Module - 19
Lecture - 26
Modes of Rectangular Waveguides

Hello and welcome to NPTEL MOOC, on Electromagnetic Waves in Guided and Wireless Media and we will continue the discussion of rectangular Waveguides that, we began in the last module.

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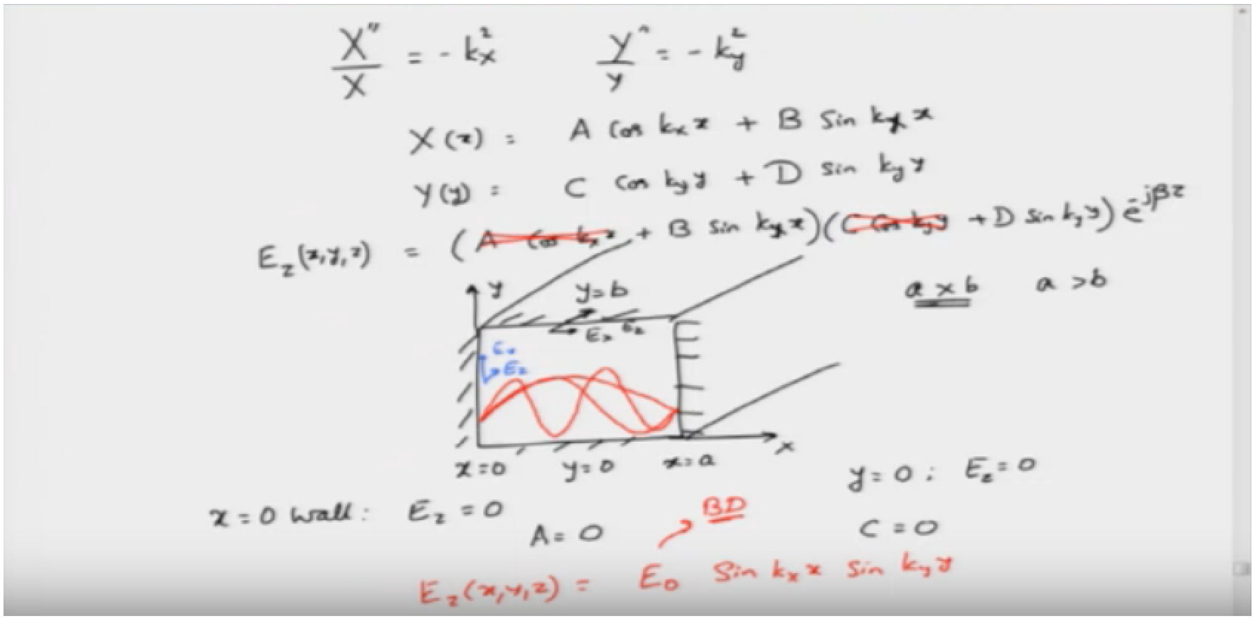

After going through several steps, I would like you to recall that we stopped at this particular equation, in which you had, this part you know, X double prime of X by X, as being a function only of X, then the next term, y double prime of Y by Y is a function only of Y, just to remind you, X double prime of X, actually means that, it is the second derivative, of the function X of X, with respect to the variable X itself. Okay? And then you have this other term, k0 square epsilon r minus beta square, as a constant. Okay? Now, we said that, one of the ways in which this equation or rather the way in which this equation, can be true, for all values of X and for all values of Y is that, this separately be equal to some constant, which we will call as, 'minus KX Square'. And this we will call it as, minus KY Square and let this also be another constant. Okay? So, what we now have is the sum of three constants, which will be equal to 0, this actually imposes the limits on beta. Okay. Of course, we still haven't decided, what would be KX and KY? they will be, you know obtained, once we have applied the boundary conditions. Okay? So, because we haven't yet done that, so for every term here is unknown, except that, k0 Square and epsilon R are known to us, because epsilon R is the material with which we have, filled the waveguide and k0 is equal to 2 pi by lambda. and lambda is the operating wave length, of the mode that we are trying to propagate inside the waveguide.

