Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
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Lecture 01
Vector properties: addition, linear combination, inner product, orthogonality, norm

Hello welcome to another module in this massive open online course on Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning. So, let us start our lecture, so we are going to start with the fundamental concepts of vectors and how vectors can be used to represent data.

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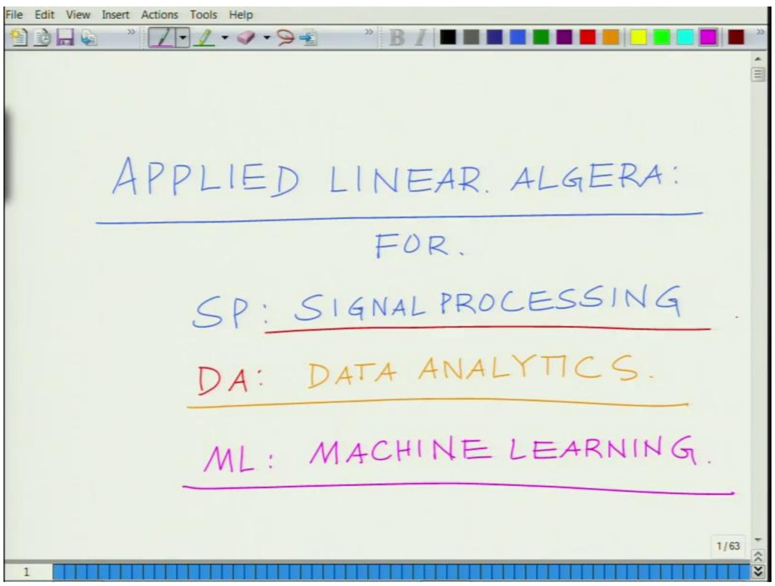

So, this is the massive open online course on Applied Linear Algebra for SP that is Signal Processing and you have the DA that is your Data Analytics and also you have the ML which is basically Machine Learning. So, as you can see essentially linear algebra and its applications, the concepts of linear algebra are so fundamental that they have diverse applications, in fact they are applied in almost all the fields of science and engineering. And, in particular, we are going to look at the interesting applications of linear algebra in signal processing, data analysis, data analytics and machine learning.

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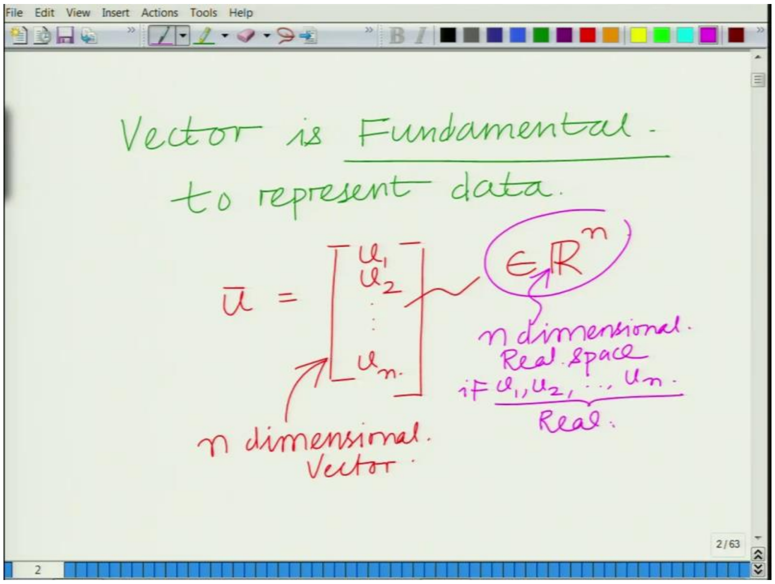

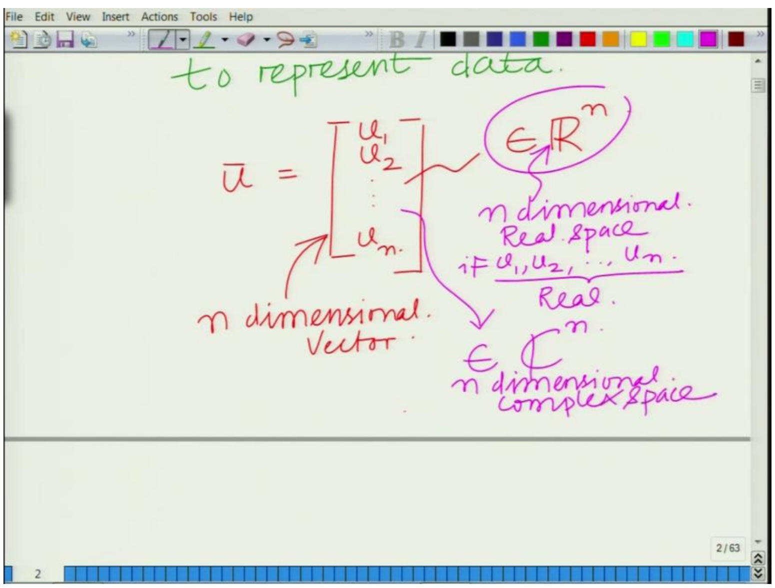

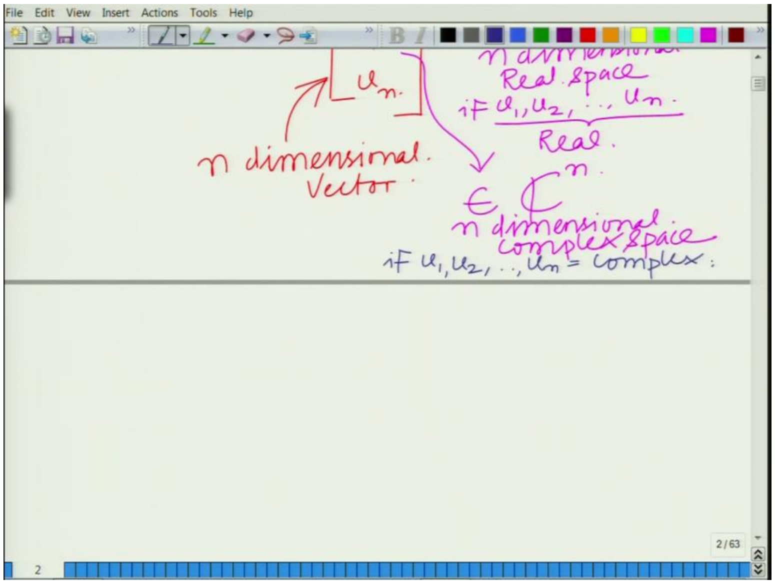

So, the notion of linear algebra, when you talk about the linear algebra the notion or concept of vectors, so vector or basically n dimensional point is fundamental. This is fundamental to represent data. For instance, we can define a vector $\bar{\boldsymbol{u}}$ as this is, an n dimensional vector comprising of the elements or we say the components u_1, u_2,.., u_n. This is we say n dimensional vector. This is fundamental to represent the data which is what we are seeing and this belongs to the n dimensional real space.

So, this notion is basically your n dimensional real space if u_1, u_2, so on up to u_n if these are real, else this belongs to the n dimensional complex space, so this is n dimensional complex space, if your u_1, u_2, u_n are, complex. So, you have the n dimensional vector $\bar{\boldsymbol{u}}$, u_1, u_2, u_n are real then this belongs to the n dimensional real space if u_1, u_2, u_n are complex this belongs to the n dimensional complex space.

