Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
Professor. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur
Lecture No. 10
Null Space of Matrix: Definition, Rank Nullity Theorem, Application in Electric Circuits

Hello, welcome to another module in this massive open online course. So, today let us discuss a new concept, that is the Null space of a matrix.

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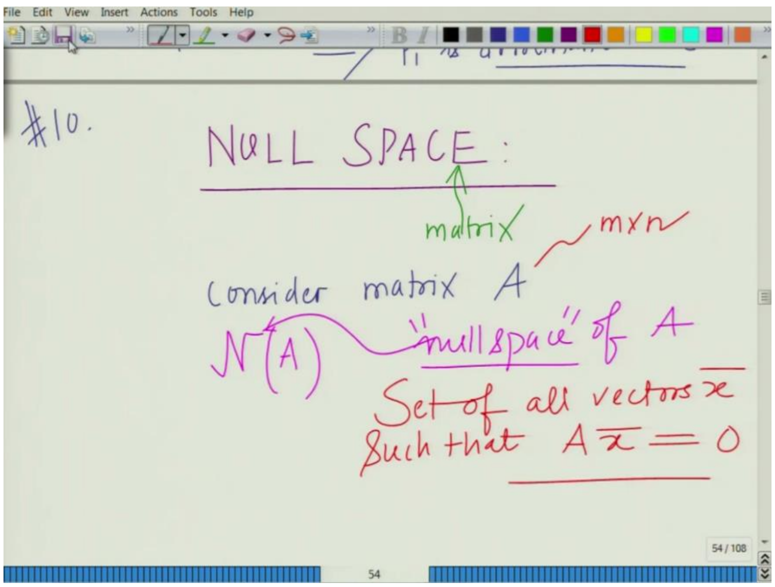

The Null space of a matrix, which is defined for a matrix A which is of size $m \times n$, then the null space of this matrix which is denoted by $\mathcal{N}(\mathbf{A})$. So, this is your $\mathcal{N}(\mathbf{A})$ that is set of all vectors $\bar{\mathbf{x}}$, such that, $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$, that is the null space, that is the set of all vectors $\bar{\mathbf{x}}$, such that $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$.

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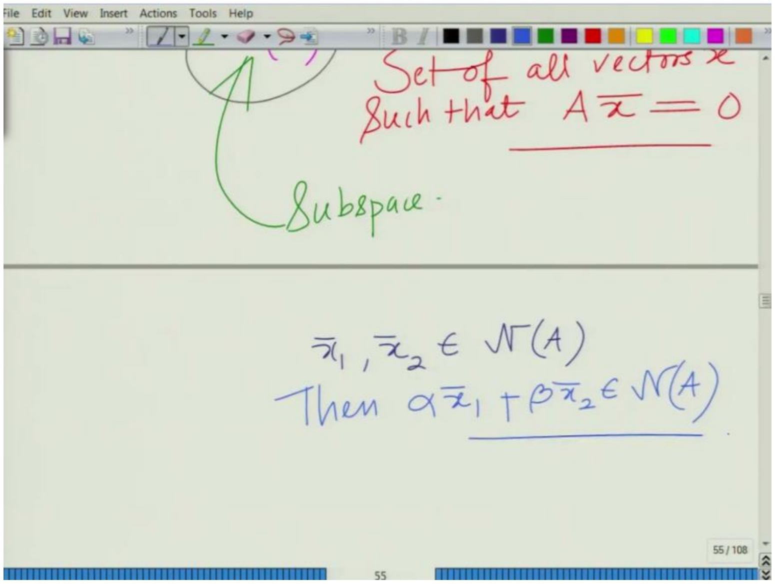

Now, remember that the terminology is justified because the null space is, in fact, it is a subspace. So, the null space is a subspace. What do we mean by a subspace? That is if we consider any 2 vectors $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2$ belonging to the null space, then their linear combination $\alpha\bar{\mathbf{x}}_1 + \beta\, \bar{\mathbf{x}}_2$, also belongs to the null space, this is not very difficult to see.

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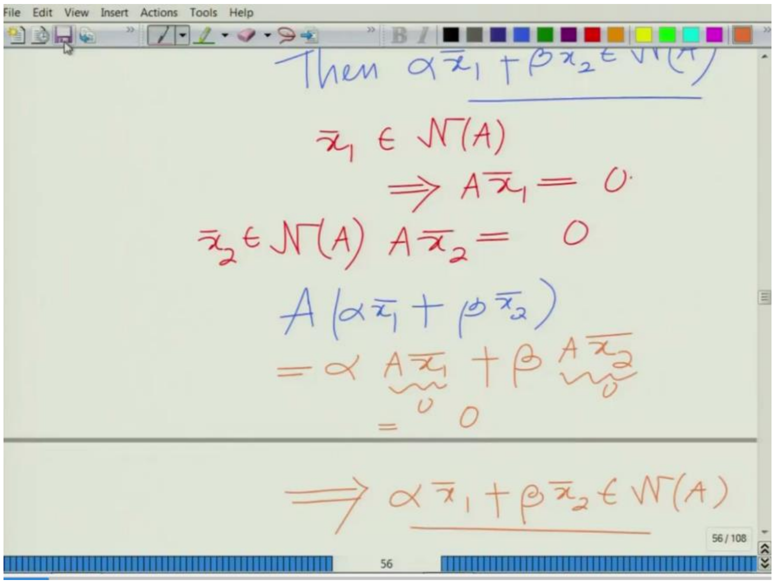

Let us take us for instance

$$\bar{\mathbf{x}}_1 \in \mathcal{N}(\boldsymbol{A}) \Rightarrow \mathbf{A}\bar{\mathbf{x}}_1 = \mathbf{0}\ .$$

$$\text{Similarly, } \bar{\mathbf{x}}_2 \in \mathcal{N}(\boldsymbol{A}) \Rightarrow \mathbf{A}\bar{\mathbf{x}}_2 = \mathbf{0}.$$

Now, $\mathbf{A}(\alpha\bar{\mathbf{x}}_1 + \beta\bar{\boldsymbol{x}}_2)$, naturally it is not very difficult to see this equal

$$\mathbf{A}(\alpha\bar{\mathbf{x}}_1 + \beta\bar{\boldsymbol{x}}_2) = \alpha\mathbf{A}\bar{\mathbf{x}}_1 + \beta\mathbf{A}\bar{\mathbf{x}}_2 = \mathbf{0} \Rightarrow \alpha\bar{\mathbf{x}}_1 + \beta\bar{\mathbf{x}}_2 \in \mathcal{N}(\boldsymbol{A}),$$

that is why it makes it a subspace, remember, what is a subspace? Subspace is nothing but whenever 2 vectors belong to the subspace their linear combination also must belong to the subspace. We call that such a set, that is, as I just said is basically a subspace. So, the null space of matrix **A** is in fact a subspace.