So to sum up, you have, K minus KX square, minus KY square, minus beta square, plus k0 square, epsilon R is equal to 0. Rearrange the equation, you can write for beta, as square root of, k0 square epsilon r minus KX square plus KY square. Okay? So, the sum which is KX square plus KY square can itself be redefined, so we can redefine this KX square plus KY square, as some KC Square. Okay? If you wish and then, the equation for beta will be square root of K 0 square epsilon R minus KC square, you can even simplify this k 0 square epsilon R, by writing this as K square, okay. With the assumption that, the permittivity of the material, has been absorbed into that definition of K itself. Okay? So, with that, beta can be simply written as, the square root of K square minus KC square. But, again please remember, we don't know what KC's? And what is beta? of course, to know that, we have to know what sort of fields, can we write for the electric field component Z, which is what we are solving here. and from the knowledge of E Z plus, the knowledge of H Z, you should be able to then write down the other field components and then apply the boundary condition. and hope is that, if everything goes well, then you will be able to determine what KX is? what is K Y? And hence, determine what KC is? And therefore, determine what would be beta, for a given wavelength or for a given mode that is propagating. Okay?

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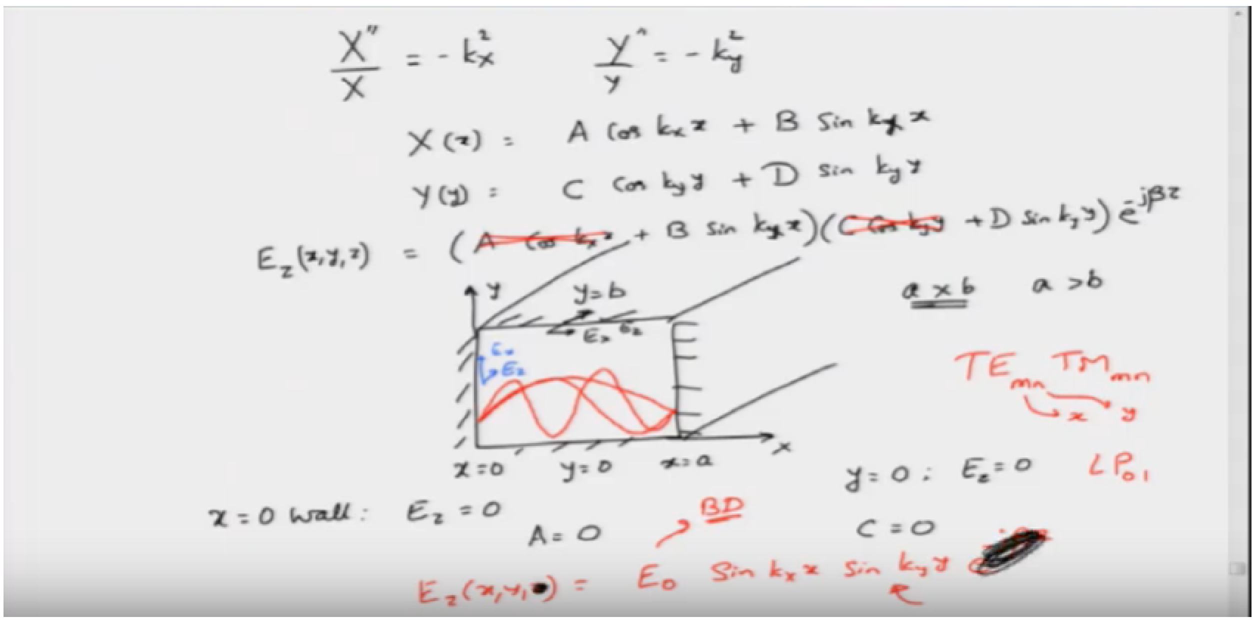

So, we will first look at this E Z, we have two equations to solve for, you have X double prime by x equals minus KX square and y double prime by y is equal to minus KY square, from your classes on differential equations, you know that, these two are two partial differential, sorry, two ordinary differential equations of second order. And the solutions, can therefore be written as, sum a cos KX, X plus B sine K, K xx where a and B are constants, which we do not know yet. But, which is fine, then you have Y of Y, which is obtained by solving this second, you know, a differential equation and that would have, additional constants. So, we will call this as say, 'C cosine k YY'. Okay? Because, this is the solution of the second equation. And D sine K y Y. Okay? So, together you can write for the Z component of the electric field, as a function of all three components X Y and Z as, A cos KX, X plus B sine K Y sorry, K X X multiplied by C cos KY y plus D sine K Y times y and what about the propagation part, yes. you have to write down the propagation part, as a power minus J beta Z. It is interesting that, after we have solved these two partial, I mean, these two differential equations, we end up, having more unknowns than what we actually started off with earlier. However boundary conditions will come to our rescue, we can apply one boundary condition right away here. Okay?

