 1 vector. This is an m cross n matrix which means this will be your m cross 1-dimensional Gaussian vector. Of course, b bar will also then be m cross 1.

So, y bar is Gaussian. What is the mean of y bar? Expected value of y bar equals expectation of A times x bar plus b bar. Expectation operator is linear. So, this is a linear operator which means I can write this as A times expected value of x bar plus b bar which is nothing but A times mu bar plus b bar. Because the expected value of x bar equals mu bar. So, you can write this as mu bar y that is the mean of y is A times the expected value of x bar. We can write this as expected value of x bar equal to we can write this as mu x bar. So, this is mu x bar and this is also going to be mu x bar. So, mu bar y equals A times mu bar x plus b bar. So, that is very simple. So, this is basically the mean of y bar.

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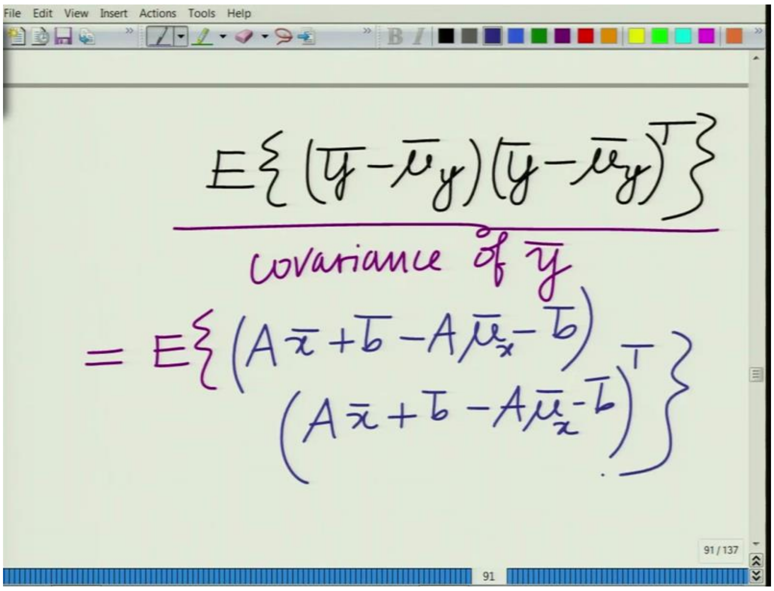

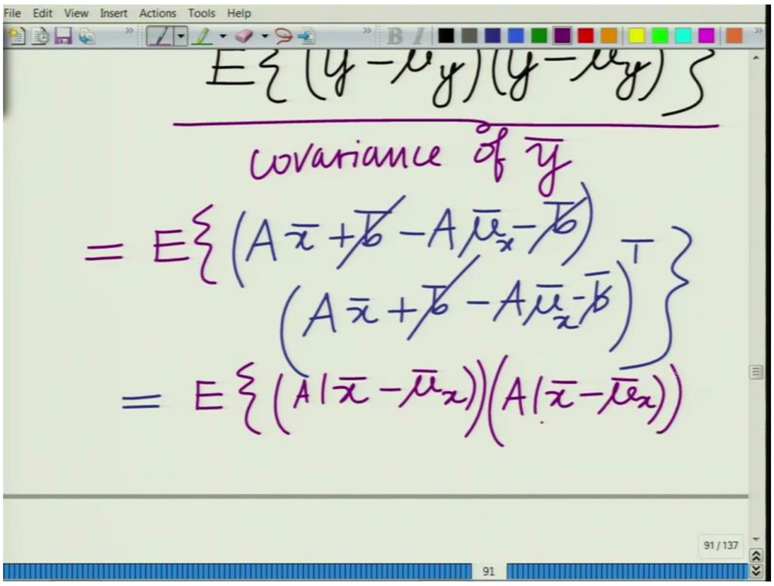

Next, we come to the covariance that is what is the expected value of y bar minus mu bar y into y bar minus mu bar y transpose. This is the covariance. This is the covariance of y bar. This is the covariance matrix of y. This is expected value of y bar minus mu y bar into y bar minus mu y bar transpose. Now, let us substitute for this quantity. So, this you can write this as expected value of y bar is A x bar plus b bar minus A mu bar minus b bar times A x bar plus b bar minus A mu bar, mu x bar minus b bar transpose which is equal to; now the b bars cancels. You can clearly see that. So, that leaves expected value of A x bar minus mu bar x times A x bar minus mu bar x transpose.