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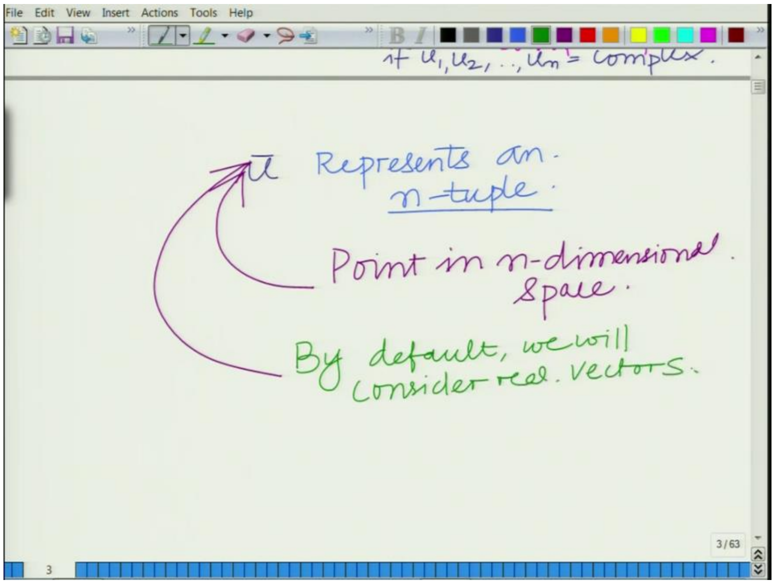

And you can also say $\bar{\boldsymbol{u}}$ represents, this essentially represents an n tuple, or essentially, you can also say this is a point in n dimensional space. So, by default we will consider real vectors. For example, a classic example of that for our course would be an n dimensional signal, n samples of a signal, so you take continuous time analog signal, you sample it and you consider n samples of the signal that would be a vector.

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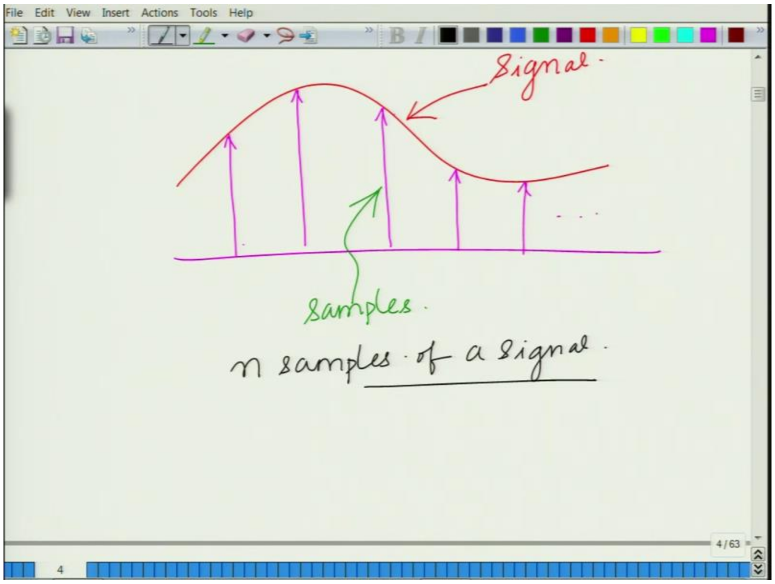

So, you take for instance a classic example, a very useful example for us. So this is your signal and then you take the n samples. So n samples of a signal in space, sorry n samples of a signal in time or we can generally say n samples of a signal, so that will be our signal vector.

So, for instance you have n samples of noise that can be your noise vector. So, vectors have a lot of applications, of course, you can consider n samples of a spatial signal such as an image, so of course, an image is 2 dimensional but you can also take the samples and put them as a vector. So, essentially vectors are very useful in representing data signals and various other quantities.

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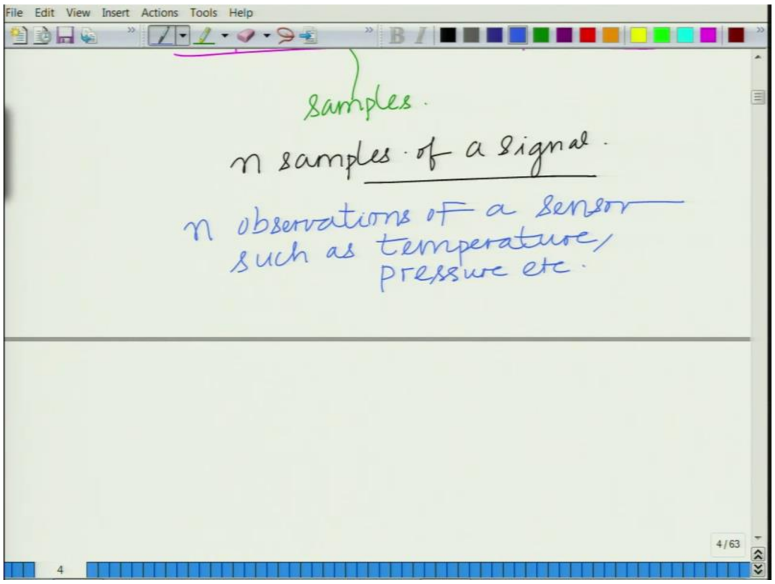

Therefore, now for instance, you can also have the n observations of a sensor, so that can also be another interesting vector. So, n observations of a sensor in time such as temperature pressure. For instance, let us write this another example, you can have n observations of a sensor such as temperature, pressure, etc so that is your, there are a lot there are basically many many examples of such vectors.

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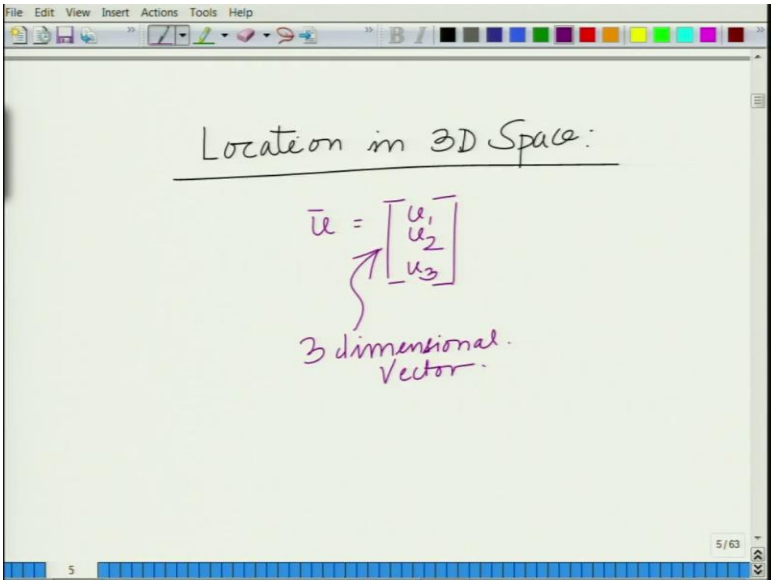

Now, for instance a location, 3D space, that can be represented by or you have a location in 3D space that can be represented by a 3-dimensional vector. Any location in 3D space that can be represented by the 3-dimensional vector u_1, u_2, u_3 or what we call as the x y z coordinates. So, this is basically a 3-dimensional vector, this is the physical space that we are used to, this is your location in 3-dimensional space.

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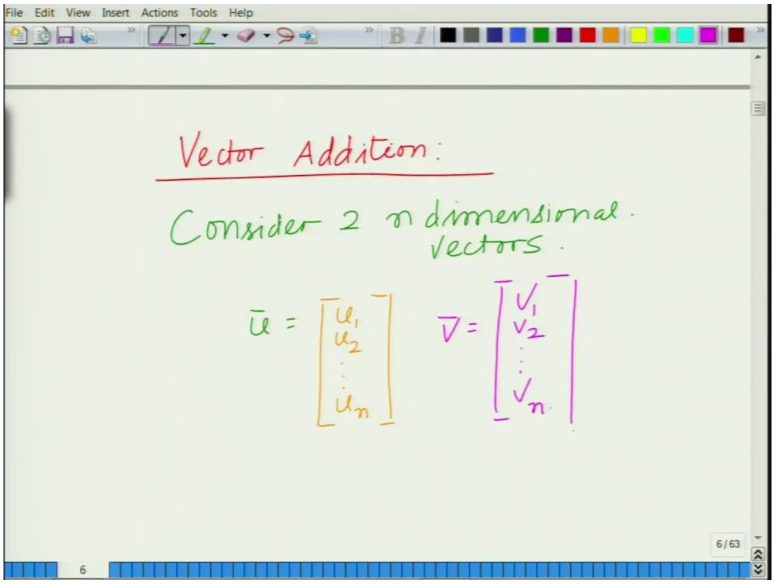