Recall the cross section that we have, we have the cross section, in terms of this x-axis and y-axis. Right? And you have two walls, one wall is X equal to zero wall and the other wall that you have, is X equal to a wall. Okay? and then you have a, bottom wall, which is y equal to zero and the top wall, which is y equal to B. Okay? The waiver of course is of cross section e cross B, it usually a greater than B. Okay? Now, with this one and we are looking at the E Z component, what kind of a component will be this E Z, when it you know, it's at the two walls. Now, we know that, the tangential components to the X equal to a wall, as well as for X equal to zero wall, will be the E Z component as well as, E Y component. you have to imagine that, I am stretching this guide, along the z axis in this manner and therefore, E Z it is kind of gliding along with this wall. so if you imagine that, this is the wall, then either it is just gliding along the wall and then e Y is increasing along this particular direction. So, you have this E Z and E Y, both will be tangential to this, will E X be tangential, no, e X will be pointing this way. Right? Where the pen is pointing, obviously this will not be tangential to the, X equal to 0 wall rather, this will be perpendicular to the wall. Right? Similar arguments will tell you that, E X again cannot be the tangential component, however EY I component, you can see that ,this is e Y and that E Y can be a tangential component, as well as, E Z is a component tangential. similarly for the y equal to b wall, the tangential components will be e x, as well as, E Z. Okay? So, you have both TX and E Z and now, we don't know, yet what is E Y I and E X? because, we don't know, what is H Z at this point? So, we can apply boundary condition for E Z, at these two or rather at these four walls and see, whether we are able to get something out of it. Right? Okay?

Let's, apply the boundary condition at X equal to zero wall. Right? The boundary condition will be that, E Z must be equal to zero. Because, this is a tangential electric field component and we don't have, you know and it's a perfect electric conducting wall, therefore the tangency electric field component must go to zero and this is the boundary condition. Right? With EZ equal to 0, at X equal to 0. Let's, go to this expression that we have written and then say, at X equal to 0 what will happen. We won't touch the y part; we won't touch the Z that is e Bar minus J beta Z. But, immediately when you put X equal to zero, this cosine of zero will be one, a will be present, whereas sine KX times zero which is sine of 0 will be equal to zero. Right? And because, this entire term must be equal to zero and the conclusion is that, a must be equal to zero. Okay? Now, we will also apply a boundary condition at, y equal to zero wall, which again says that, E Z must be equal to zero. and then, following the same arguments, you can show that, C will be equal to zero. these are not very complicated to find, you just have t substitute the value of x and y in the s expressions and then see, which component has to be made equal to zero. Right? So, when you do that, you will easily find out that, a is equal to zero and C is equal to zero and with that done. Right? You can eliminate these terms, from the possible solution set, interesting. Right? and so, that the overall solution that, you now have for E Z will be, E Z of XY and Z, will be some B times d ,which itself could be a constant. So, we will have to call this some other constant. So, what shall we do? we will call this as, 'some E 0'.okay? So, E Z Rho is a constant, which is basically the product of B into D, because B and D both were constants, we call the new constant as E 0. And then what you have is, sine K X x, times sine K Y y. Right? So, if you take Y is equal to constant plane and then, actually plot what you are going to get for ,the electric field as a function of Z, you would see that, this electric field would actually go something like this, it would be sine, so it would be zero at the center. And it would actually go to sorry, it would be zero at the edges and it would go to a Maxima at the center, if this is the lowest, half wavelength that we can fit. Okay? Of course, you can also have additional type of solutions, in which case you will have this one, you can have even more number of points. But, all the time please understand that, as you move along X, the solution will be in the form of a, sinusoidal wave, with it going to 0 at the two edges and having multiple Maxima, in fact you know that, this is, this kind of a mode function was exactly what we had for a, parallel plate waveguide. Okay? That we tackled in the earlier module and there, we gave different mode numbers. Right?