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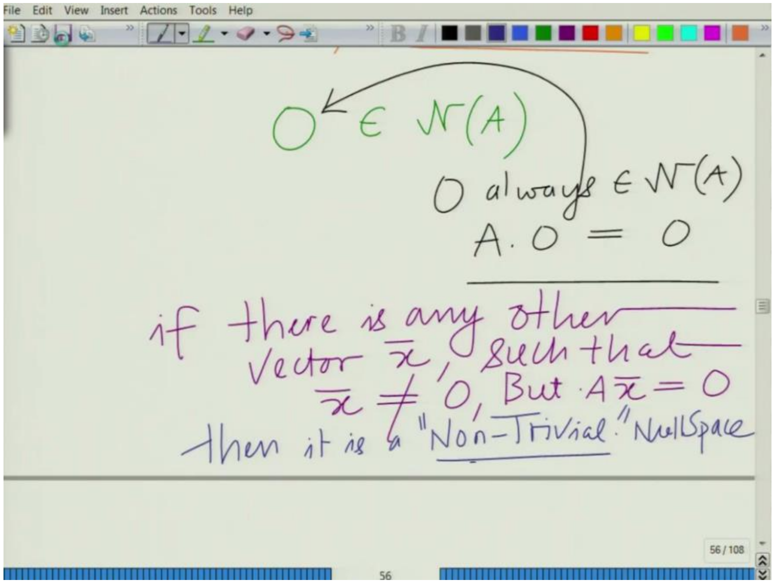

Now, the interesting thing that you observe over here is that **0** always belongs to the null space of any vector. So, **0** always belongs to the null space of any matrix that is easy to see because

$$\mathbf{A0} = \mathbf{0}.$$

Now, if there is any other vector $\bar{\mathbf{x}}$, such that, $\bar{\mathbf{x}}$ not equal to $\mathbf{0}$ but $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$ then we call it a non-trivial Null Space. Then it is frequently known as non-trivial, non-trivial means non obvious, because **0** is the element that belongs to the null space of any matrix.

Now, if there is even a single vector in the null space other than the **0** vector, then it is known as a non-trivial Null Space. Now, obviously, the null space cannot contain a single vector because if

$\bar{\mathbf{x}}$ belongs to the null space, then $\alpha\bar{\mathbf{x}}$**,** also belongs to the Null space. So, it either contains the trivial null space that is $\mathbf{0}$ which contains only a single element, whereas the non-trivial null space always contains an infinite number of vectors.

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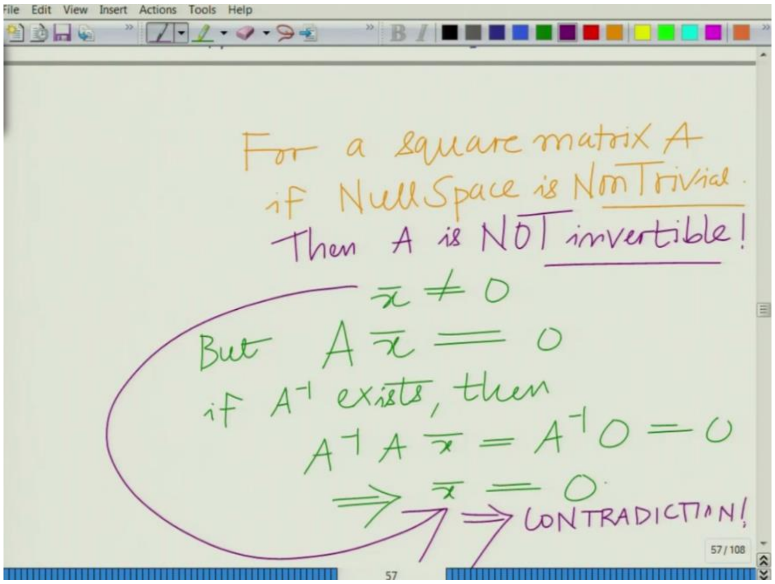

Another interesting thing, an interesting result for a square matrix $\mathbf{A}$ if the null space is non-trivial, then $\mathbf{A}$ is not invertible. It is very easy to see. For instance, if there exists

$$\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}, \text{ where } \bar{\mathbf{x}} \neq \mathbf{0}.$$

Now, if $\mathbf{A}^{-1}$ exists then we must have multiplying both sides by $\mathbf{A}^{-1}$, we must have

$$\mathbf{A}^{-1}\mathbf{A}\bar{\mathbf{x}} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0} \Rightarrow \bar{\mathbf{x}} = \mathbf{0}.$$

But we started with the assumption $\bar{\mathbf{x}} \neq \mathbf{0}$. So, if you go back and look at the assumption $\bar{\mathbf{x}} \neq \mathbf{0}$ we are able to show that this is basically a contradiction. Therefore, $\bar{\mathbf{x}} \neq \mathbf{0}$ this implies $\mathbf{A}^{-1}$ does not exist.

If there is a contradiction arising because we are assuming $\mathbf{A}^{-1}$ exists this implies $\mathbf{A}^{-1}$ does not exist. So, if a square matrix has a non-trivial null space, that is, there is an $\mathbf{A}\bar{\mathbf{x}} \neq \mathbf{0}$ but $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$ then the square matrix is not invertible. The other condition for singular that is $\mathbf{A}$ is non invertible or $\mathbf{A}$ is basically singular the other condition we saw is basically determinant.

So, these are all equal conditions and we can show that if the determinant is 0, then basically there also exists a non-trivial null space. All these are another way to say that is a singular matrix. If it is an $n \times n$ square matrix then the $rank(\mathbf{A}) < n$, means it is a rank deficient matrix. So, all these are different properties of basically a singular matrix.

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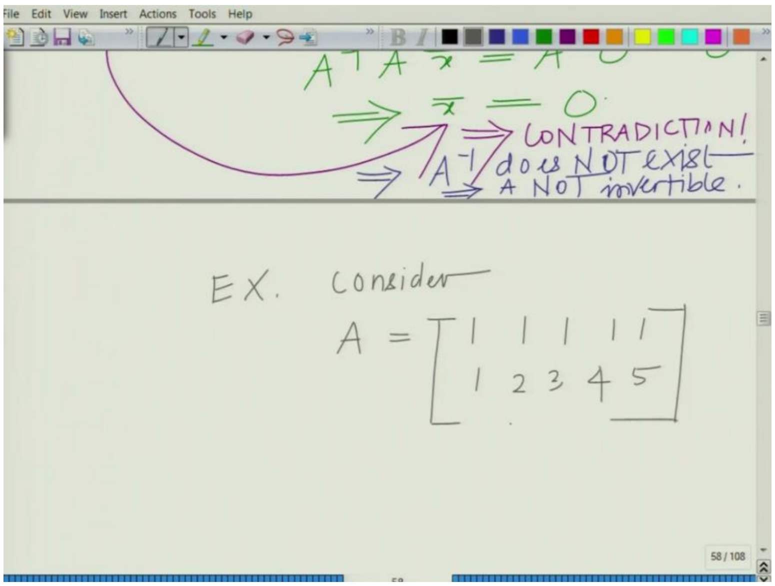

