Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
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Lecture No. 11
Gram-Schmidt orthogonalization

Hello, welcome to another module in this massive open online course. So, today let us continue our discussion with Gram-Schmidt procedure for orthogonalization or ortho-normalization.

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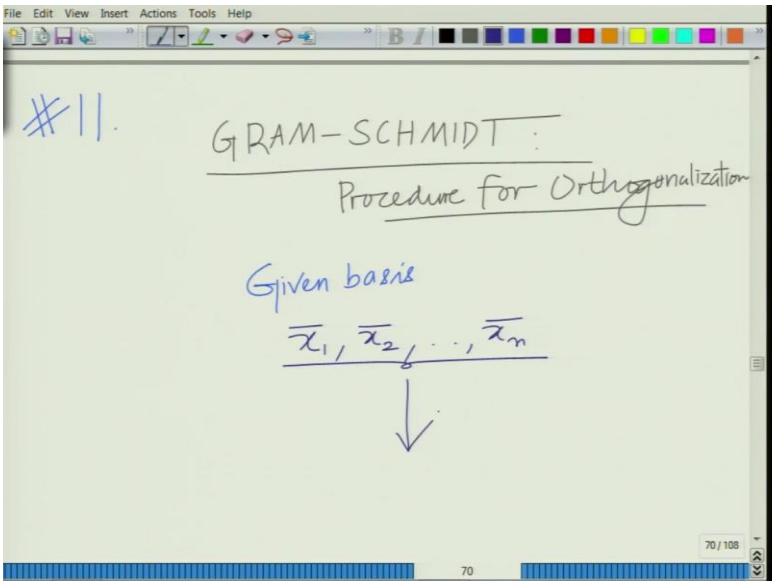

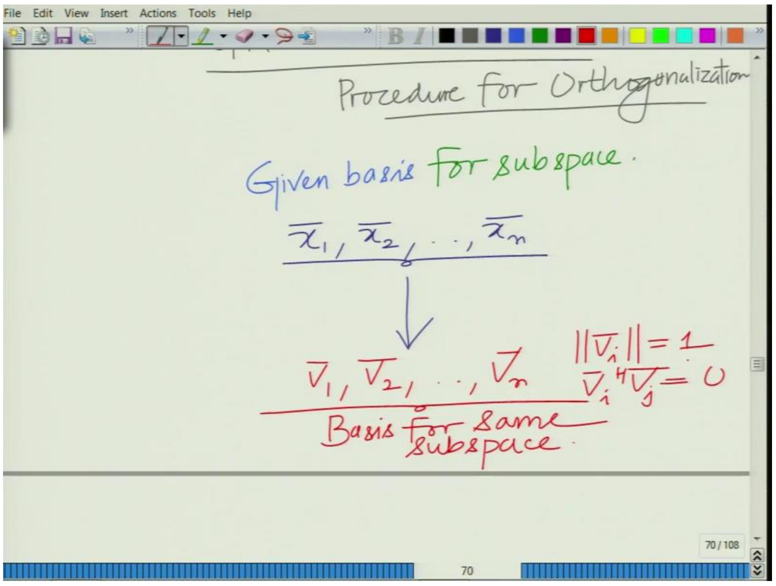

So, I am going to describe more about it so, let us talk about the Gram-Schmidt procedure. This is a procedure for orthogonalization the procedure for orthogonalization, what is the

Gram-Schmidt procedure. The procedure is basically follows let us say we have a basis for a subspace that is basically given by the vectors x1 bar x2 bar so on xn bar this is a basis.

Now, from this, this can be any basis now, from this we obtain for the same sparse subspace given basis for subspace let us say given basis for a subspace. Now, for the same subspace we obtain another basis v1 bar v2 bar vn bar such that now, this is a basis for the same subspace such that, nor vi bar equal to 1 that is each vector has unit norm and vi bar Hermitian vj bar equal to 0 that is a vectors are orthogonal so, the vectors are orthonormal.