So, if you had only one Maxima, it was T E one, if you had two Maxima, it was T E two and so on and so forth. Right? So, in much of the same way, you can actually have, a numbering scheme for, this one as well. Except that, you now have to field, two dimensional field distribution, is not just it is varying along X, but because there are top and bottom walls, there will be a sinusoidal variation along, Y as well. Right? Which is what you can obtain? or which you can see from this sign K by Y term. Right? So, there will be a sinusoidal variation in the X, Y direction, there will be sinusoidal variation in the X direction. Therefore, all modes whether we are dealing with the transverse electric mode or transverse magnetic mode, they will come with, two numbers with them. Right? these numbers can refer to the variations along X and this n will refer to the variations along Y and in contrast to that parallel plane waveguide, you now have to specify every mode by two numbers. this not something new, for the fiber we have already done that, so you had for the fiber, lp0 one mode, LP 1 1 mode, LP 2 1mode and so on. So therefore, this makes sense that, you actually have two, different you know, indices to go with, whether you are dealing with the TE mode or a TM mode. Okay? So, anyway with that, as E Z overwrite, let's also put down E Bar minus J beta Z. But, because the E Bar minus J beta Z is a common thing, I won't carry it further, okay. So, I will just remove this one, but please remember that, that would always be there. Okay? So, we have found E Z and we have seen that, it would be in the form of sine and sine. Can we try and find out, what would be the solution for H Z.

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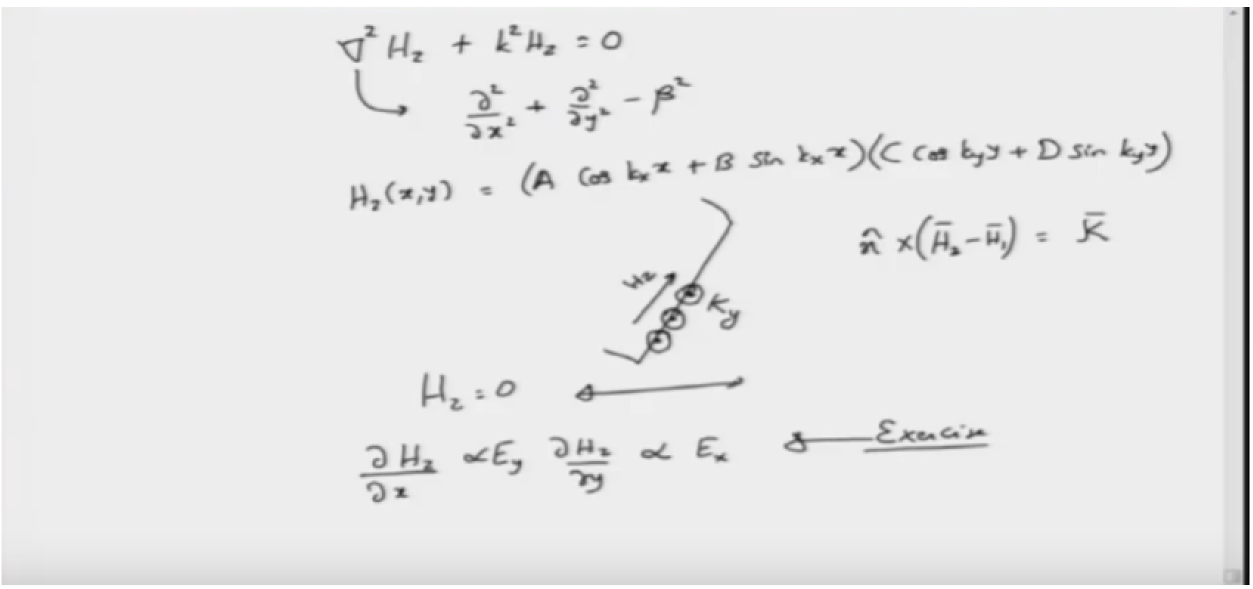