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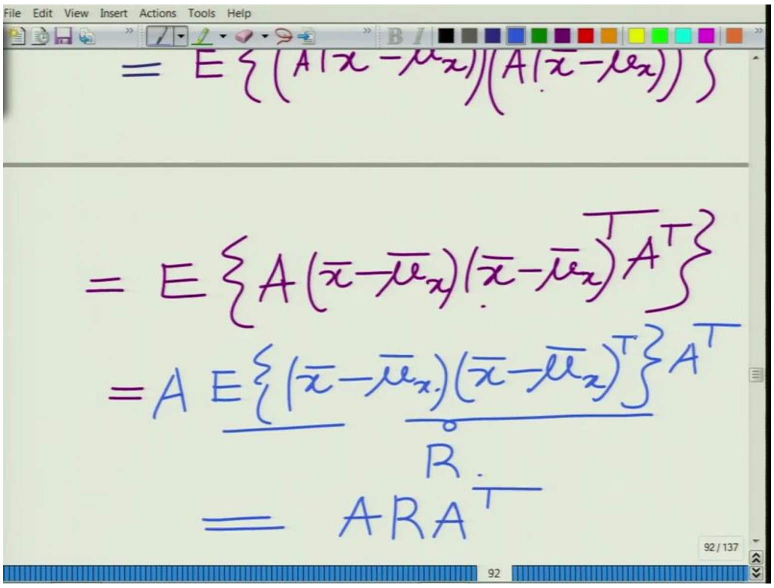

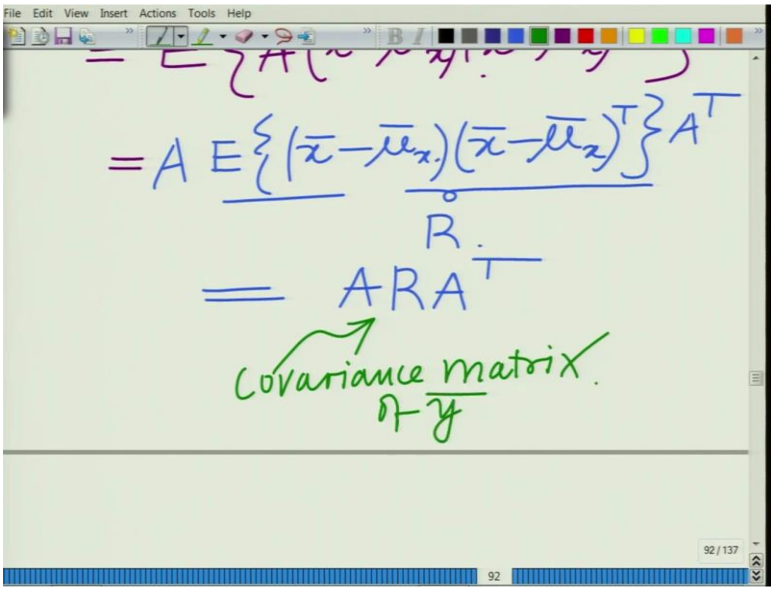

Which essentially if you look at this, this is nothing but expected value of A opening the brackets x bar minus mu bar x times x bar minus mu bar x transpose into A transpose which is now A; these are constant matrices. So, bring them out of the expectation operators. So, this would be A expected value of x bar minus mu bar x times x bar minus mu bar x transpose into A transpose. And this is nothing but the covariance matrix of x. This is R. This is R. So, I can write this as the ARA transpose. So, this is basically your covariance matrix of; so, this is basically the covariance matrix of y bar.

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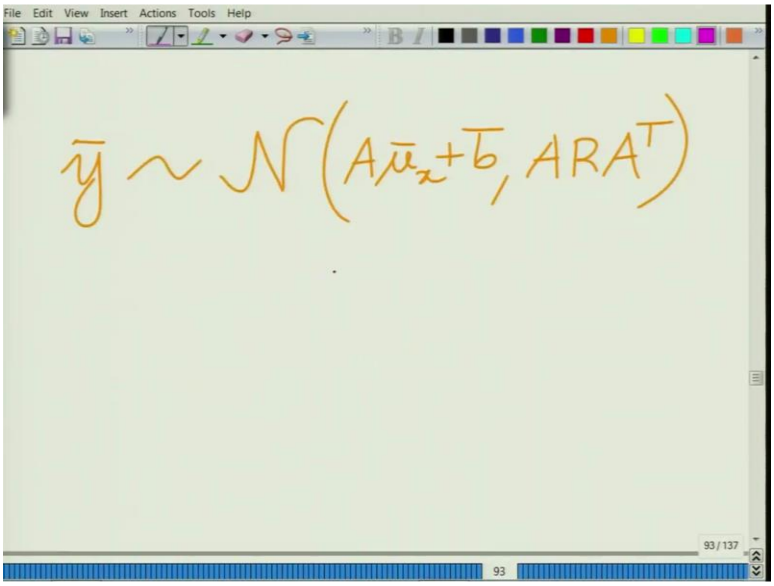