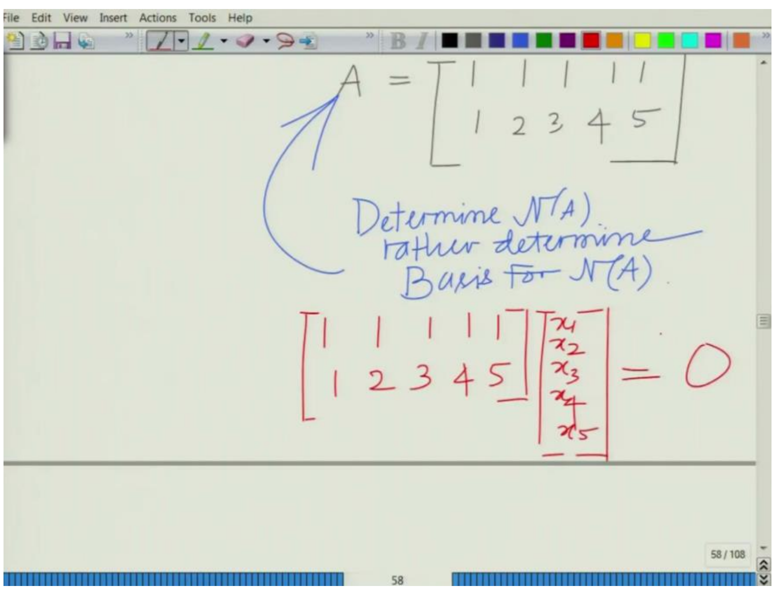

Now, let us look at a simple example to determine the null space. consider **A** equals

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

Now, we want to determine the $\mathcal{N}(\mathbf{A})$ or rather determine a basis for $\mathcal{N}(\mathbf{A})$. So, any vector that belongs to the null space, we must have

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \mathbf{0}.$$

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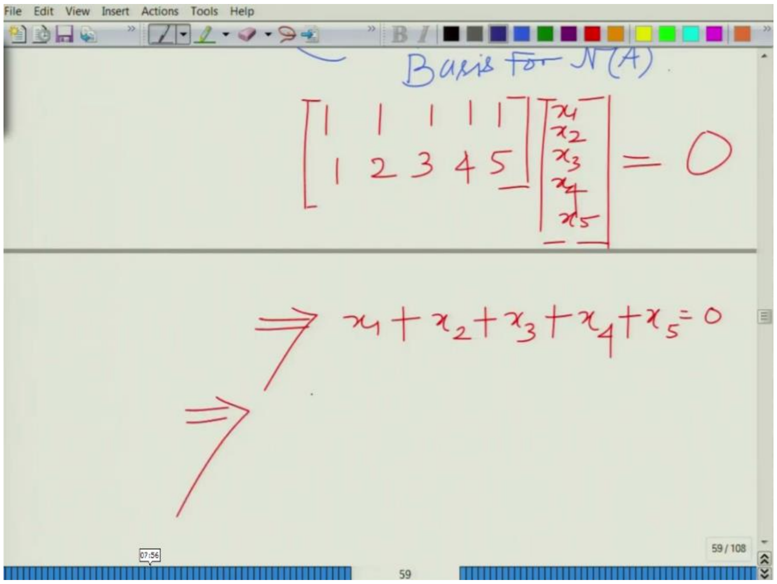

Therefore, now, let us expand the first condition this implies that

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0.$$

Let us first start with the row reduced form then it becomes more convenient.

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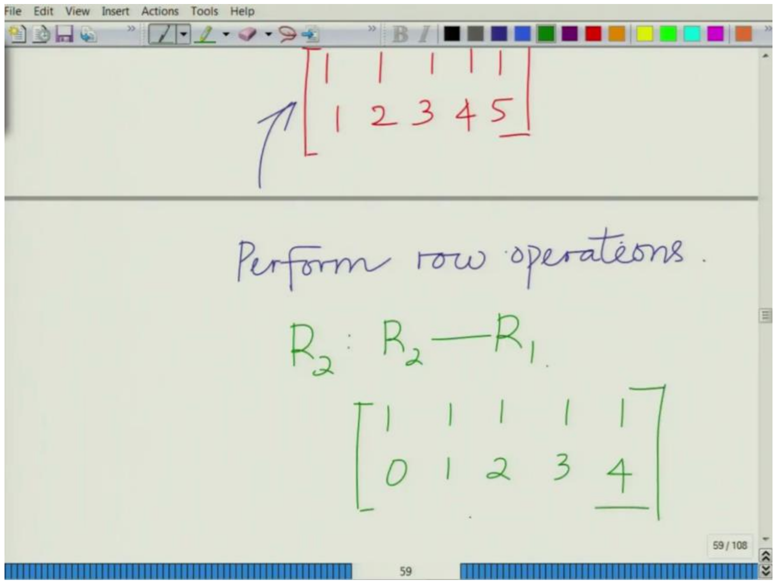

Now do row operations, let us perform row operations first, let us replace $R_2 \leftarrow R_2 - R_1$, then we have the matrix equivalently becomes

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

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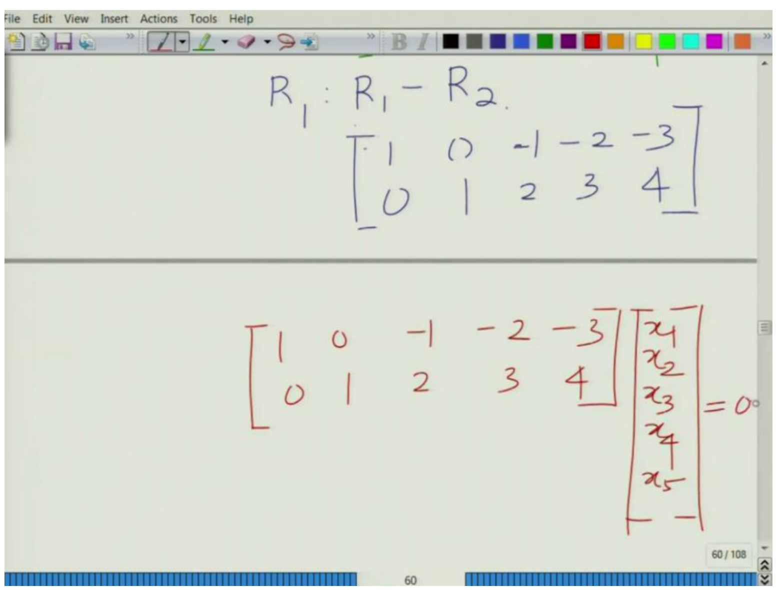