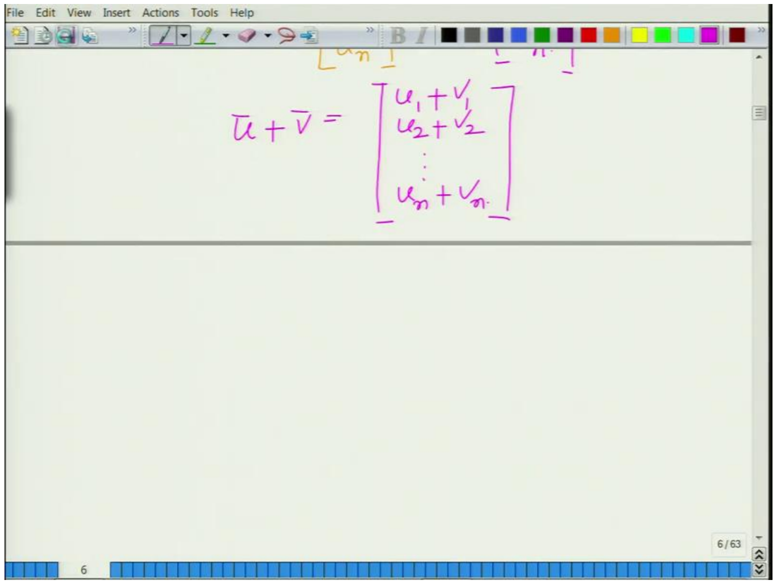

Now, consider 2 n dimensional vectors, now let us look at vector addition. Consider 2 n dimensional vectors that is we have, let us say, $\bar{\boldsymbol{u}}$ equals $u_1, u_2, .., u_n$ and we have $\bar{\boldsymbol{v}}$ equals $v_1, v_2, \dots, v_n$ then the vector addition is simply an element-by-element addition of these 2 vectors.

So, essentially $\bar{\boldsymbol{u}}$ plus $\bar{\boldsymbol{v}}$ will also be an n dimensional vector with the first element u_1 plus v_1, second element u_2 plus v_2, so on, nth element u_n plus v_n.

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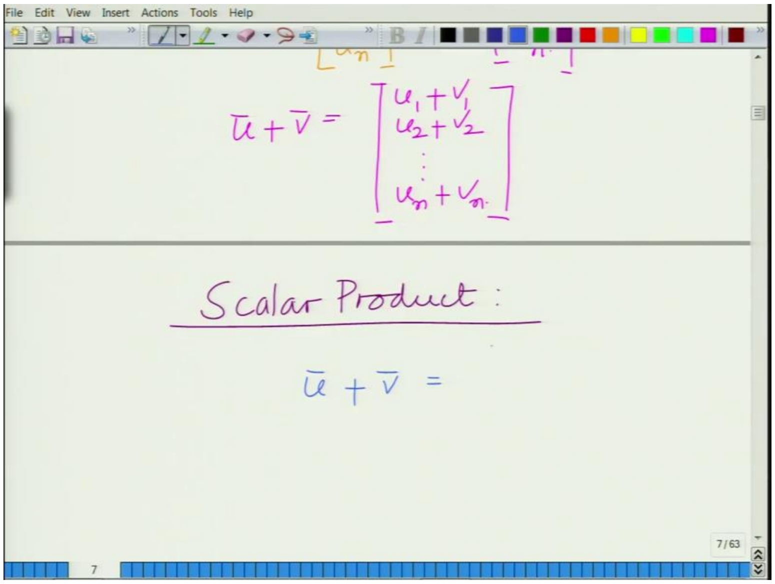

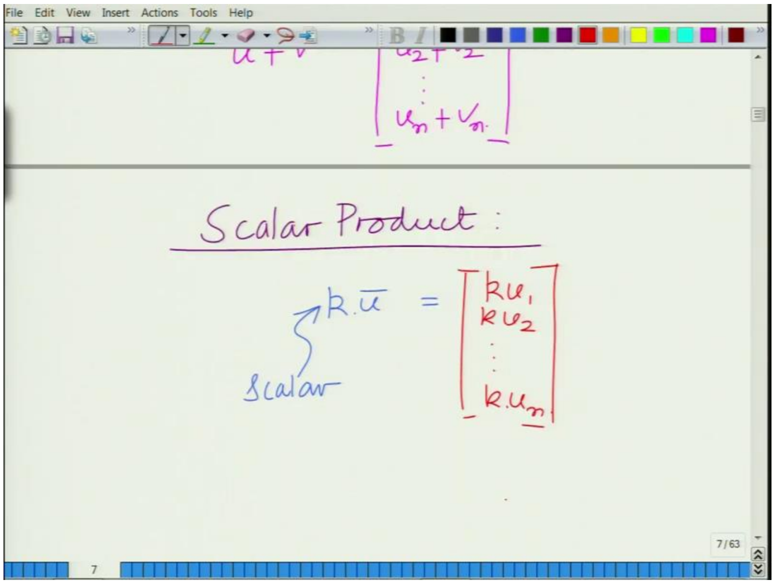

And let us now consider the scalar product, let us consider the scalar product you have $\bar{\boldsymbol{u}}$ plus $\bar{\boldsymbol{v}}$ or you have k that is a scalar, so k is a scalar coefficient, so k times $\bar{\boldsymbol{u}}$ you take the scalar multiplied by each element of the vector, so this will be ku_1, ku_2 so on up to ku_n. And then we have the notion of a linear, so this basically completes the basic operations that is your what we call as your vector addition and the scalar product. And now let us look at another fundamental operation of vectors that is basically a linear combination.

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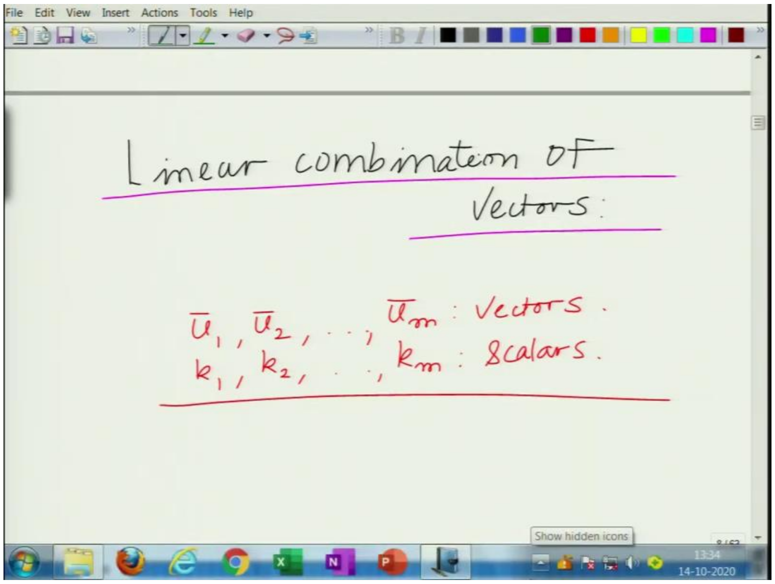