And therefore, y bar we can say, y bar is also Gaussian with mean. Let me write this down clearly: y bar is distributed as a Gaussian with mean A times mu bar x plus b bar covariance ARA transpose. This is the general result. So, this is an interesting. So, this is basically your linearly transforming Gaussian random vector Ax bar Ax bar plus b bar. It remains a Gaussian. What is the mean of the resulting Gaussian vector y bar? A mu bar x plus b bar. What is the covariance matrix? ARA transpose where R is the covariance matrix of the vector x1.

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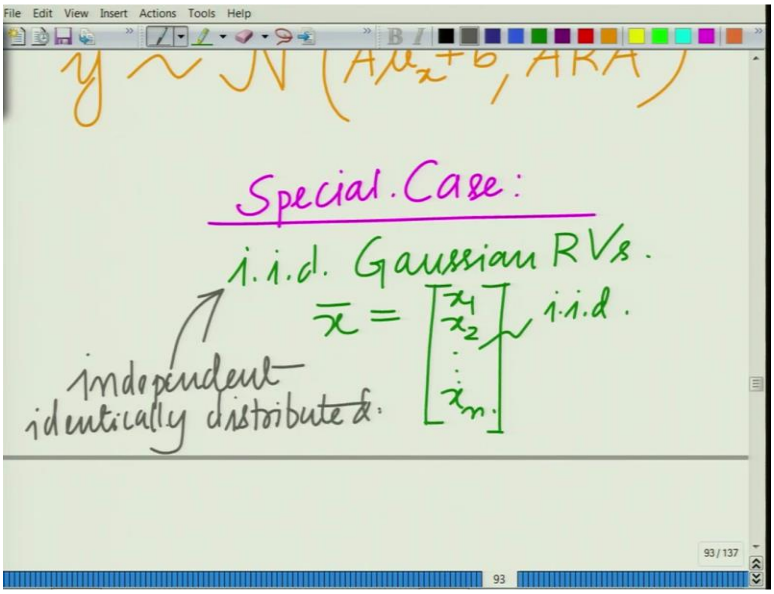

Let us look at a special case. Let us consider a special case. Let us consider a special case. So, let us consider a special case. What is a special case? Special case we want to consider is let us consider i.i.d Gaussian random variables that is x bar comprises of x1, x2, xn which are we already seen that independent identically distributed. You might remember what i.i.d means? Recall this means independent, independent. This means independent identically distributed. Very good.

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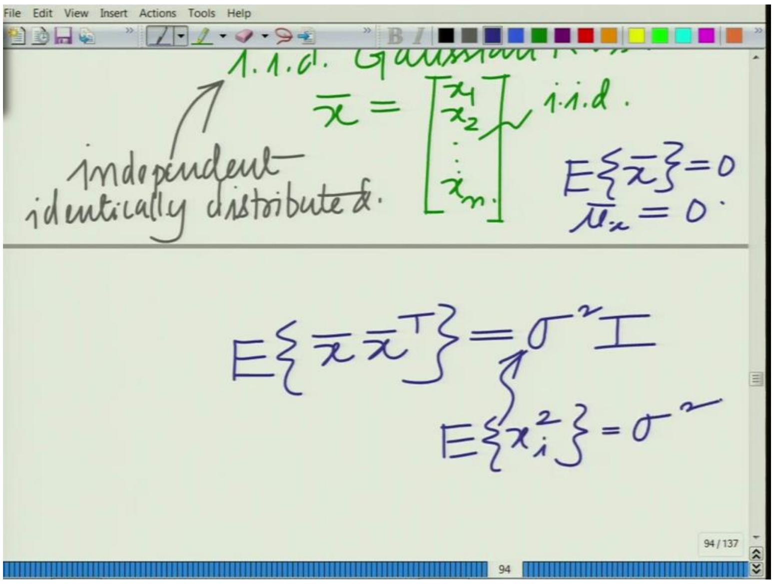

And let us also consider the mean 0; mu x bar equal to 0. Expected value of x bar equal to 0. That is essentially what that means is mu x bar equal to 0. And it is equally distributed. So, if you look at the covariance matrix that is expected value of x bar into x bar transpose, this is proportional to identity; sigma squared times identity. What is sigma square? Sigma square we have seen this is nothing but expected value of x i square equal to sigma square.