Now, let us do another row operation, let us do $R_1 \leftarrow R_1 - R_2$. Now, the matrix becomes

$$\begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

Now, we are in business now, let us find the null space of this Row Reduced matrix. So, of course, you can see there are 2 pivots, so, basically the $rank(\mathbf{A}) = 2$, that is not very difficult to see. Now, let us determine the null space. So, you have

$$\begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \mathbf{0}$$

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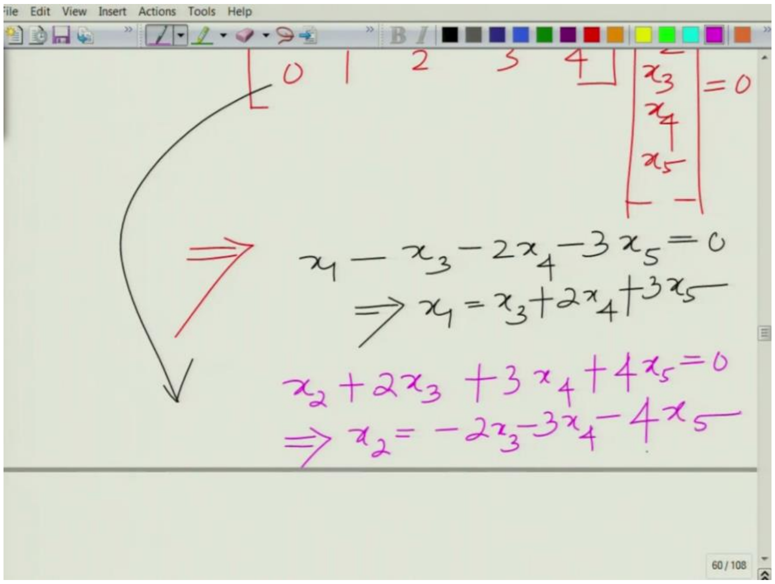

This implies now let us expand the conditions

$$x_1 - x_3 - 2x_4 - 3x_5 = 0$$

$$x_1 = x_3 + 2x_4 + 3x_5.$$

Let us write the second condition from the second row. Now, this implies that we must have

$$x_2 + 2x_3 + 3x_4 + 4x_5 = 0$$

$$x_2 = -2x_3 - 3x_4 - 4x_5.$$

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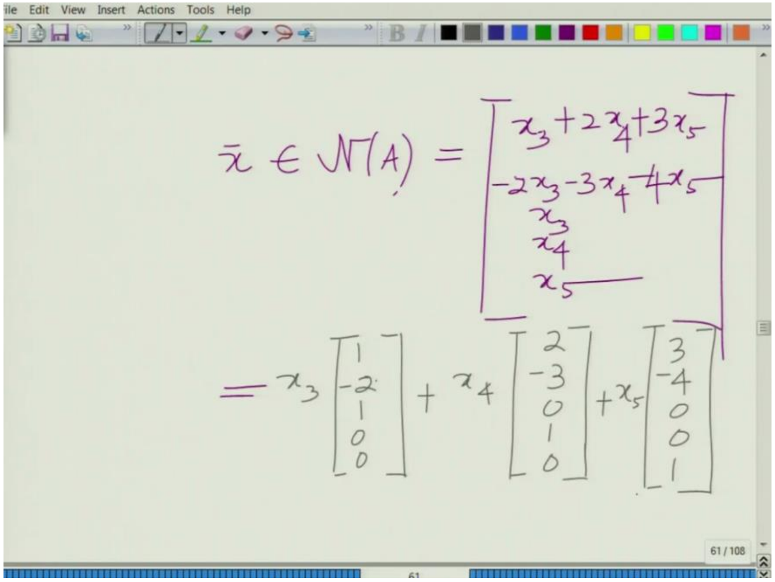

Therefore, the element $\bar{\mathbf{x}}$ is of the form, that is, any $\bar{\mathbf{x}} \in \mathcal{N}(\mathbf{A})$**,** the $\mathcal{N}(\mathbf{A})$ is of the form that you have the free ones $x_3, x_4,$ and x_5 and then you have $x_2 = -2x_3 - 3x_4 - 4x_5$, and then you have x_1 which is basically $x_1 = x_3 + 2x_4 + 3x_5$, and now, you can also write it interestingly as

$$\bar{\mathbf{x}} \in \mathcal{N}(\mathbf{A}) = \begin{bmatrix} x_3 + 2x_4 + 3x_5 \\ -2x_3 - 3x_4 - 4x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

So, any element of the null space is a linear combination of these 3 vectors.

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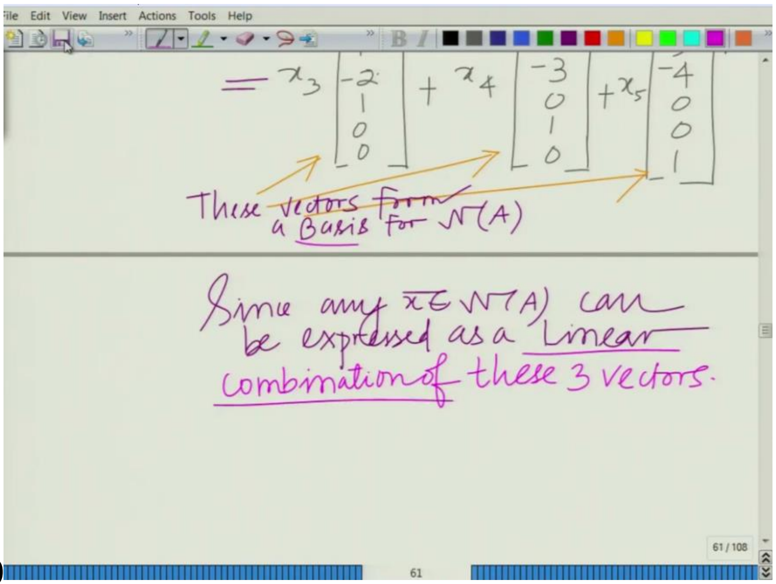

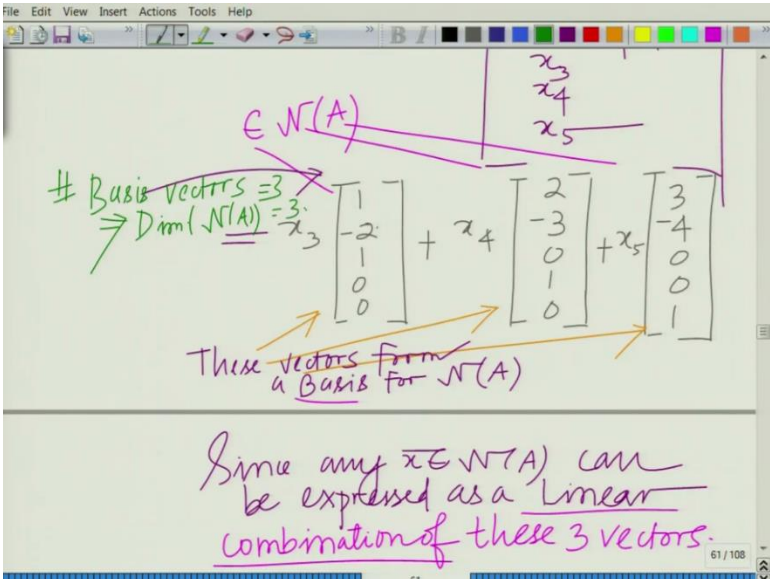

Now you can see very interestingly, you have $x_3, x_4,$ and x_5, these are the free variables and any vector belonging to the null space can be expressed as a linear combination that is

$$x_3\begin{bmatrix}1\\-2\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}2\\-3\\0\\1\\0\end{bmatrix}+x_5\begin{bmatrix}3\\-4\\0\\0\\1\end{bmatrix}.$$

So, basically, these vectors $\begin{bmatrix}1\\-2\\1\\0\\0\end{bmatrix},\begin{bmatrix}2\\-3\\0\\1\\0\end{bmatrix}$, and $\begin{bmatrix}3\\-4\\0\\0\\1\end{bmatrix}$ form a basis for the $\mathcal{N}(\mathbf{A})$. So, any vector $\bar{\mathbf{x}}$ belonging to null space can be expressed as a linear combination of these 3 vectors. So, these vectors form a basis and you can see being the basis, they themselves are, that is each of these also belongs naturally to the $\mathcal{N}(\mathbf{A})$, you can quickly check that.