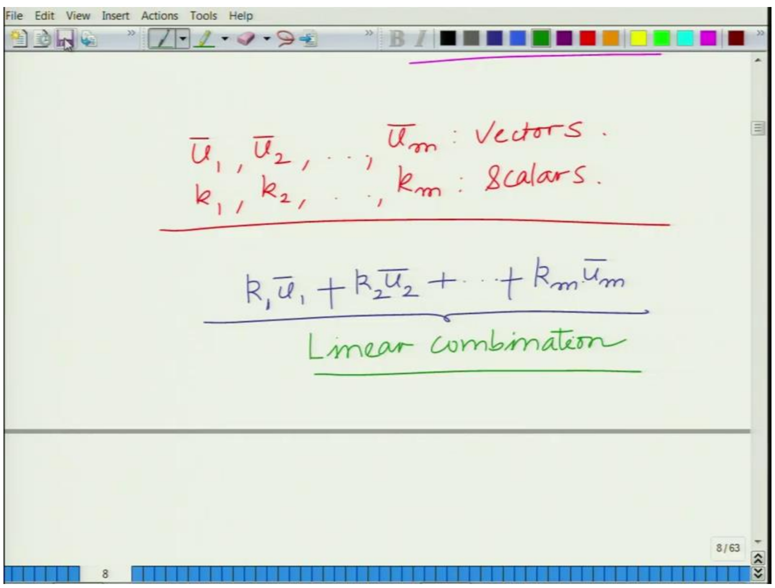

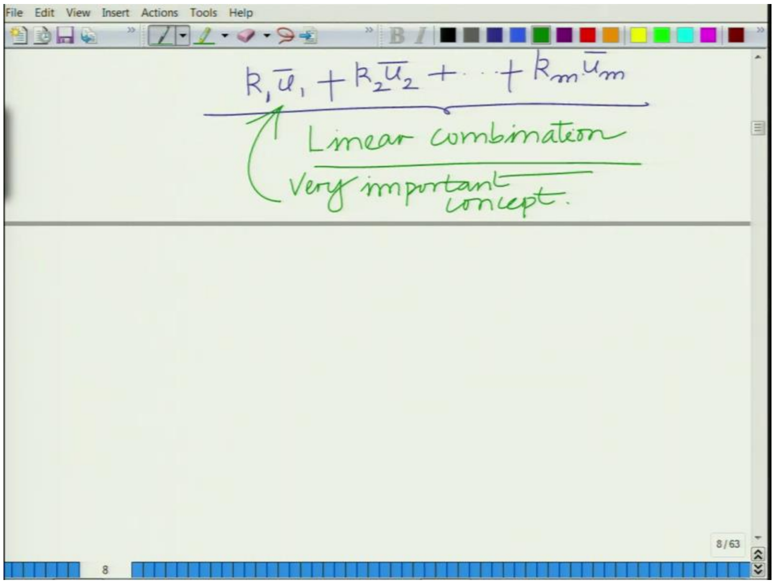

So, let us now look at a linear combination, let us now look at a linear combination of vectors, so you have the vectors $\bar{\boldsymbol{u}}_1$, $\bar{\boldsymbol{u}}_2$,.., $\bar{\boldsymbol{u}}_m$. So, consider the vectors $\bar{\boldsymbol{u}}_1$, $\bar{\boldsymbol{u}}_2$,.., $\bar{\boldsymbol{u}}_m$, these are the vectors and then you have the scalar quantities $k_1, k_2, \ldots, k_m$ these are your scalars. Then your linear combination of vectors, the linear combination is basically given as multiply each scalar coefficient by the corresponding vector.

So, this is essentially the concept of the linear combination of the vectors. Then we have the notion of an inner product, so that is basically and this is a very important concept the linear combination of the vectors.

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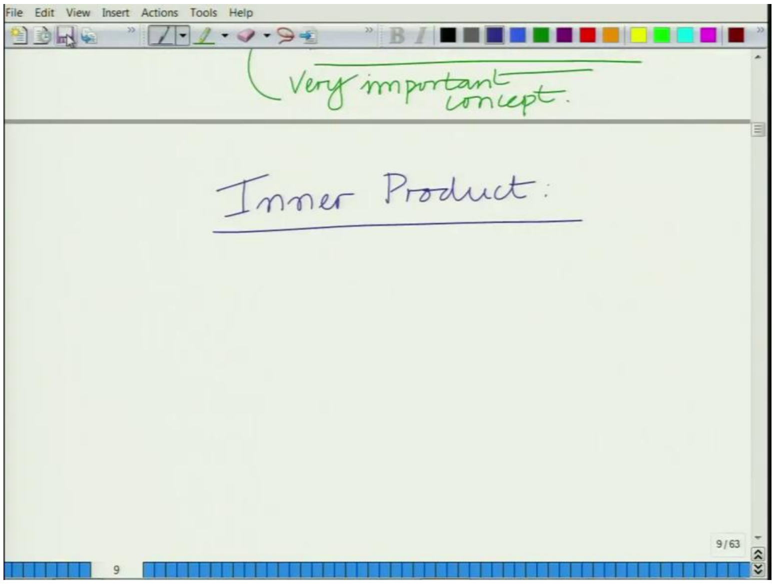

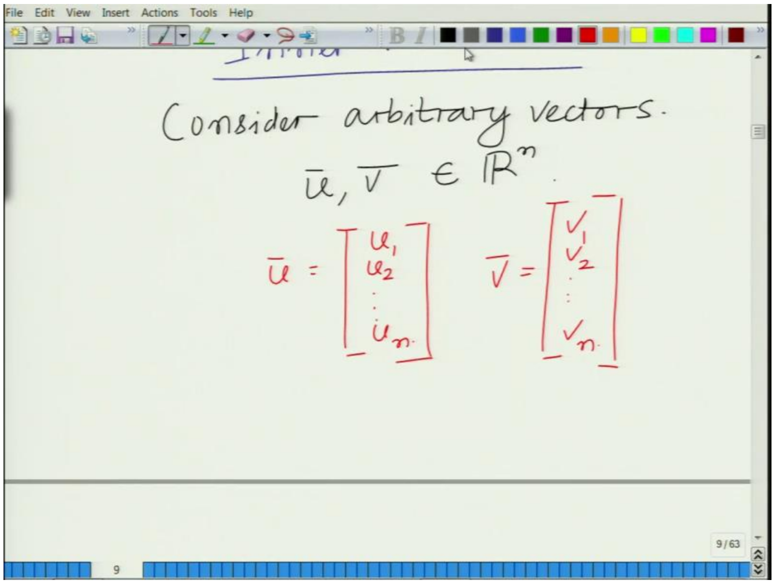

Now, let us consider another concept that is the inner product, what about the inner product and this is another fundamental concept that is the inner product when you have 2 n dimensional vectors what happens to the inner product of these 2 vectors. So, consider arbitrary vectors $\bar{\boldsymbol{u}}$, comma $\bar{\boldsymbol{v}}$ that belong to the n dimensional space of real vectors that is you have $\bar{\boldsymbol{u}}$ equals u_1, u_2, up to u_n, and $\bar{\boldsymbol{v}}$ equals v_1, v_2, up to v_n.

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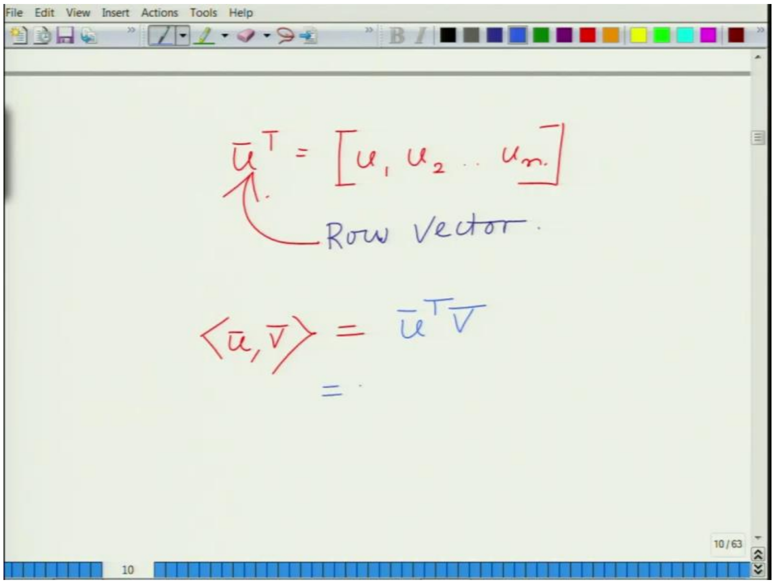