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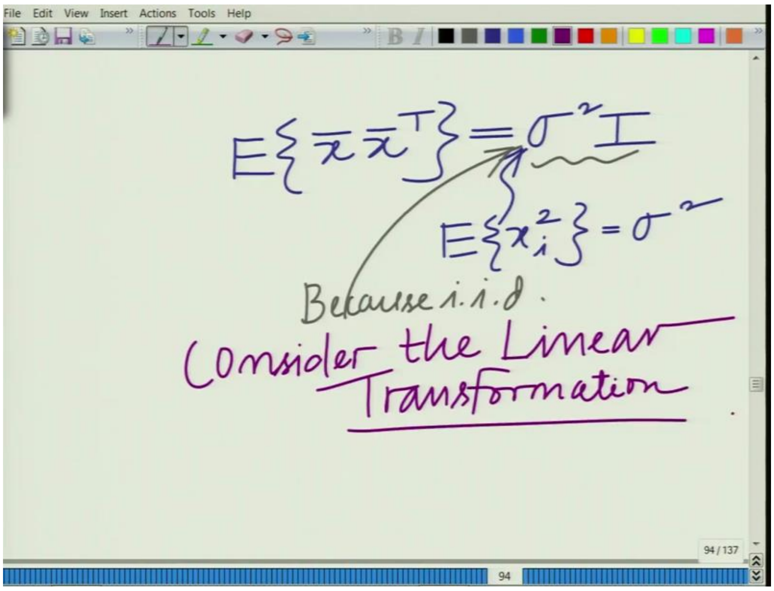

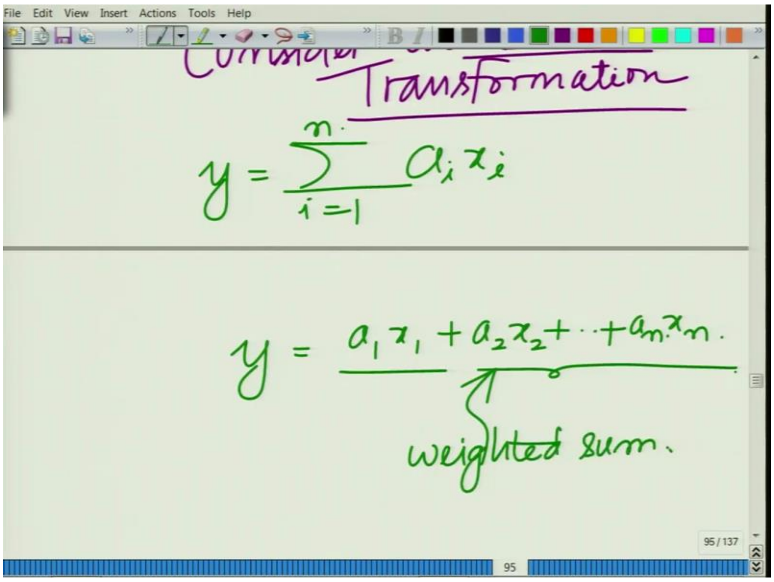

So, we start with this. So, the covariance matrix because these are i.i.d because i.i.d. Because these are i.i.d, the covariance matrix is proportional to identity, it is diagonal and in fact, it is also proportional to identity; some sigma square times identity where sigma square is the variance of each component of the vector x bar.

Now, let us consider the transformation. Consider the linear transformation; consider the linear transformation i equal to 1 to n ai Xi. So, we are forming a single; so y is a single scalar quantity. Y equals summation ai xi which is basically a1 x1, a2 x2 plus an xn. That is you are taking a linear, a weighted linear combination. This is what is called a weighted sum. And this arises in several contexts.