See, the Helmholtz equation for H Z will be exactly the same equation, as that off for the E Z component .So; you have Del square H Z plus k-square. Right? h z is equal to 0, I can write this del square, as del square by Del X square plus del square by Del Y square minus beta square, so the equation essentially goes back to the same thing. So, in following the same steps and removing the Z dependence or suppressing the Z dependence, H Z as a function of x and y, would be again, some a cos KX x plus, b sine KX x times c cos KY y plus D sine K Yy . Right? Okay? Now, you have to apply boundary condition. Now, here is where it gets a little bit of a tricky thing. Because, there is no boundary condition for the tangential H Z, when there is a conductor. Right? Because, you have a conducting wall here. Right? If you consider H Z component here, then there will be a current sheet that would be present, which would be along the Y direction. Right? Which would be along the Y direction, in this particular case? Okay? So, and you will of course have to also imagine that, there will be an edge wipe, along a particular wall and because of that H Y, there will be another current sheet .Right? So, basically there will be this surface currents .Okay?

which would be propagating or which would be present or which would be induced the walls and because, those currents are present, you cannot take the time mention magnetic field components to be equal to zero, in the case of a, you know, perfect electric conductor to dielectric interface, something that we already have seen, in the equation when we read, when we wrote, head N cross, H 2 minus H 1 to be equal to K, where K was the current sheet and hedge 2 minus H 1 was the, difference in the tangential electric field. Of course you take H 2 equal to 0, but you still have to deal with the fact that, the boundary condition for the tangential electric field is now, discontinuous or predict that, there is a discontinuity in the magnetic field and this discontinuity is actually taken up, by the current sheet K. Okay? Depending on, whether you are dealing with H Z or H Y or H X, the direction of the current sheet will also differ, because there is a curling operation out there. Right? So, will not worry about that, but the main point to note here is that, I can't make H Z equal to 0, at any of the two walls, then what should I do? Well it turns out that. since H Z is or since the derivatives of H Z del H Z by del X and del H Z by Del Yare respectively proportional to, some E Y, as well as, e X component, depending on the wall you have to invoke, the derivative being equal to 0. Right? Because, e Y and DX are tangential components, E Y is tangential at X equal to 0 and at X equal to a, E X is tangential at y equal to 0 and y equal to B. So, based on which wall you are looking at, you take the derivative of H Z, whose expressions we have already turned down and then set that to 0. Okay? To eliminate the constants a, b, c and d. and then, to simplify the solution. as before i will leave this as an exercise to you. So, i am going to not deal with this, but it is important that you actually carry it out. Okay? So, what I wanted to say is that, compared to how easily we were able to evaluate E Z, you know and eliminate certain constants, it is not so simple to do that for H Z keys .Okay? However, we are not done with the E Z yet, I mean, we did apply the boundary conditions at X equal to 0 wall and attic, at y equal to zero wall. But, we still have two additional walls to, work out, that is X equal to a wall and y equal to B wall. Right? So, let us apply the boundary condition to X equal to a and y equal to B, for this expression. Okay? So, when you write down that.

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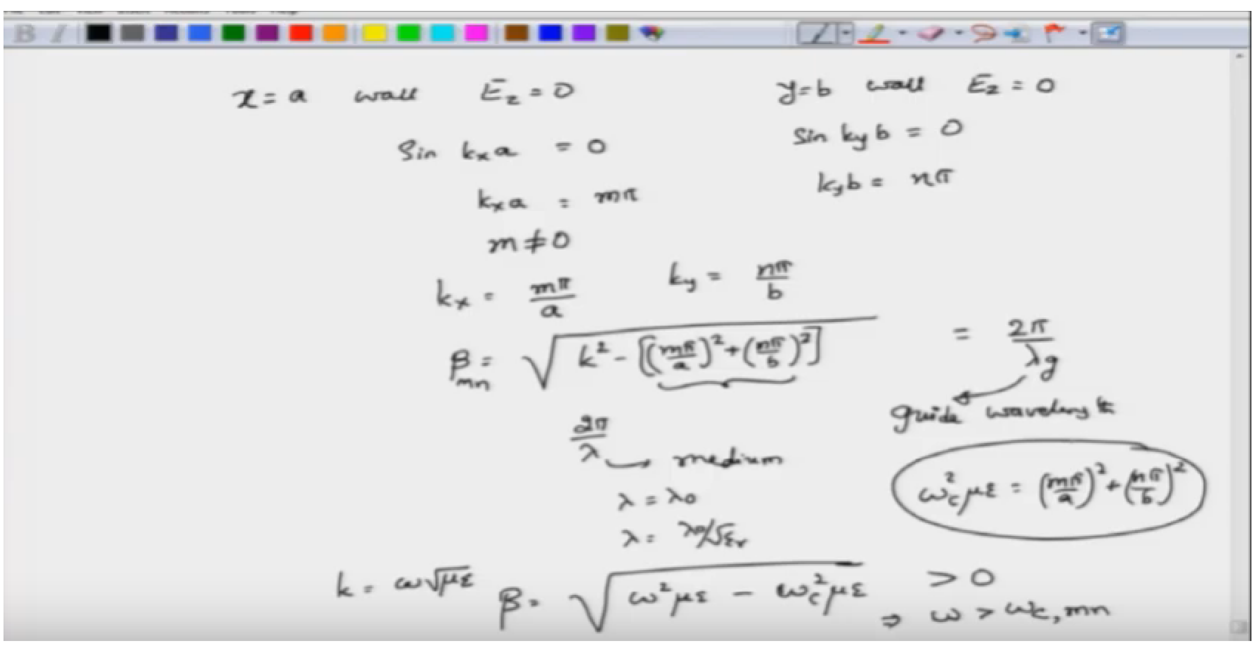