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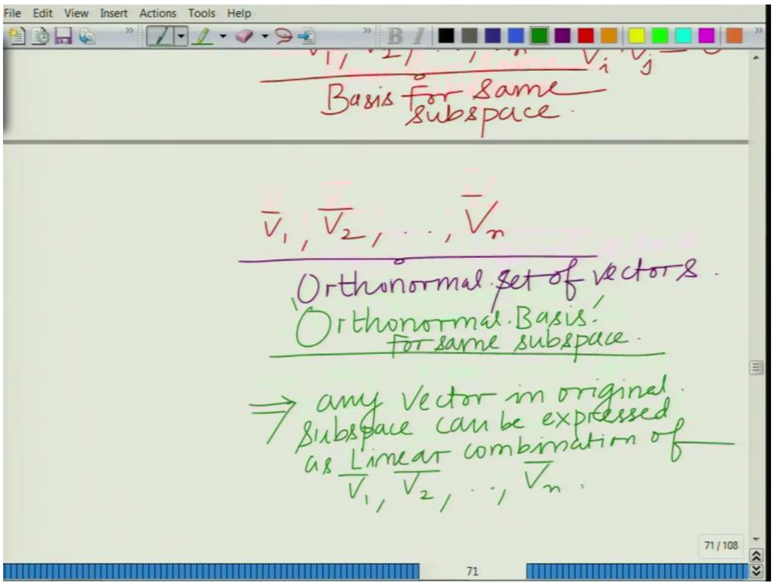

So, we say v1 bar v2 bar so on vn bar these are orthonormal set of vectors orthogonal and unit norm. So, this is an orthonormal set of vectors and more importantly this basis is an ortho normal basis so, what do we mean by an orthonormal basis? It is a basis for the same subspace as x1 bar x2 bar xn bar that is essentially the subspace you might recall is basically the span of all these vectors so, it is spanned by all these vectors essentially it is formed by the linear combinations of x1 bar x2 bar xn bar.

So, what we have said so, what the Gram-Schmidt procedure gives you is another equivalent basis v1 bar v2 bar vn bar but, with the property that, these vectors are have unit norm and they are orthogonal to each other that is it is an orthonormal basis and spans the same subspace that is any vector in the original subspace can be expressed as a linear combination of v1 or v2 bar. So, orthonormal basis for same subspace for the same this implies original vector original subspace can be expressed as linear combination of v1 bar v2 bar vn bar.

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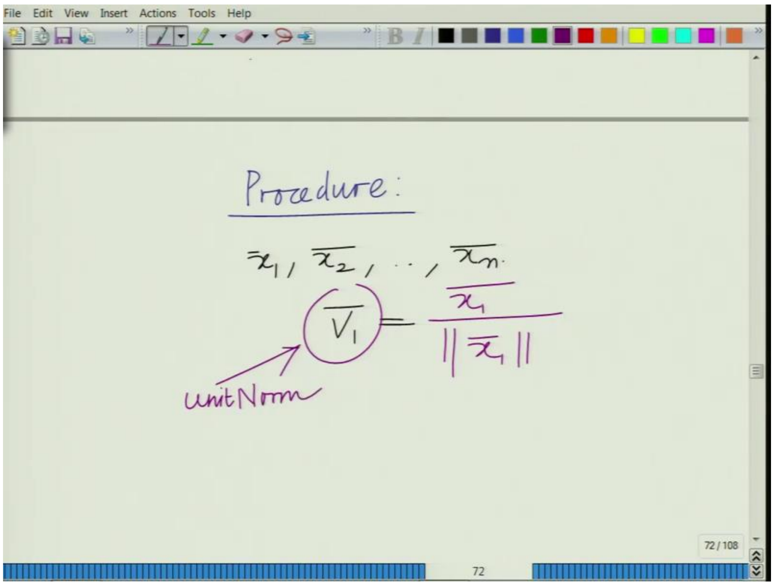