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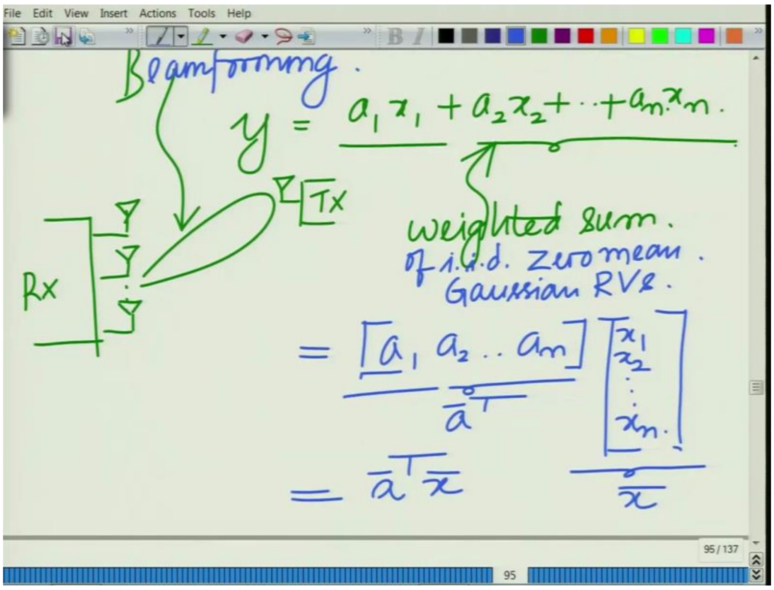

For instance, one of the very interesting contexts this arises in is what we call as beamforming in a wireless system. Probably we looked at this which is essentially you have the receiver of a wireless system; they form a beam in the direction of the transmitter. So, for instance, this is what is termed in wireless communication, this is what is termed as beamforming. This is an interesting and very important problem; this problem of beamforming.

So, this forms arises. So, we are forming a weighted sum of i.i.d Gaussian RVs of in fact i.i.d I would also add 0 mean Gaussian random variables. And this I can write this as we have written in summation ai xi. You can also write this as the row vector a1 a2 an times x1 x2 xn which if you look at this, this is a bar transpose. This is x bar. So, this is a bar transpose times x bar. So, this is a bar transpose x bar where a bar is the vector. a bar transpose is the row vector: a1 a2… an, x bar is of course, the column vector: x1 x2 xn.

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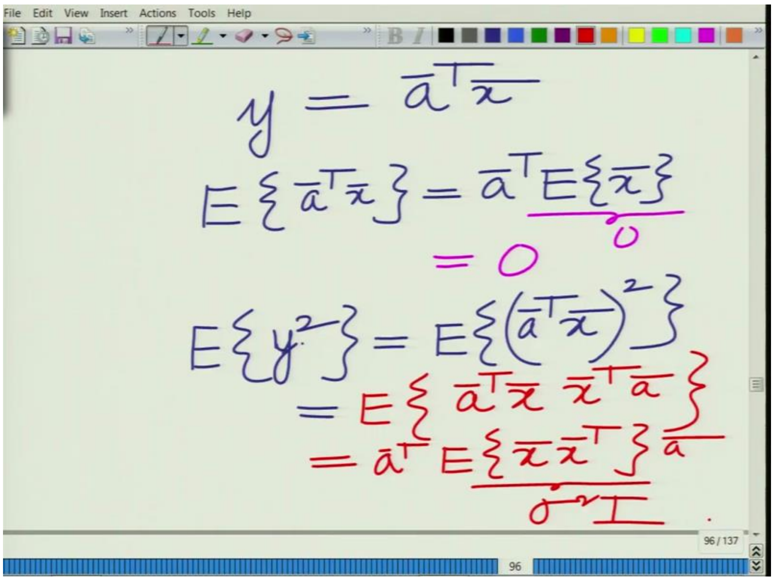

And so, y bar is basically we are saying is given by the linear transformation a bar transpose x bar. Therefore, if we look at now expected value of a bar transpose x bar that is a bar transpose expected value of x bar. But we have said these are 0 mean. So, expected value a bar transpose x bar is 0. Expected value of x bar is 0 so this is simply 0. So, y bar is, this is simply y not y bar. This scalar quantity, y equals a bar transpose x bar.

So, expected value y is 0 and expected value of y square that is the variance of y if you look at this that will be expected value of a bar transpose x bar whole square which I can write interestingly as the following thing. I can write this as expected value of a bar transpose x bar times x bar transpose a bar. Because it is a scalar quantity. Both a bar transpose x bar and x bar transpose a bar are the one and the same. Now, if you bring the a bar transpose outside, this becomes expected value of x bar x bar transpose times a bar. We know this is sigma square times identity.