So, let us go to a new one. So, when you write down, the boundary condition at x equal to a wall. and demand that EZ must also be equal to 0, what you are actually demanding is that? this sine KX x, evaluated at sign K X x, evaluated at X equal to a, the equal to 0. Okay? Similarly when you write down, boundary condition at y equal to B wall, for E Z and say that, EZ must be equal to 0. The condition on the y dependent function would be, sine K Y B must be equal to 0. Now, when can sign of a function or sign of an argument go to 0, whenever this argument, I mean, whenever this is a multiple of Pi, sign will go to0. Because, sine 0 is 0 sine, PI is 0 sine2, pi is 0 sine 3, pi 0 and so on. Right? So, M is an integer, of course M must not be equal to 0, because when M is equal to 0, every field will be actually equal to 0. Okay? So, that condition we don't allow, similarly sine K y B equal to 0, implies that K Y B must be equal to n pi. Right? From which, you can immediately write down what is K X? which is M PI by a and K Y will be equal to n PI by B. and therefore, beta can be obtained, even without finding the other field components, in a very simple manner, as K square minus M PI by a whole square plus, N PI by B whole square. Okay? So, this is very important thing to note down. And we can also write down beta as, 2 pi by lambda G, to denote ourselves that, lambda G is what is called as the guide wavelength? Okay? That is the wavelength, along the Z direction, in the waveguide where as K, which would be in general given by 2 pi by lambda or 2 pi by lambda, where lambda can be called as the,' Medium Wavelength'. Right? So, if the material is filled, if the waveguide is filled with air, then lambda will be equal to lambda 0 that is, it's free space value, otherwise lambda will be filled by, lambda divided by epsilon R, our lambda 0 divided by square root epsilon R, we can kind of find this thing out. Right?

So, you have found beta, but again because you have these different values of M and n possible, you put a subscript on MN and you put a subscript M N and for beta as well, indicating that, not just one more is possible, but there are many modes which are actually possible, for us to find. Right? There's another relationship that, one can actually obtain, because K is actually equal to Omega square root mu Epsilon. Okay? Where Absalom will be equal to epsilon 0, epsilon R. and this M pi by a square plus n PI by B Square, can also be rewritten, so we will do that one, so beta can be written as Omega square, mu epsilon minus Omega C square mu Epsilon. Okay? So, I can write beta as, Omega square mu epsilon minus Omega C square mu epsilon ,where Omega C square of mu epsilon, will be equal to M PI by a whole square plus n PI by B whole square. And obviously beta will be positive, you want beta to be positive, otherwise modes will not propagate, but they will simply decay out. So, beta has to be greater than zero, indicates that, Omega has to be greater than Omega C. Okay? And what is the Omega C? Well again, Omega C also needs to be written with the subscript M and n, because you have two possible values of M and n that would satisfy this particular equation. Okay? So, this expression, Omega C square mu epsilon, which is equal to this thing.