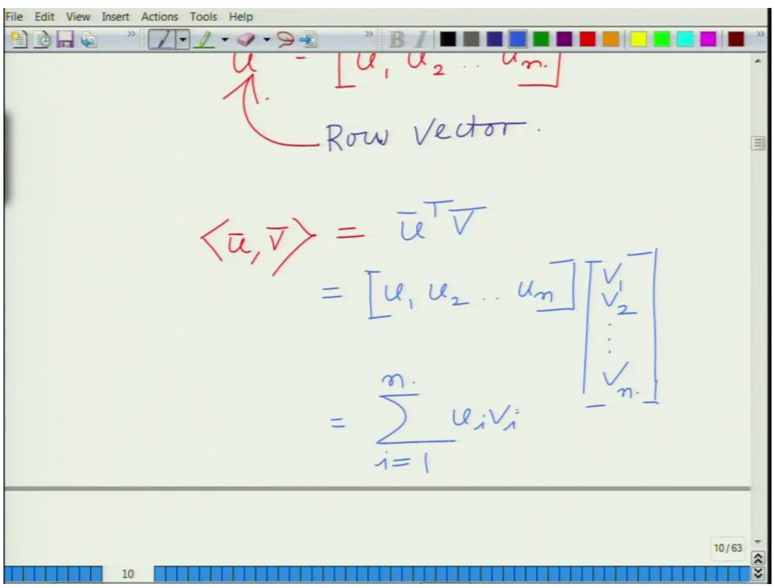

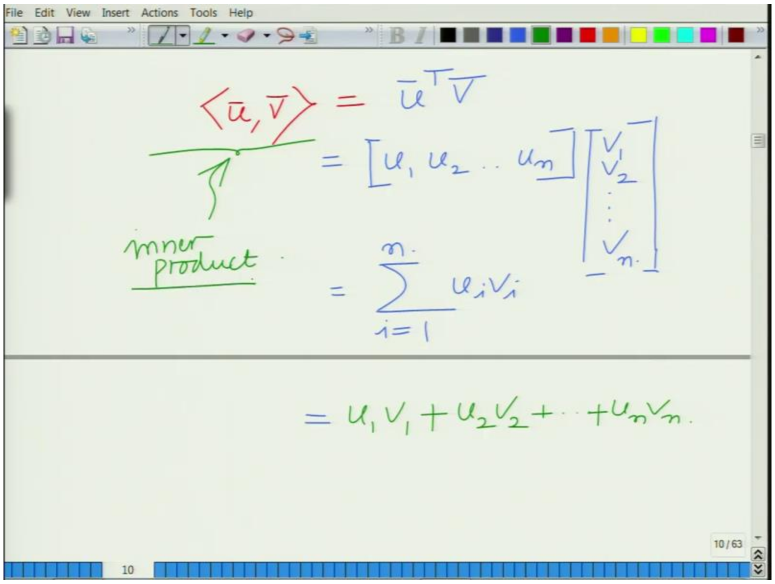

Then we have $\overline{\boldsymbol{u}}^T$, this becomes a row vector $\overline{\boldsymbol{u}}^T$ when you take the transpose of a column vector that becomes a row vector. So this is $\overline{\boldsymbol{u}}^T$ and then so this is $\overline{\boldsymbol{u}}^T$is essentially a, this is essentially a row vector and then we define the inner product, this is a very important definition, we define the inner product of $\overline{\boldsymbol{u}}$, $\overline{\boldsymbol{v}}$, we define it as $\overline{\boldsymbol{u}}^T\overline{\boldsymbol{v}}$ which is essentially if you look at this, this is the product of the row vector $u_1, u_2, \ldots, u_n$ times the column vectors $v_1, v_2, \ldots, v_n$ which is equal to summation i equal to 1 to n, I can write it as $u_i\ v_i$ or essentially this is u_1, v_1, write it in the expanded form $u_1\ v_1$ plus $u_2 v_2$, plus $u_n v_n$ so this is basically the notion of the inner product.

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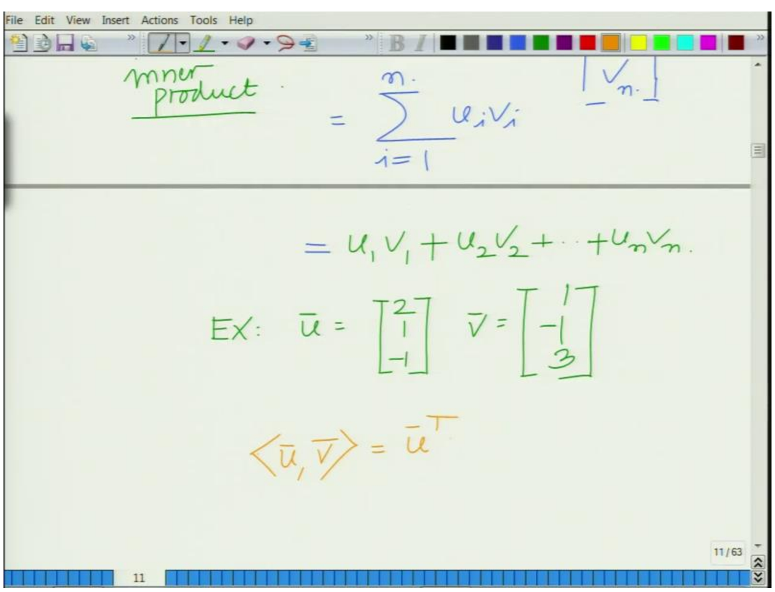

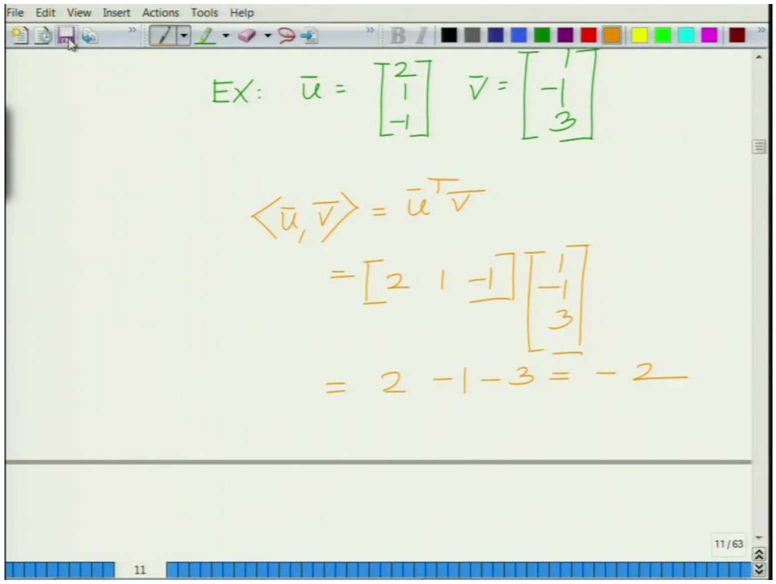

So, this is the inner product, this is a very important concept, the concept of an inner product between 2 vectors. For example, let us take these 2 vectors, example $\bar{\boldsymbol{u}}$ equal to 2 1 minus 1 and $\bar{\boldsymbol{v}}$ equal to 1 minus 1 3 and then what you can say is you have $\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}$ the inner product is $\bar{\boldsymbol{u}}^T \bar{\boldsymbol{v}}$ which is basically equal to 2 1 minus 1 and 1 minus 1 3 which is essentially equal to 2 minus 1 minus 3 which is equal to minus 2, so that is a simple example for the inner product which is essentially you can also think of it as an element wise multiplication of 2 vectors and then an addition that is $u_1 v_1$ plus $u_2 v_2$ so on until $u_n v_n$.