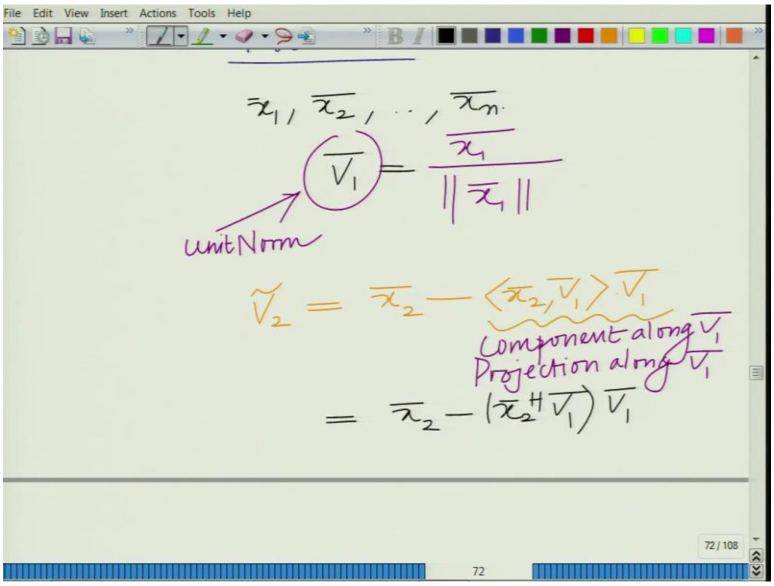

So, how do we obtain this orthonormal subspace what is the procedure, the Gram-Schmidt procedure is as follows we start with so, we are given x1 bar x2 bar up to xn bar, we start with v1 bar equals x1 bar divided by norm of x1 bar so, this is the first one you can see v1 bar now, this v1 bar you can see this will be unit norm because, you are taking x1 bar dividing it by norm of x1 bar so, you can take so, you can clearly see this will be unit norm.

Now, the second vector has to be now, v2 bar now, first we will form this vector v2 tilde v2 tilde is formed by taking x2 bar and subtracting from x2 bar the component along v1 bar this is important 2x bar which is given by the inner product x2 bar v1 bar times v1 bar this is essentially what we are calling as the component or projection along v1 bar, we will mark or the projection along v1 bar which is equal to x2 bar minus x2 bar, this is essentially nothing but, x2 bar hermitian v1 bar times v1 bar this is the projection of x2 bar along the v1 bar.

So, this is essentially the so you are removing the component and therefore, what you can show so, what you are doing is from v2 tilde from x2 bar, you are removing the component that is along v1 bar so, what is remaining will be perpendicular or orthogonal to v1 bar.

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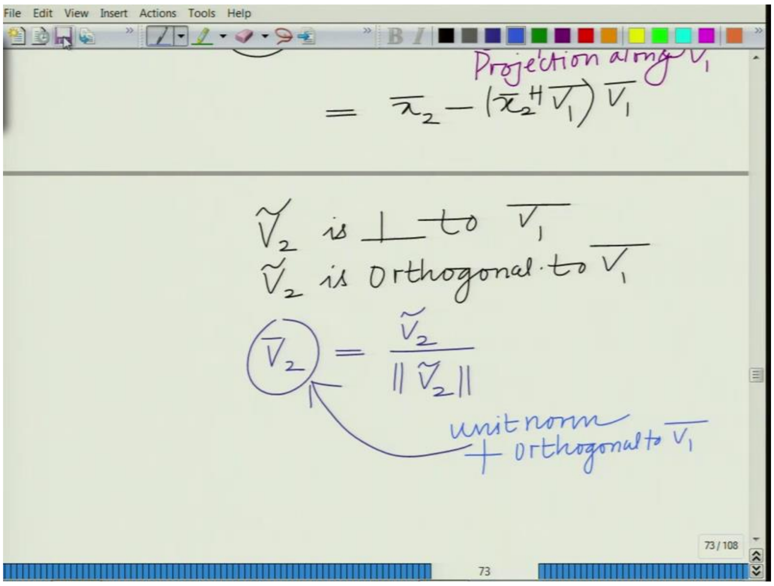

So, v2 tilde you can verify v2 tilde is perpendicular to v1 bar or v2 tilde is essentially orthogonal to v1 bar v2 tilde is orthogonal to v1 bar. Therefore, now so, you have v2 tilde is orthogonal to v1 bar, now, we need the unit norm v2 tilde. So, unit norm v2 tilde now, you get the v2 bar is simply taking v2 tilde and dividing this by the norm of v2 tilde so, that will make way to tilde unit norm as well as orthogonal to v1 bar plus because, v2 tilde is all already orthogonal to v1 bar, v2 bar will also be orthogonal to.

So, at every step you are removing the projections are long the orthonormal basis already found which makes the resulting vector or orthogonal to the previously determined orthonormal vectors and then finally you divided by the norm to make it unit norm.