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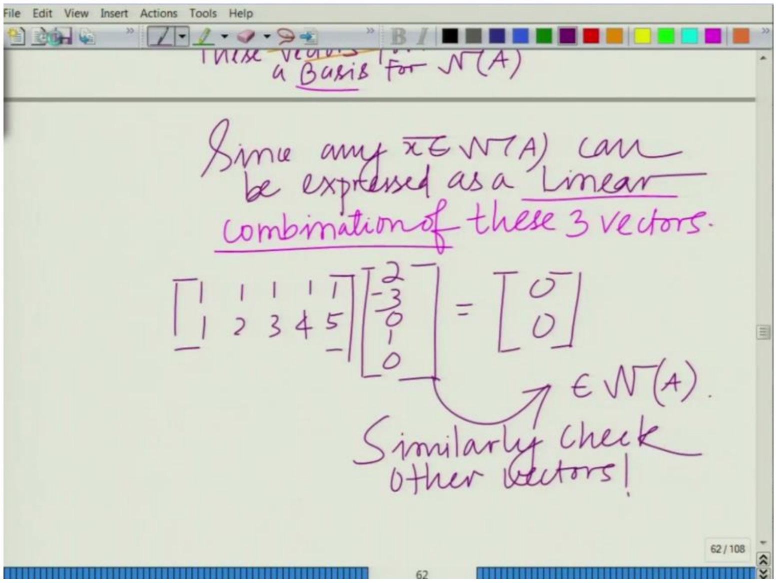

For instance, if you consider $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So, this indeed belongs to the $\mathcal{N}(\mathbf{A})$ and similarly you can check for the other vectors. Similarly, this is an exercise for you. Similarly, what you see very interestingly is the number of elements in the basis, that is the rank. So, you can see from here number of basis vectors equal to 3 implies dimension of $\mathcal{N}(\mathbf{A}) = 3$. Now, this dimension of $\mathcal{N}(\mathbf{A})$ is also termed as the nullity.

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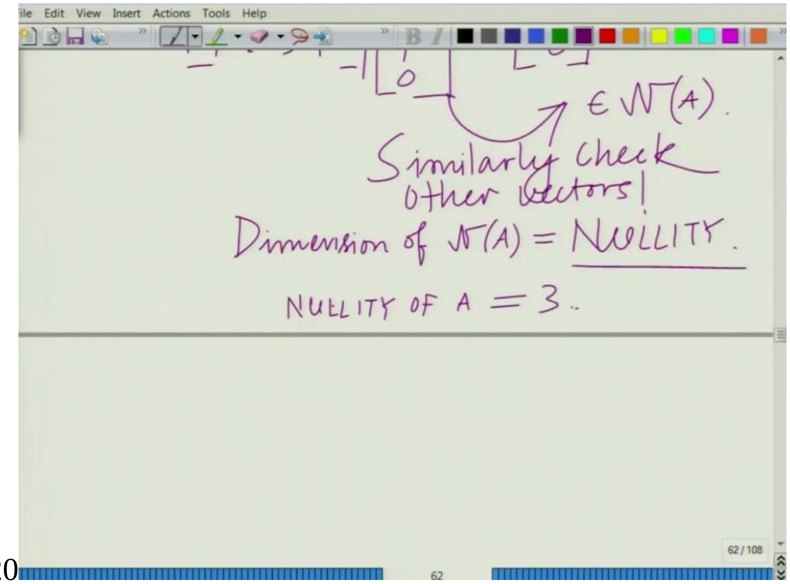

This is also termed as the nullity of the matrix. And now, you can also see, which in this case equal to 3 for nullity of **A** equal to 3 and nullity because the number of elements in the basis is equal to 3, which implies the nullity of **A** is equal to 3. If you can look at this, now, you go all the way back and you look at the $rank(\mathbf{A})$ it is not very difficult to see you have 2 pivots.

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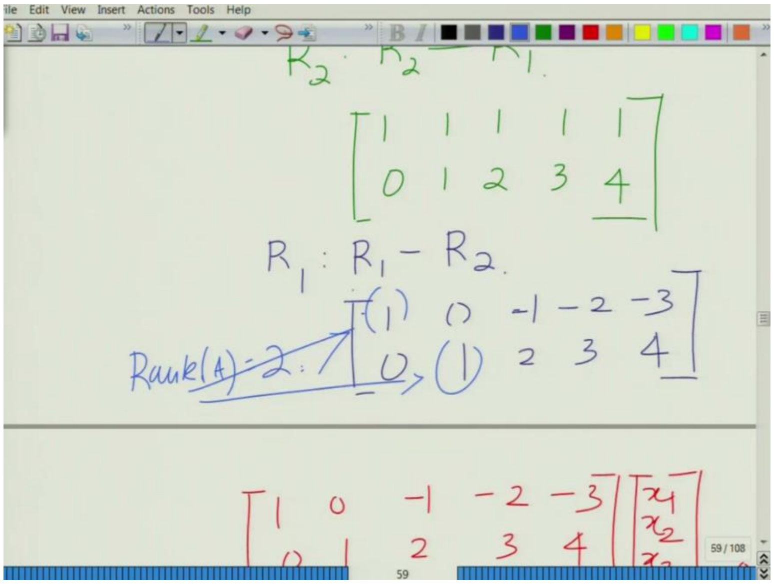