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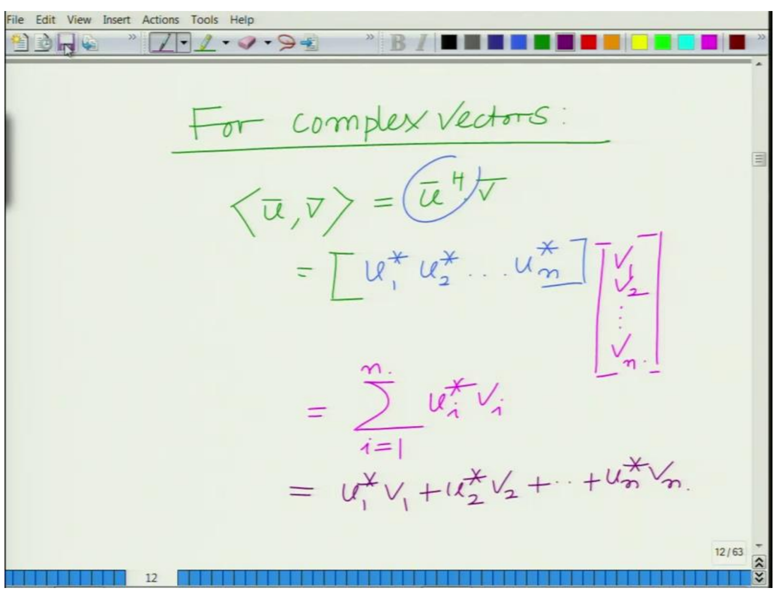

Now, for complex vectors the definition becomes we have $\bar{\boldsymbol{u}}$, the inner product of $\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}$**.** We will define this as $\bar{\boldsymbol{u}}^H \bar{\boldsymbol{v}}$ that is essentially you have to take the transpose that is when you look at the Hermitian you essentially have to take the transpose and also the complex conjugate $u_1^*, u_2^*, \dots, u_n^*$ times $v_1, v_2, \dots, v_n$ that is sigma equal to 1 to n that is $u_i^* v_i$, so this is the notion of inner product for complex vectors u_1^* to v_1 plus $u_2^* v_2$ plus $u_n^* v_n$.

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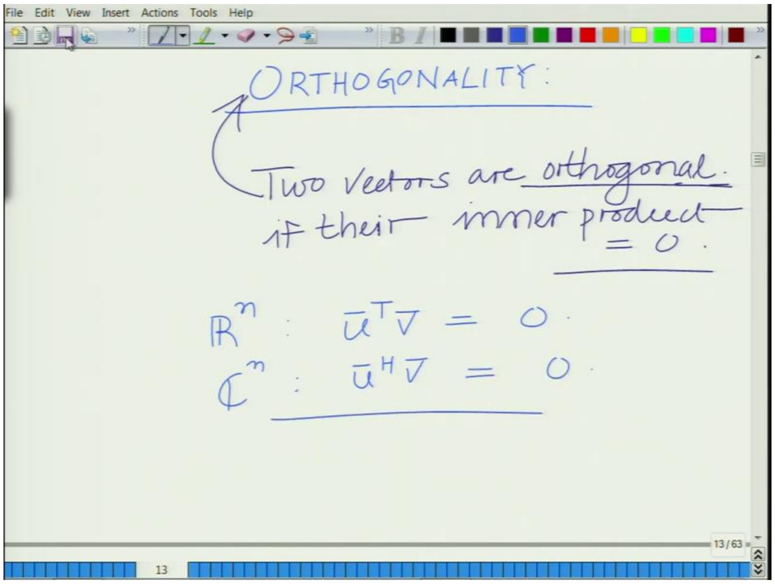

Further now there is a notion of orthogonality which is very important, the notion of orthogonality of 2 vectors, 2 vectors are, and this is a very important property 2 vectors are orthogonal if their inner product is 0, so orthogonally of 2 vectors are orthogonal or perpendicular, orthogonal, I am sorry orthogonal if their inner product equal to 0.

So, the 2 vectors are orthogonal that is in real dimensions we have $\bar{\boldsymbol{u}}^T \bar{\boldsymbol{v}}$ equal to 0 that is for the r dimensional space, for the n dimensional real space, for the complex space we have $\bar{\boldsymbol{u}}^H \bar{\boldsymbol{v}}$ equal to 0, so this is essentially the definition of orthogonality.

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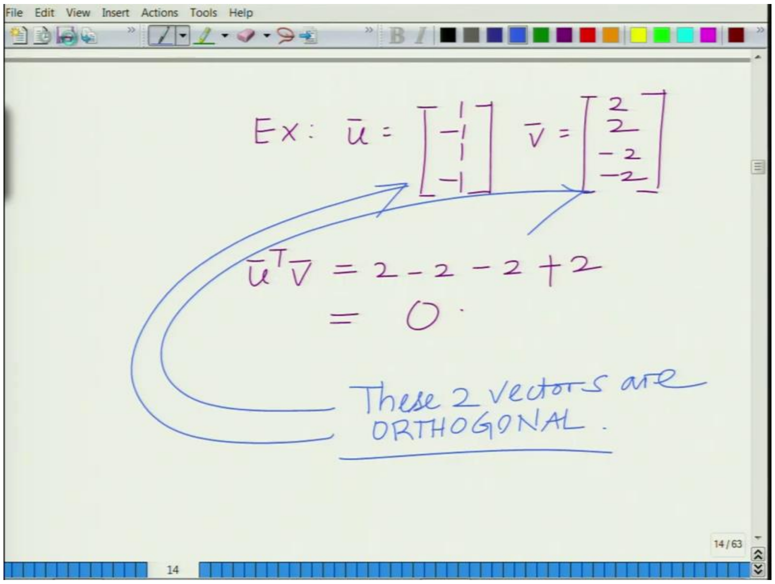

Let us take a look another again at another simple example $\bar{\boldsymbol{u}} = [1, -1, 1, -1]$ and $\bar{\boldsymbol{v}} = [2, 2, -2, -2]$ and we have $\bar{\boldsymbol{u}}^T \bar{\boldsymbol{v}}$ if we do $\bar{\boldsymbol{u}}^T \bar{\boldsymbol{v}}$ this will be $2 - 2 - 2 + 2$, which is equal to 0.

So, these 2 vectors that you have over here, so these 2 vectors are orthogonal, essentially what we also say informally as that as these 2 vectors are perpendicular and that makes more sense in 3-dimensional space that is we have 2 vectors which are at angle with respect to each other, more generally the notion is these 2 vectors are orthogonal because their inner product is 0. Let us look at another interesting example and this time with respect to complex signals and complex sinusoids.

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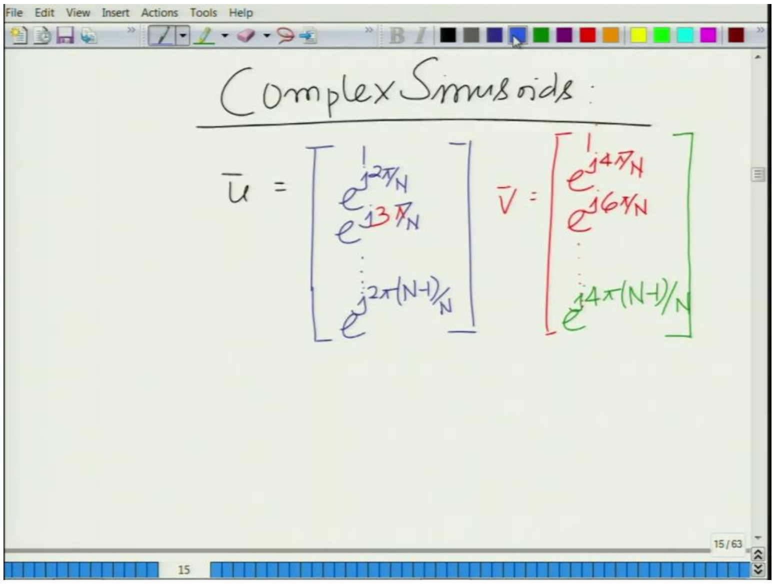