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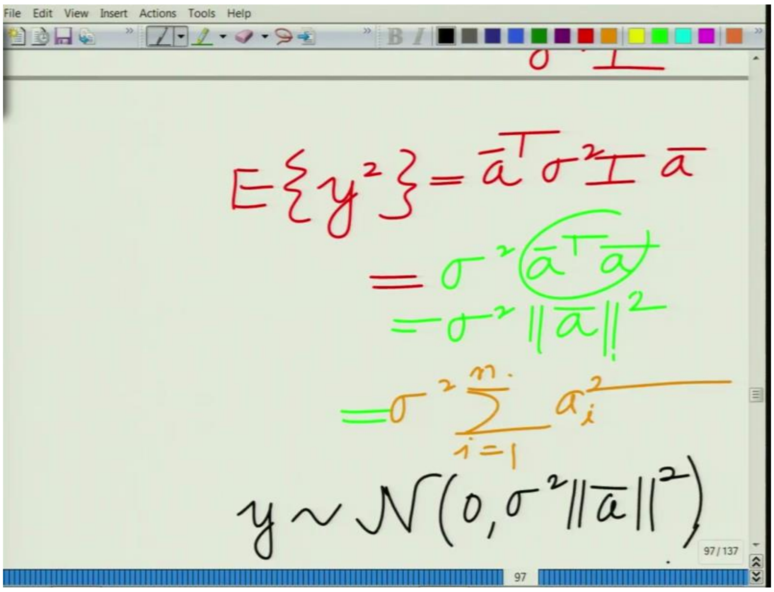

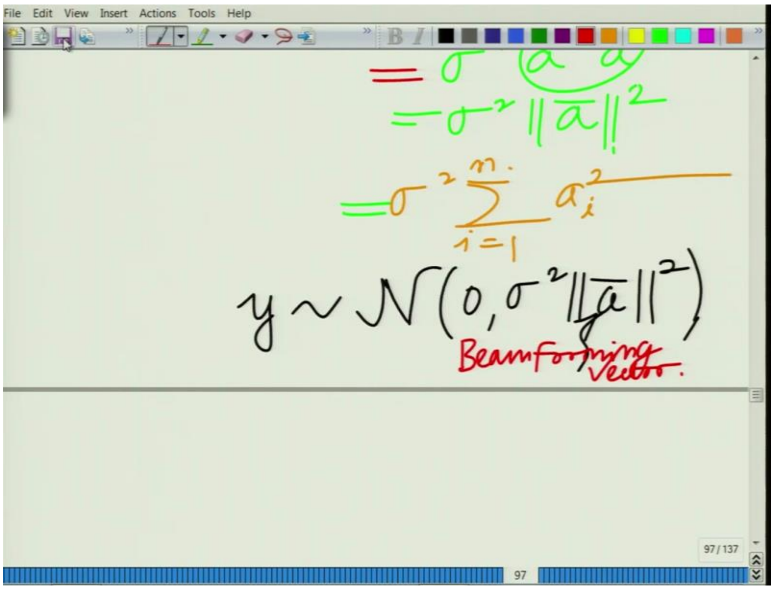

So, therefore, expected value of y square that is the variance this becomes a bar transpose sigma square identity times a bar which is essentially sigma square a bar transpose a bar which is sigma square nor a bar square which is essentially sigma square norm of because we have a bar transpose a bar. And more precisely, this is going to be sigma square summation i equal to 1 to n ai square. So, interestingly y is Gaussian with mean 0, variance sigma square times nor a bar square. And this has interesting applications as I said. This has an interesting application in beamforming.

So, basically this a bar you can think of this as the beamforming vector. So, we can think of a bar as essentially the beamforming vector. You have an antenna array. You are trying to form a beam in a particular direction by using this weighted sum. And what are the weights that you use? The weights that are used are these coefficients in a which is also can also be termed as the beam formula.

So, there are a lot of interesting implications of this. As I already told you linear system theory several times with very high frequency also involves the linear Gaussian random variables, Gaussian transfer Gaussian random vectors. In particular linear transformation of Gaussian random vectors, this arises in estimation; this arises detection; this arises in classification; this arises everywhere. So, this is a very very important application of linear algebra. So, let us stop here. Let us continue in the next module. Thank you very much.