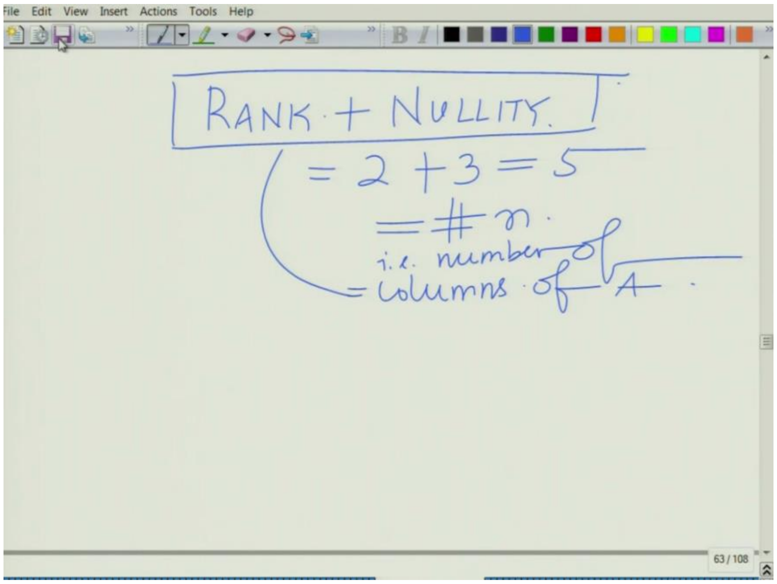

It means, $rank(\mathbf{A}) = 2$, and therefore, you have the famous rank plus Nullity theorem. So you have $rank + nullity = 2 + 3 = 5$, which is essentially equal to n, that is, number of columns of **A**. Rank nullity properties are very interesting, there is a dimension of the rank plus the dimension of the null space must equal the number of columns of the matrix **A**.

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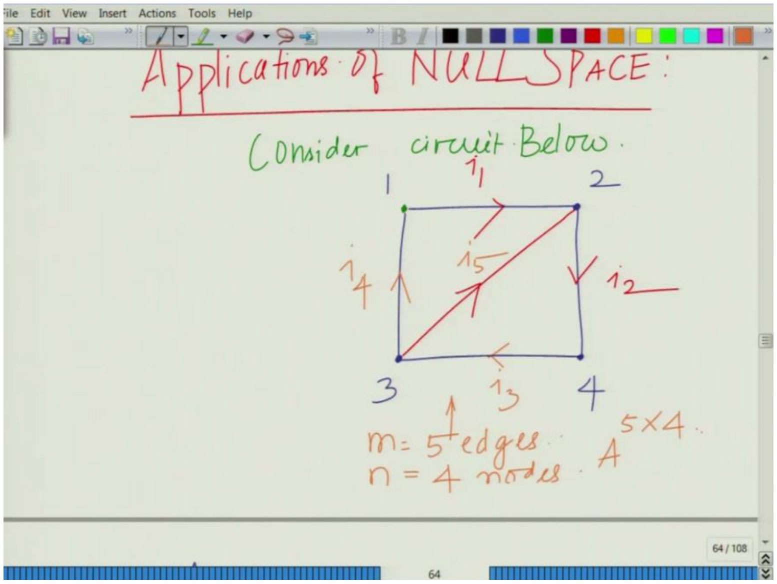

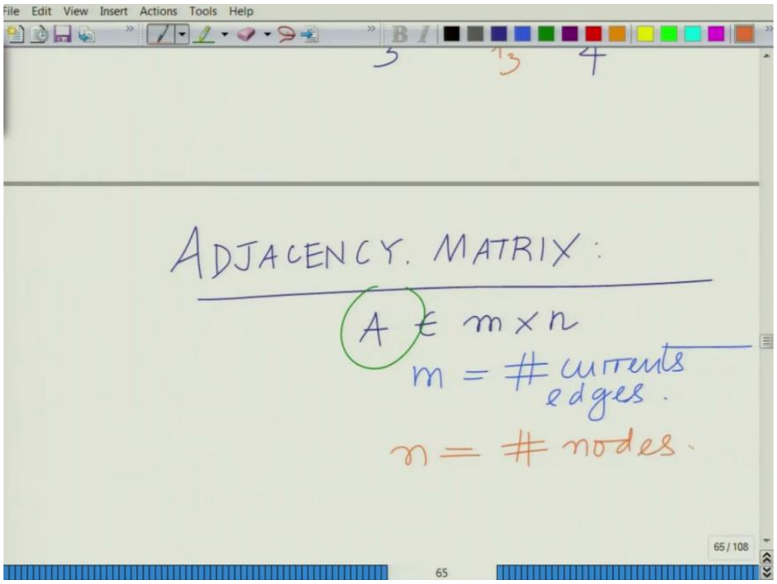

Let us look at an application of the null space. The null space is again another very interesting concept which has several applications. So, now another interesting application is in circuits. Consider the simple circuit above in the screenshot. I am going to draw it as simply a set of, so these are nodes 1 2 3 and 4. So, we are going to call this as the current from 1 to 2 as i_1, 2 to 4 as i_2, 4 to 3 as i_3, 3 to 1 as i_4 and 3 to 2 as i_5.

Now, let us define for this circuit, what is known as the Adjacency Matrix. What do we mean by the Adjacency Matrix? We have the matrix **A** is of size $m \times n$ which is the adjacency matrix, what is m? m equals basically the number of currents or the number of edges in a circuit. What is n? n

equals the number of nodes, therefore now, in this graph, we have 5 edges. That is, you have 5 edges, which means $m = 5$ and $n = 4$ nodes. So, **A** is of size 5×4. Now, what is the property of **A,** how do we construct **A**?

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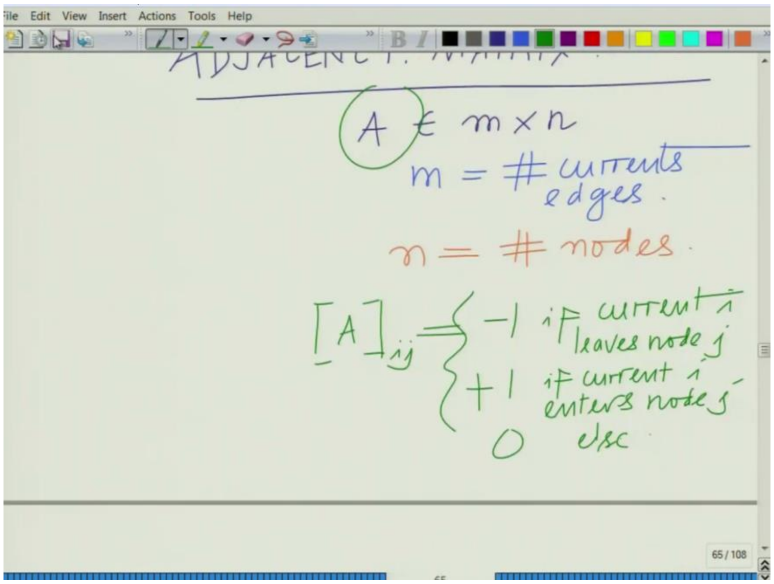

We have

$$[\mathbf{A}]_{ij} = -1, \text{ if current } i \text{ leaves node } j,$$

$$[\mathbf{A}]_{ij} = 1, \text{ if current } i \text{ enters node } j,$$

$$[\mathbf{A}]_{ij} = 0, \text{ else.}$$

Let us construct, construct the adjacency matrix for this circuit. That will illustrate this again.