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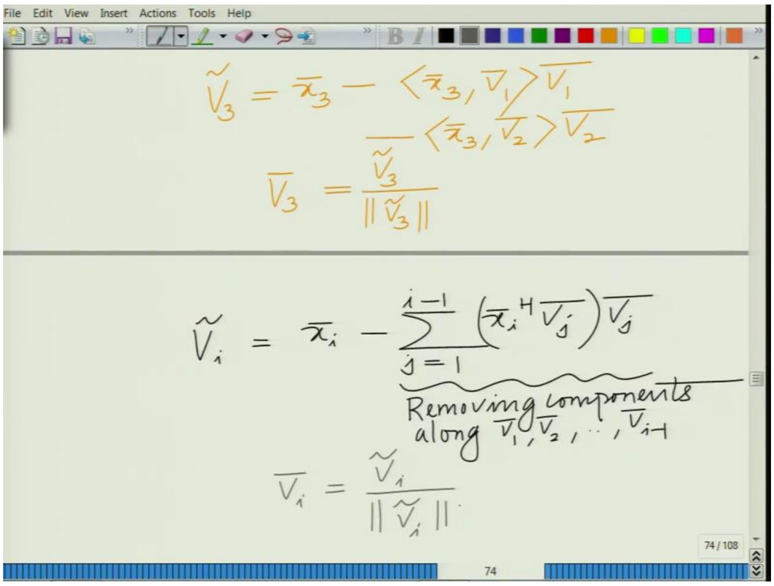

So similarly, we will have again going through the same procedure you will have v3 tilde this is equal to x3 bar minus the inner product of x3 bar comma v1 bar with v1 bar minus x3 bar comma v2 bar with v2 bar and therefore, your v3 bar will be equal to v3 tilde divided by the norm of v3, v3 tilde divided by the norm of v3 tilde.

And similarly we will have vi tilde is equal to xi bar minus summation j equal to 1 to i minus 1 xi bar hermitian vj bar times vj bar xi bar hermitian vj bar minus vj bar so, you are removing the projections along or removing components along v1 bar v2 bar v bar i minus 1 and finally, we have vi bar equals vi tilde divided by norm vi tilde so, this essentially eventually so, this eventually results in the orthonorm.

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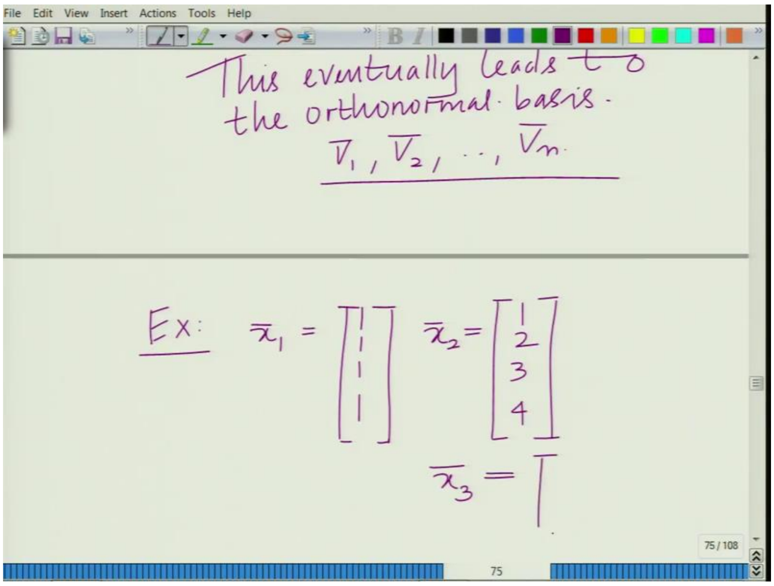

So, this leads to the orthonormal basis v1 bar v2 bar this eventually leads to the orthonormal basis v1 bar v2 bar vn bar. So, essentially at every stage you are removing the projections along the previously determined orthonormal vectors and then finally, you are dividing it by its own norm to make it so, this makes it orthogonal to the previously determined orthonormal vectors and then finally you divided by its norm to make it unit norm.

Let us take a simple example to understand this better so, let us take a simple example, we have x1 bar equals 1 1 1 1 and x2 bar equals 1 2 3 4.