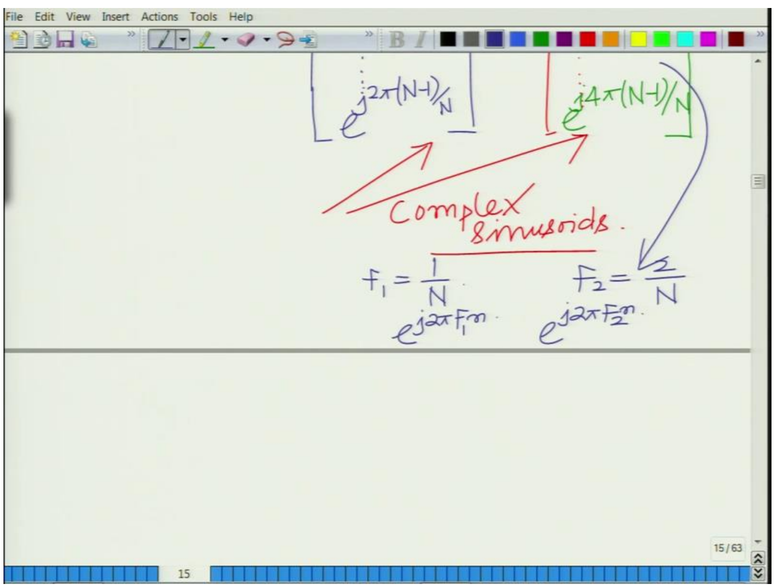

So, another very important example I would also say consider the complex sinusoids for instance you have the sinusoids $\bar{\boldsymbol{u}}$, let us call this as your $1, e^{\frac{j2\pi}{N}}, e^{\frac{j4\pi}{N}}$ so on and so forth $e^{\frac{j(N-1)\pi}{N}}$, so this is your first sinusoid. And then you have another which is essentially $1, e^{\frac{j3\pi}{N}}, e^{\frac{j6\pi}{N}}$ and so on and so forth $e^{\frac{j4(N-1)\pi}{N}}$.

So, you can see these are essentially 2 complex sinusoids, I am sorry let me make this again, this has to be $e^{\frac{j4\pi}{N}}, e^{\frac{j8\pi}{N}}$ and essentially what you have, yeah, essentially these are 2 complex sinusoids, these are 2 complex sinusoids and if you look at their inner product so these are 2 complex sinusoids corresponding to the frequencies if you look at the frequency of this that is f_1 this will be equal to 1 over n and this will be a complex sinusoid of frequency f_2 this will be equal to 2 over n, so you are talking about the complex sinusoid $e^{j2\pi f_1 n}$ and $e^{j2\pi f_2 n}$.

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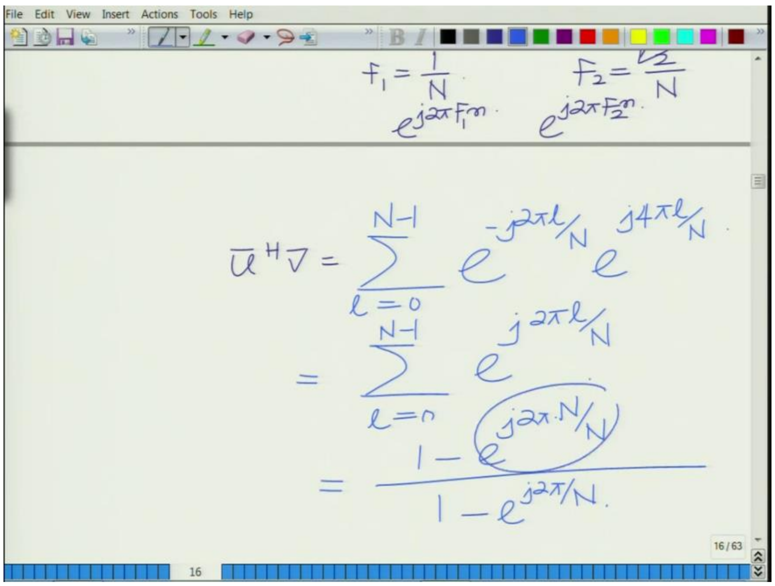

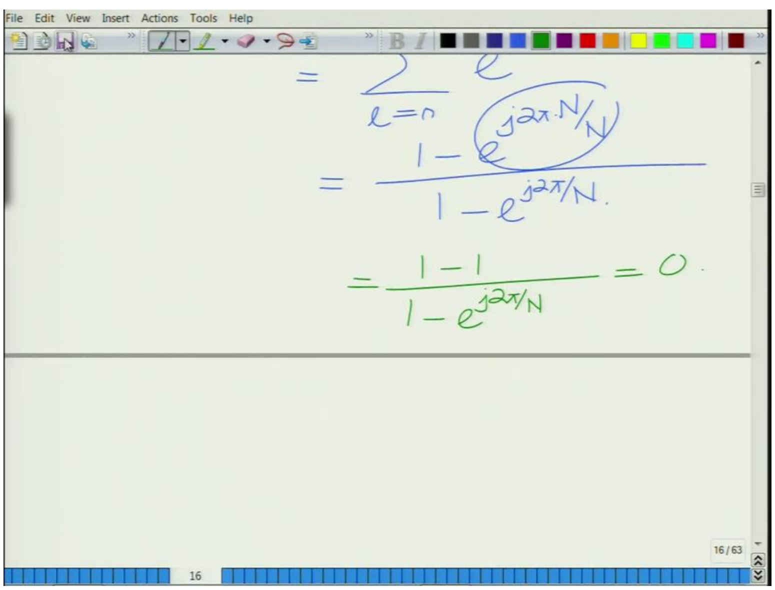

And when you look at their inner product, their inner product $\bar{\boldsymbol{u}}^H\bar{\boldsymbol{v}}$ is basically if you look at this that will be the summation l equal to the $\sum_{l=0}^{N-1} e^{-\frac{j2\pi l}{N}} e^{\frac{j4\pi l}{N}}$. And therefore, this is going to be the $\sum_{l=0}^{N-1} e^{\frac{j2\pi l}{N}}$ which is essentially $\frac{1-e^{\frac{j2\pi N}{N}}}{1-e^{\frac{j2\pi}{N}}}$.

And if you look at this $e^{\frac{j2\pi N}{N}}$ is nothing but $e^{j2\pi}$which is 1. so this is $\frac{1-1}{1-e^{\frac{j2\pi}{N}}}$. So, therefore, you have this interesting property where you have the complex sinusoids at the frequency $0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N-1}{N}$, these are essentially orthogonal.

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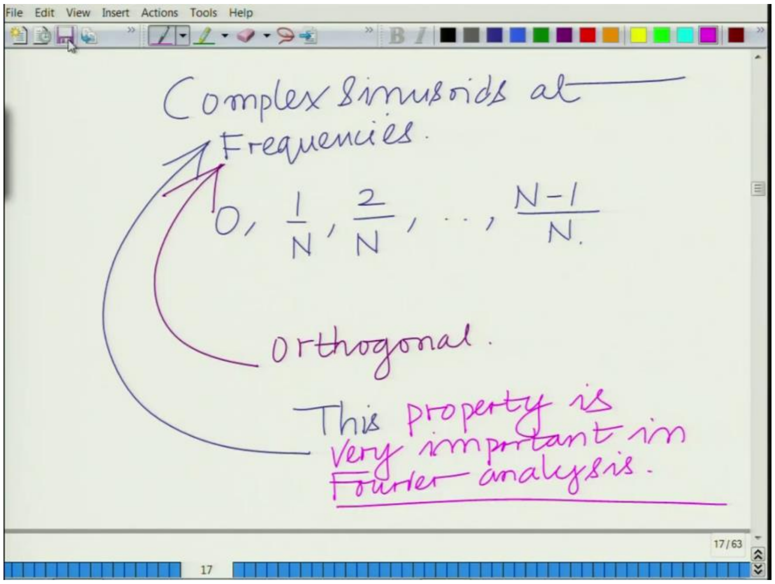

So, we have the interesting property complex sinusoids at frequencies, so you have complex sinusoids at the frequencies, at the frequency $0, \frac{1}{N}$, so these frequencies, these are orthogonal, complex sinusoids edges and this property, this orthogonality of this complex sinusoids, this property is extremely important in Fourier analysis, this forms the basis of all Fourier analysis.

So it is going to be very important. Naturally it is important for all of signal processing because remember the Fourier transform or the spectrum of a signal, the decomposition of the signal is very important in signal processing and in general, of course, naturally when you apply to images and audio signals and so on, it becomes also very important in machine learning and also naturally in data analytics, all of these are interrelated.