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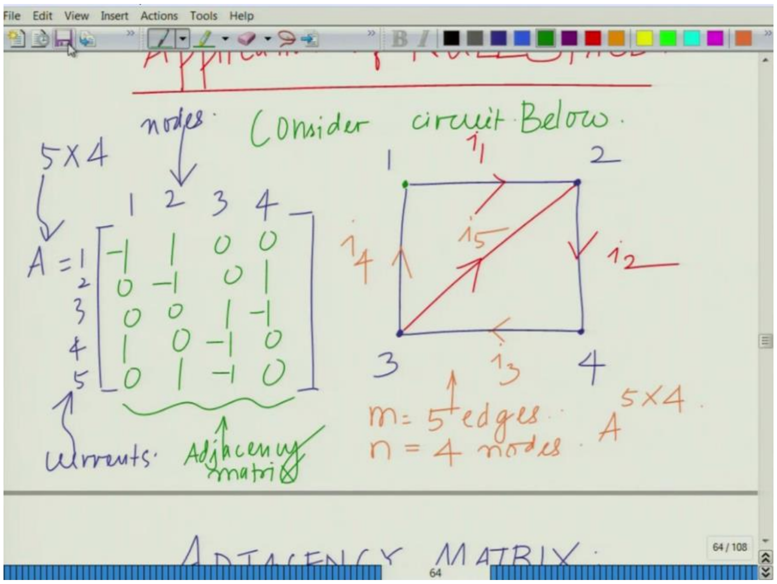

Let us go back. Let us take a look at this circuit **A** equals we already said this is a 5×4 matrix. So, one row for each current and one column for each node.

$$\mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \end{array}$$

So, this is the Adjacency Matrix. Previously, we have seen the Vertex matrix this is basically different this is the Adjacency matrix.

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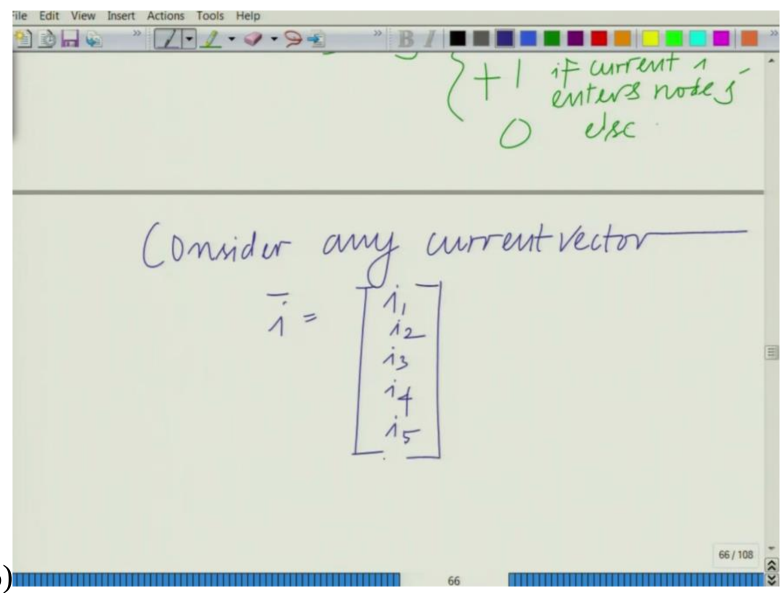

So, now consider the current vector $\bar{\boldsymbol{\iota}}$. Now consider any current vector

$$\bar{\boldsymbol{\iota}} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix}$$

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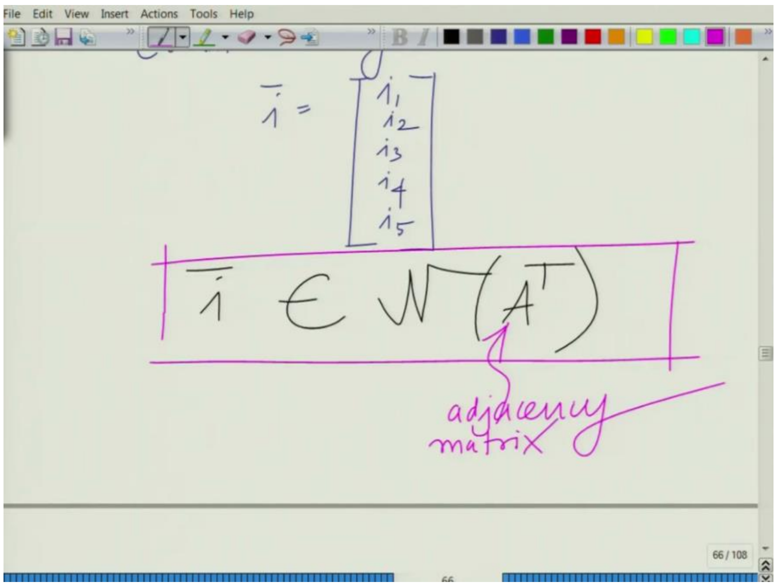
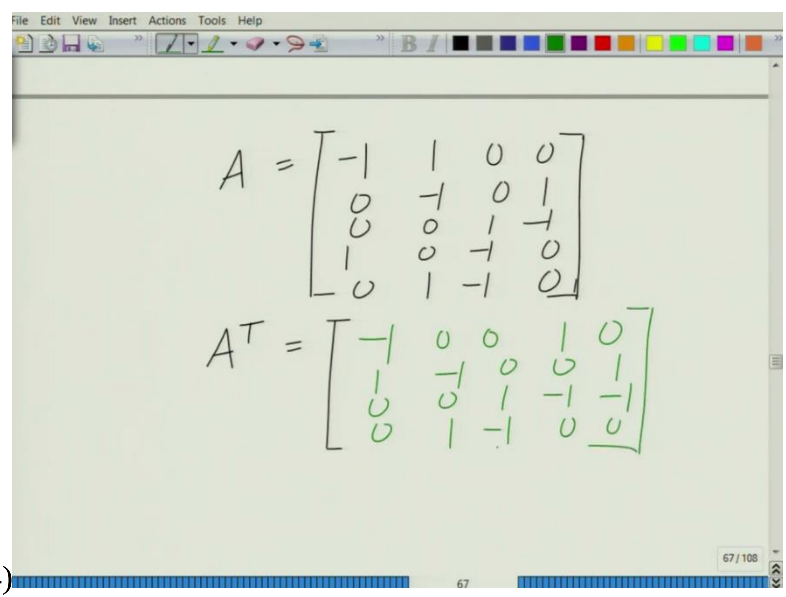


Then the property of the null space says that $\bar{\boldsymbol{\iota}} \in \mathcal{N}(\mathbf{A}^T)$ where **A** is the Adjacency matrix. So, any legitimate current vector of the circuit must belong to the $\mathcal{N}(\mathbf{A}^T)$, let us verify that.