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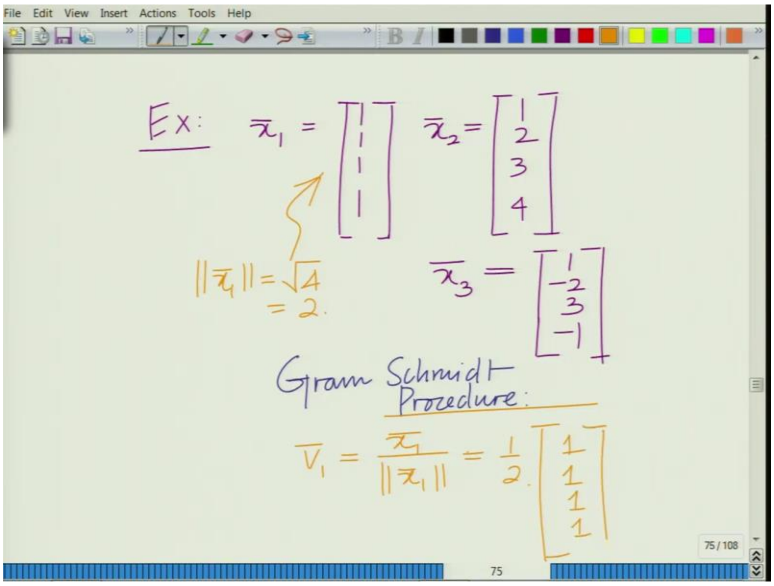

And x3 bar equals 1 minus 2 3 minus 1 and therefore, we have now to start the Gram-Schmidt procedure. Now, we want to do the Gram-Schmidt procedure. Now therefore, we have, remember the first one is very simple x v1 bar equals simply take x1 bar divided by its norm to make unit norm we have x1 bar equal to 1 1 1 1 so, norm of x1 bar if you look at this is equal to square root of 4 equal to 2. So, v1 bar will be 1 over 2 times the vector x1 bar which is essentially this is your v1 bar, what is your v2 tilde?

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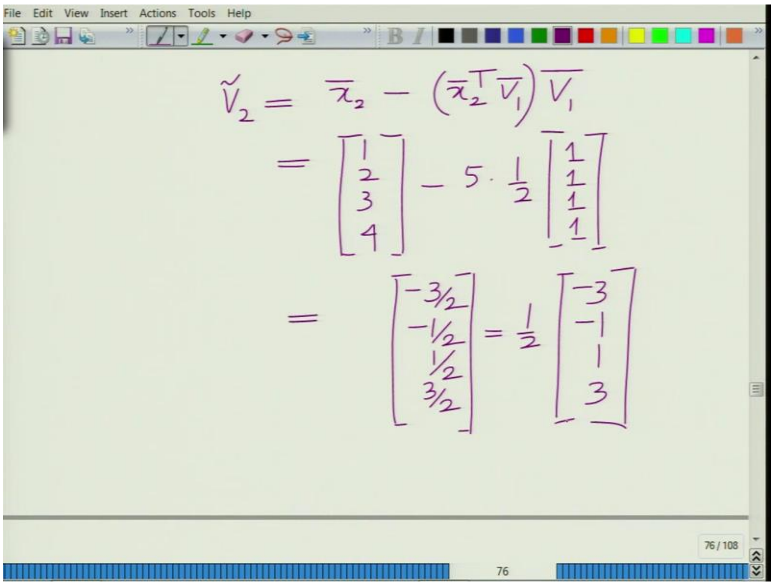

Now, let us form the v2 tilde equals remember, you have your x2 bars since, these are real vectors I am going to simply write x2 bar transpose v1 bar times v1 bar which is nothing but, x2 bar is 1 2 3 4 minus this will be x2 bar transpose v1 bar so, that will be 7 2 9 5 times v1 bar which is half 1 1 1 1 so, if you look at this, this is going to be basically 1 2 3 4 this and this you can do you can check this is going to be minus 3 by 2 minus half, half, 3 by 2 which is basically if you look at it half times minus 3 minus 1 1 comma 3 so, this is the vector and now, you obtain v2 bar by taking v2 tilde dividing it by its norm so, take v2 tilde divided by its norm.