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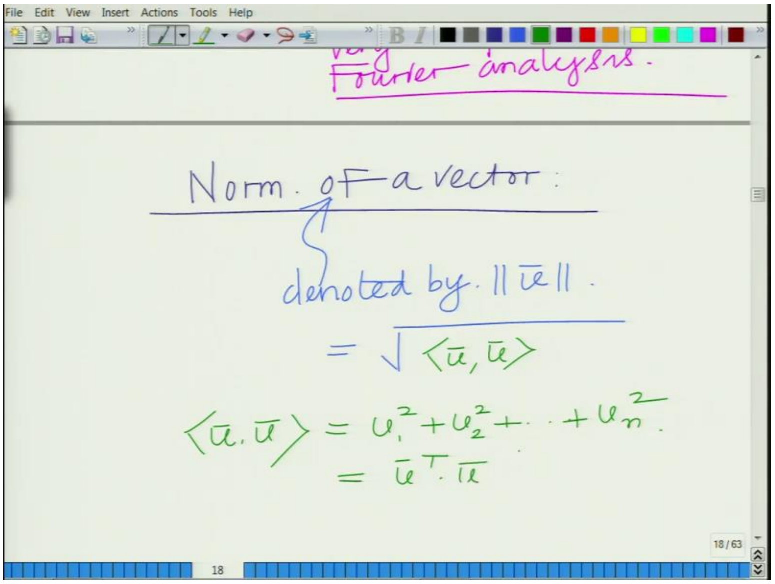

Let us now go to this notion of the norm of a vector, so what we mean by the norm of a vector? So, the norm of a vector this is denoted by $||\boldsymbol{\bar{u}}||$, which is nothing but the square root of the inner product of a vector $\boldsymbol{\bar{u}}$ with itself which is essentially for a real vector, you can clearly see the inner product of $\boldsymbol{\bar{u}}$ with itself for a real vector $\boldsymbol{\bar{u}}$ inner product with itself this is $u_1^2, u_2^2, \ldots, u_n^2$, which is nothing but $\boldsymbol{\bar{u}}^T \boldsymbol{\bar{u}}$.

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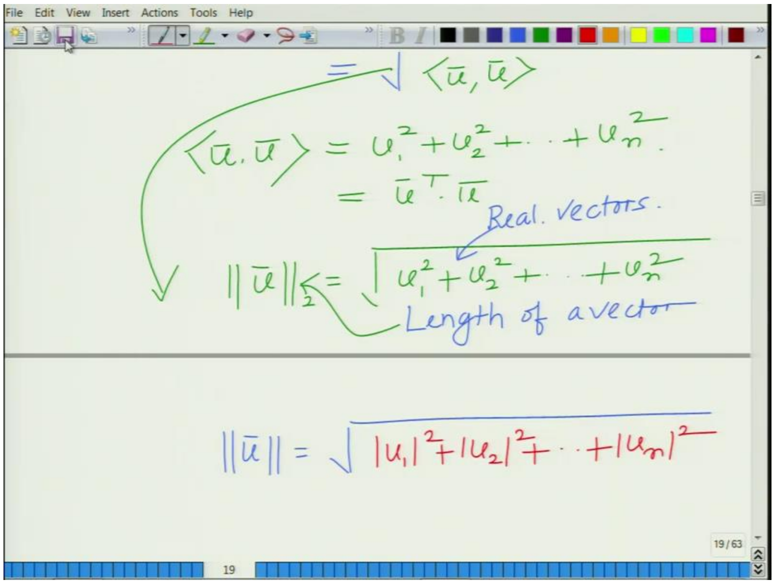

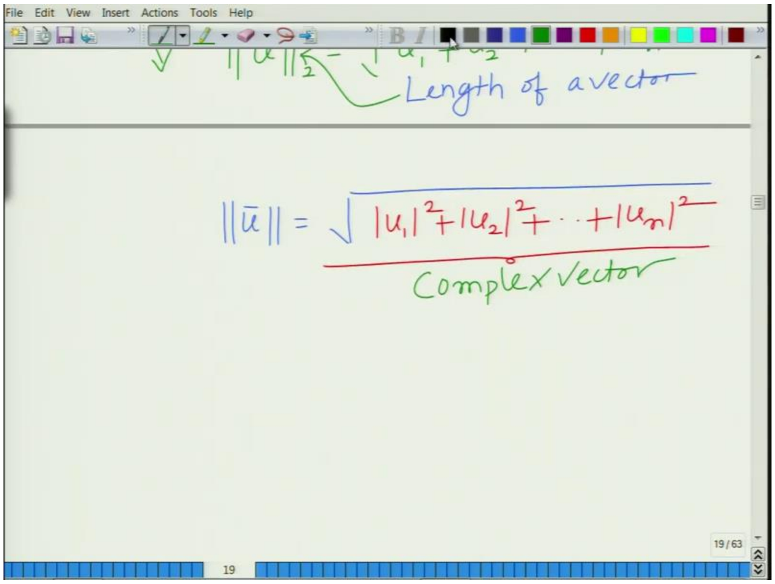

And therefore, this $||\bar{\boldsymbol{u}}||$ or what you also call as $l2$ norm of $\bar{\boldsymbol{u}}$ which is what we are going to use by default is going to be $\sqrt{u_1^2 + u_2^2 + .. + u_n^2}$. And this is also what we call as the length of a vector in 3-dimensional space or in general length, this is essentially the length of a vector.

And for complex vectors, so this is for real vectors where $u_1, u_2, \ldots, u_n$ are real, for complex vectors this will be simply the magnitude, this is simply the $\sqrt{|u|_1^2 + |u|_2^2 + .. + |u|_n^2}$. And so, this is for a complex vector, this definition is for a complex vector.

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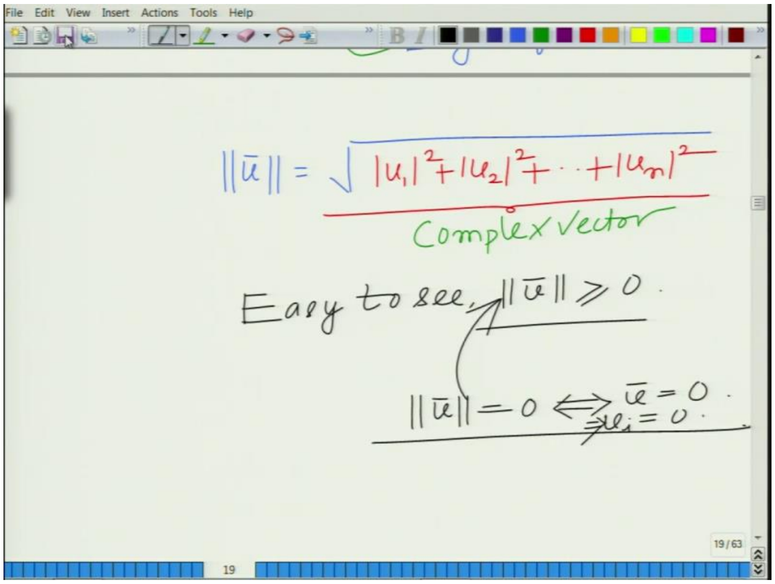

And it is very easy to see $||\bar{\boldsymbol{u}}||^2$ that is $||\bar{\boldsymbol{u}}||$ is greater than equal to 0 and this equal to 0, $||\bar{\boldsymbol{u}}||$ equal to 0 if and only if this is implies and is implied by $\bar{\boldsymbol{u}}$ equal to zero that is the norm or length of a vector equal to 0 only if $\bar{\boldsymbol{u}}$ equal to 0 which means every u_i which means essentially every u_i equal to 0.

So, this is essentially the norm of a vector which is defined as $\sqrt{u_1^2 + u_2^2 + .. + u_n^2}$ for a real vector and $\sqrt{|u|_1^2 + |u|_2^2 + .. + |u|_n^2}$ for a complex vector. And this is always greater than equal to 0 except for a vector $\bar{\boldsymbol{u}}$ which is identically equal to 0, that is $u_1, u_2, \ldots, u_n$ all the elements are 0. So, let us stop this module here and let us continue in the next module with further discussion on these various concepts of applied linear algebra. Thank you very much.