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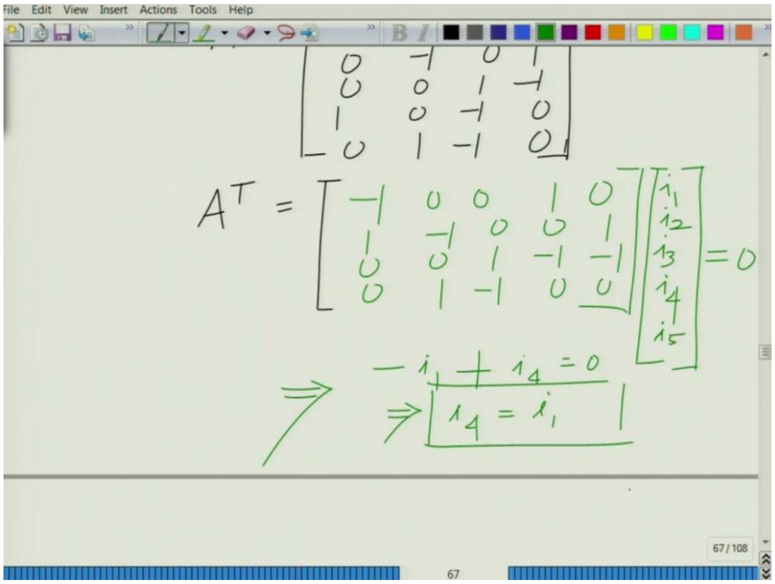

We have once again, **A** equal to

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

What is $\mathbf{A}^{\mathrm{T}}$**?** It equals

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix}.$$

Now, determine the null space of $\mathbf{A}^{\mathrm{T}}$. Let us write

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{i_1} \\ \boldsymbol{i_2} \\ \boldsymbol{i_3} \\ \boldsymbol{i_4} \\ \boldsymbol{i_5} \end{bmatrix} = \mathbf{0} \Rightarrow -i_1 + i_4 = 0 \Rightarrow i_1 = i_4.$$

So, current entering node 1 must be equal to current leaving node 1 so this is the Kirchhoff's current law for the first node.

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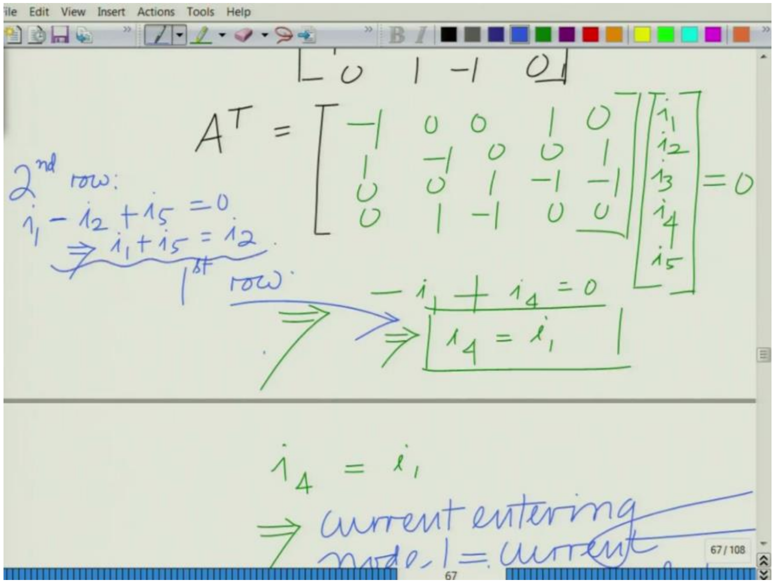

Similarly, if you look at the second one that says $i_1 = i_2 + i_5$.

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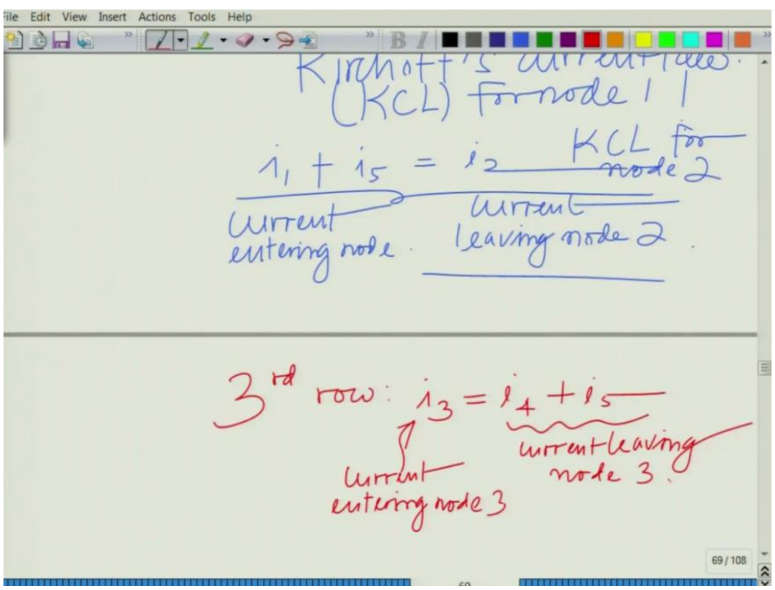

Similarly, if we expand the third row, you will get $i_3 = i_4 + i_5$. And finally, from the 4th row, you have

$$i_2 = i_3.$$

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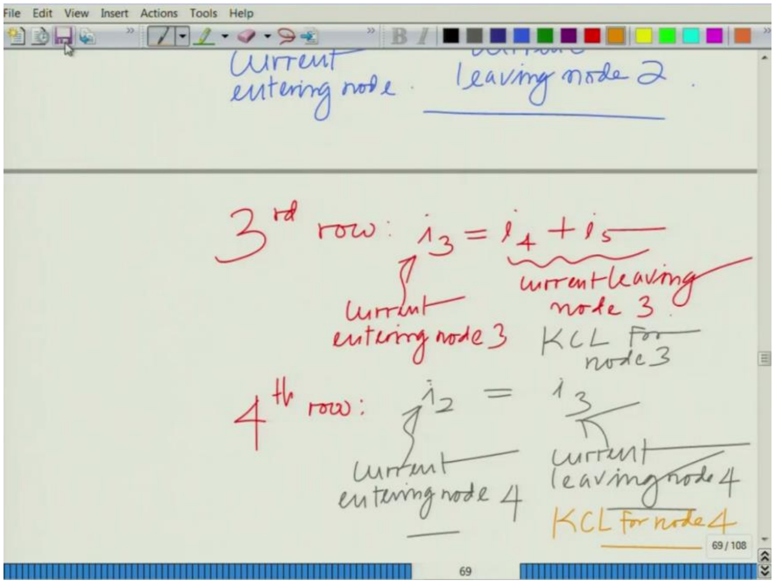

So, you write the Adjacency matrix of the circuit, then any legitimate current vector, because it has to satisfy the KCL equations for every node, this has to belong to the null space of $\mathbf{A}^T$.

And these set of equations translate to nothing but the KCL or the Kirchhoff's current law to every node. So, that is a very, very interesting property. And the null space in general has many applications as already seen, I have already shown null space. If a square matrix has a non-trivial null space, then it shows that the matrix is basically not invertible. It is a singular matrix. So, it is a very important property of a matrix. So, let us stop our discussion here and continue in the subsequent modules. Thank you very much.