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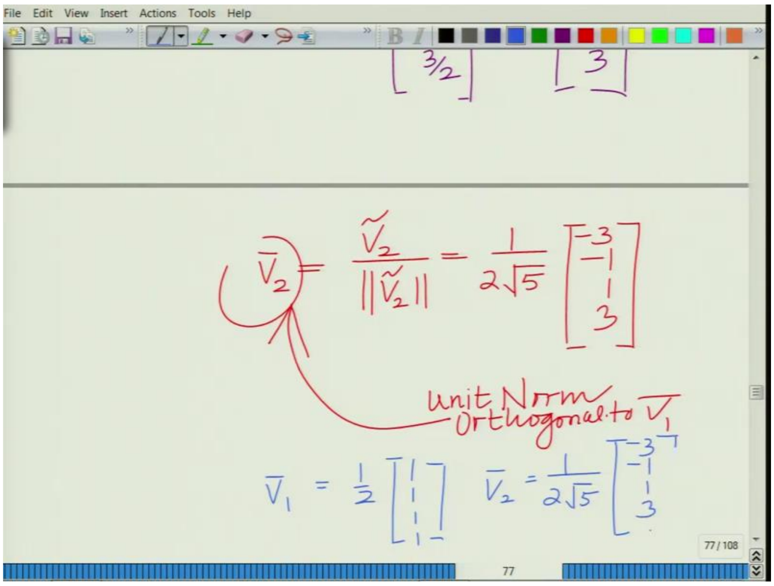

So, this will be v2 bar which is equal to v2 tilde divided by its norm, this will be v2 tilde that is correct so, this is going to be 1 divided by and the norm of this it is not very difficult to see the norm of this if you divided by its norm we will obtain 1 over twice square root of 5 times minus 3 minus 1 1 comma 3 so, this is your v2 tilde and this is essentially unit norm. Now, let us recall the property, this is basically unit norm and orthogonal to v1 bar you can quickly check it you can see for instance v1 bar equals half you have 1 1 1 1 v2 bar equals 1 over twice square root of 5 minus 3 minus 1 1 comma 3.

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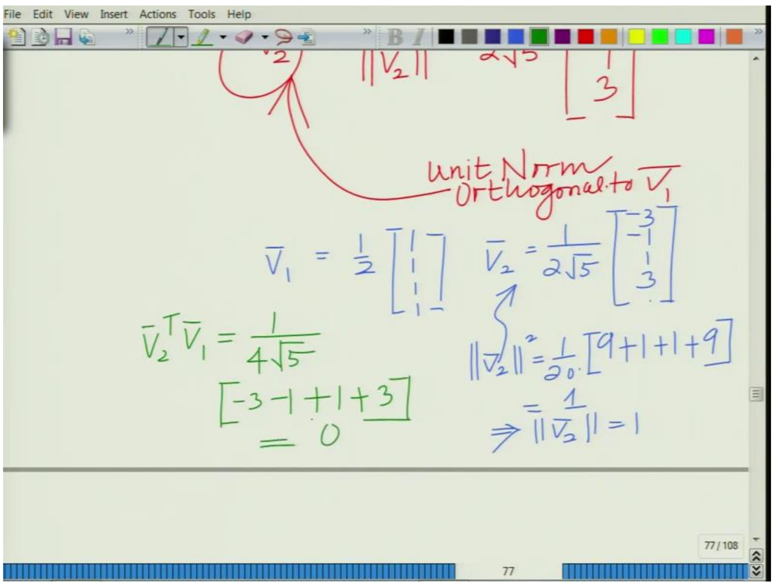

So, you can first see norm v2 bar square equals 1 or 20 times 1 times 9 plus 1 plus 1 plus 9 equals essentially 1 which implies norm v2 bar square equal to 1 plus v2 bar transpose v1 bar or v2 bar hermitian v1 bar this will be equal to you can see this is equal to well you can write the constants but, it does not matter 1 over 4 square root of 5 times minus 3 minus 1 plus 1 plus 3 which is equal to 0.

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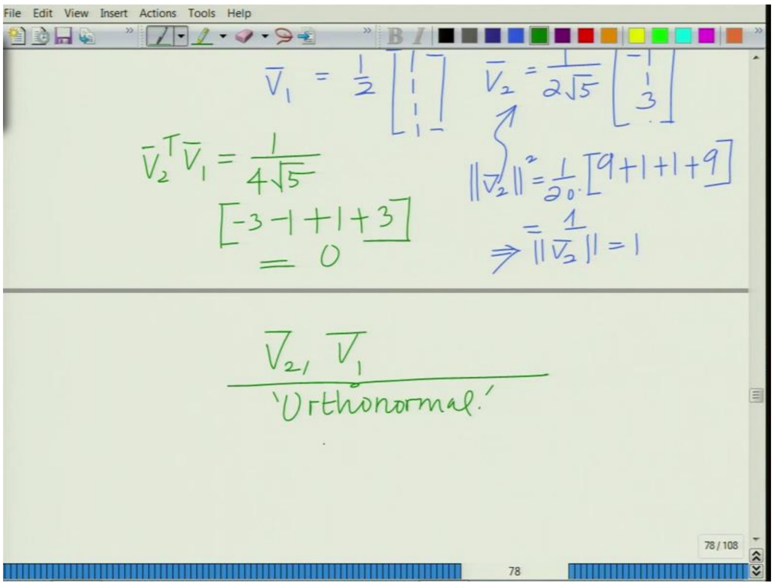

So, v2 bar v1 bar so, you can see v2 bar v1 bar are orthonormal because, they are orthogonal to each other and the both of them have unit norm. Now, let us again find v3 bar again the same procedure take x3 bar remove the components along v1 bar v2 bar and divided them you get v3 tilde divided by its norm to get v3 bar.

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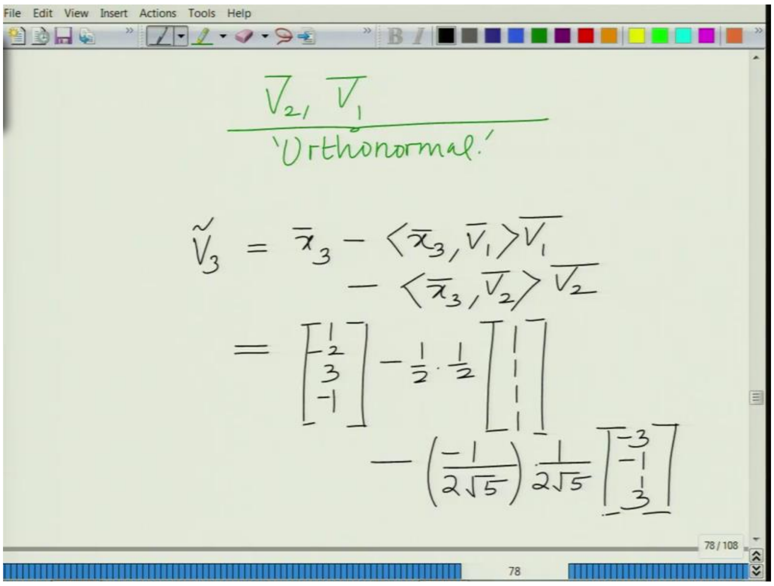

So, the procedure is again very so, you have v3 tilde which is basically you start with x3 bar minus you take x3 bar comma v2 bar or v1 bar minus x3 bar v2 bar which is basically now, you take 1 minus 2 3 minus 1 minus half into half 1 1 1 1 minus or this will rather be you can check minus or I am going to write what is this, this projection is going to be minus 1 over twice square root of 5 1 over twice square root of 5 times minus 3 minus 1 1 3.

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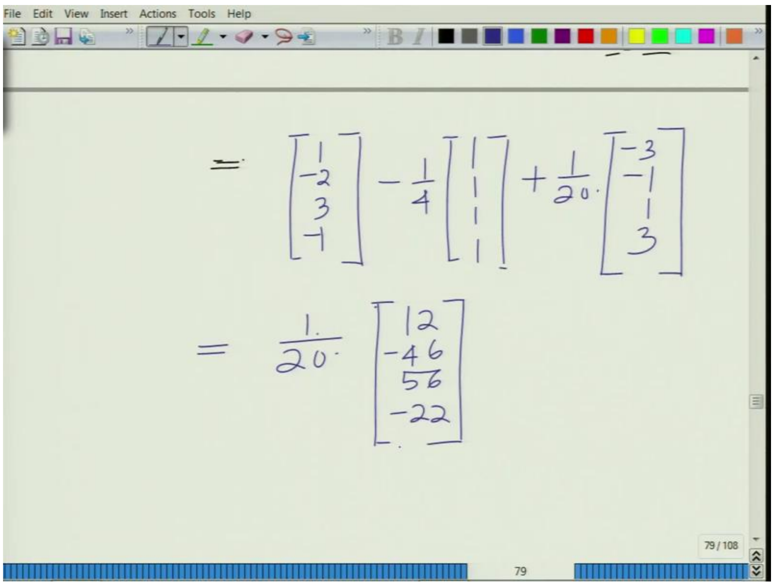

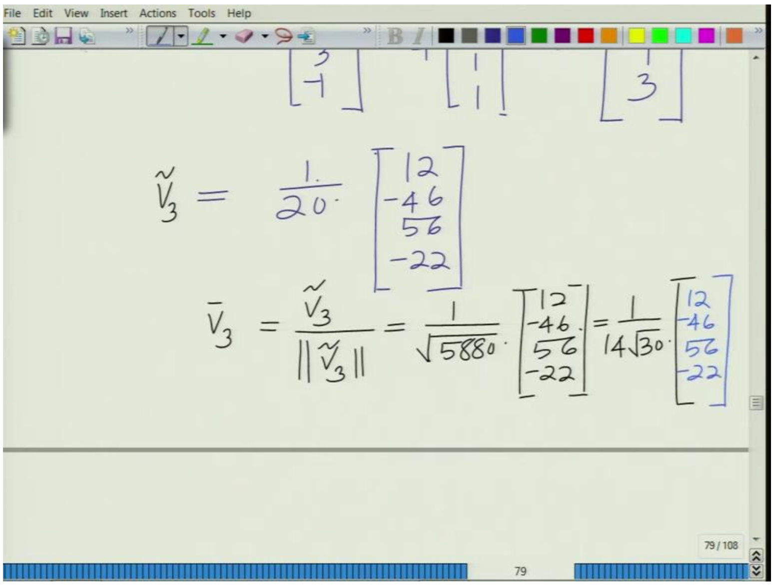

Which you can write essentially now, as 1 minus 2 3 minus 1 minus 1 over 4 1 1 1 1 plus 1 over 20 minus 3 minus 1 1 comma 3 and then you can also write this as the following, you can write this as 1 over 20 and you can simplify this it is not very difficult to see this will be 1 over 20 12 minus 46 56 minus 22 and then, this is your v3 bar or this is your v3 tilde and now, v3 bar which is equal v3 tilde divided by the norm of v3 tilde, which is equal to essentially 1 over you can calculate the square root of 5880 times 12 minus 46 56 minus 22 equals 1 over 14 square root of 30 times 12 minus 46 56 minus 22. So, this is essentially what you have so, that is essentially that brings us so, essentially now, if you look at it, we have the original set of vectors x1 bar x2 bar x3 bar.

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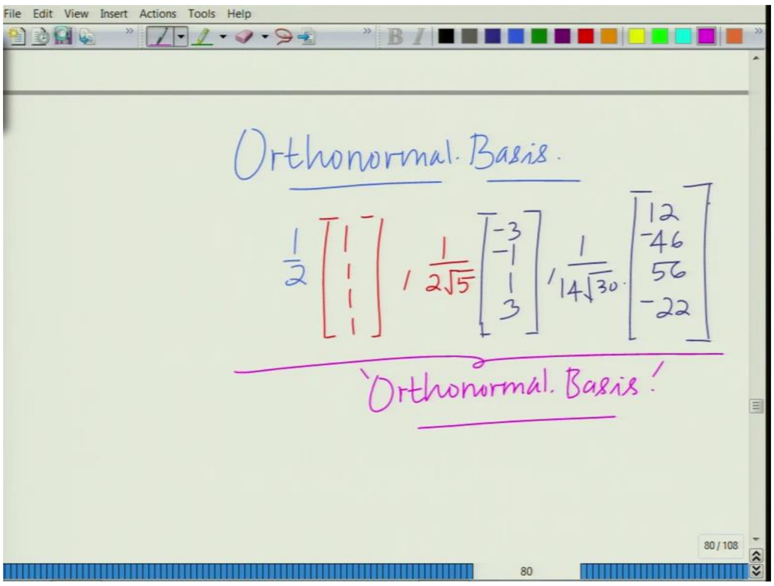

Now, after the Gram-Schmidt procedure, the orthonormal basis is given as half 1 1 1 1 comma 1 over 2 square root of 5 minus 3 minus 1, 1 comma 3 and 1 over 14 square root of 30 12, minus 46, let me just write it here, 12 minus 46 56 minus 22 so, this is essentially you are as we have already said, this is a orthonormal basis that is each vector in this basis as unit norm and the vectors are orthogonal to each other.

So, the Gram-Schmidt procedures has helped us derive this orthonormal basis starting with the basis that is a general basis that is x1 bar x2 bar x3 bar. So, let us stop this module over here and we will continue in the subsequent modules. Thank you very much.