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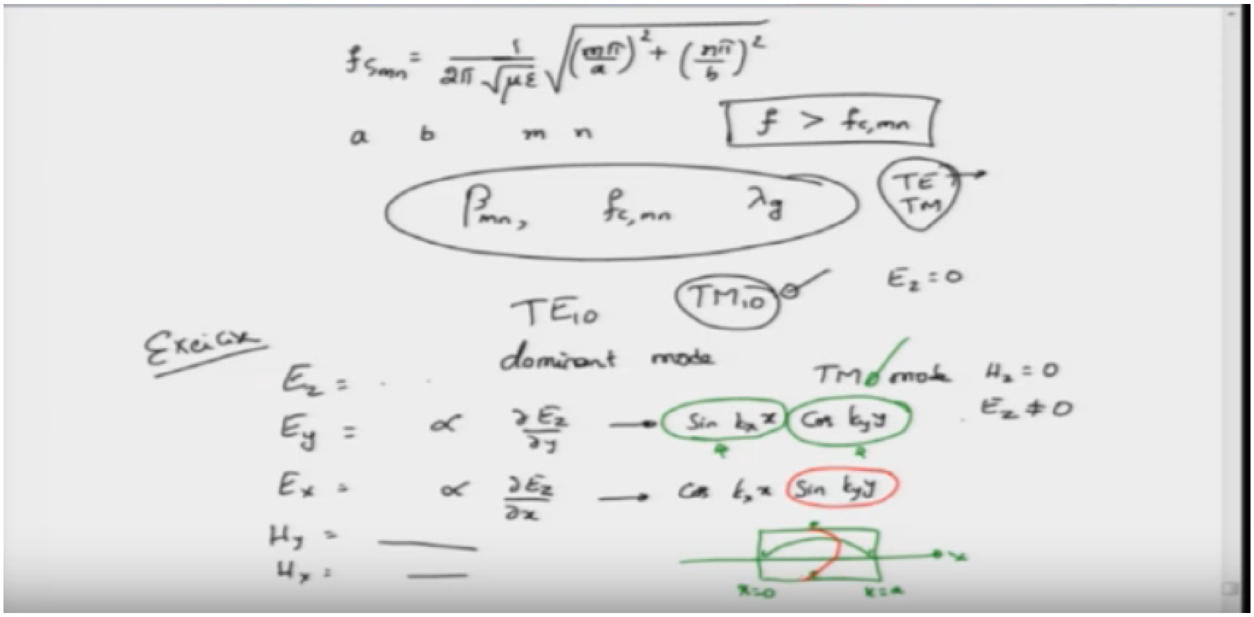

Can be rearranged, to find out, what exactly is Omega C? Omega C will be 1 by square root mu epsilon, M PI by a whole square plus n PI by B whole square under root. And because, I don't like Omega, I like F, I can even write this as FC, as 1 by 2by square root mu epsilon times, square root of this. So given A, given B. Right? You can keep changing m and n, to determine the cutoff frequencies for different modes, because the corresponding frequency F, at a particular mode, which you want to propagate, must always be greater than this minimum value FC. Okay? Because, with F is less than FC, the component here or the term here, Omega C square mu epsilon, will actually be more than Omega square mu epsilon, causing beta to become complex. But, because your solutions are X the form E Bar minus J beta Z. and beta becoming complex means that, the waves actually decayed down, as you propagate e along the waveguide, you don’t want that, you want ways to actually propagate through the waveguide. So for that, to happen beta has to be positive and for positive value of beta, Omega must be greater than Omega C and because, Omega C itself is different, for different modes, at a given frequency only a certain number of modes can actually propagate. Okay?

So, you can actually cut off many different frequencies, for example, let's say we find out what are the two fundamental modes in this waveguide, I haven't talked about it, but we talked about it later on. So, I will find out two fundamental, I mean, two modes of this one. where the first one is the, first value the smallest value of FC that you can get for, M and N. and then, there is a next mode. Okay? whose value FC also will find out. Now, if you want your wave way to operate, only in the single mode regime that is, it must carry only one particular mode, from point A to point B, then you can actually choose the frequency F, to be such that, it is greater than the first mode, which you want to propagate. But, it must be less than the mode, which you do not want to propagate. However, if you want a lower order mode, to not propagate, by lower order we mean the mode, which has a lowest cutoff frequency, if you don’t want that to propagate; there is nothing that you can do. Because, as you increase the frequency, anything that would be you know, that mode would be present whose cutoff frequency is actually less than, the operating frequency, will begin to propagate. So, in that case we say that, the waveguide is multi molded. and multimode in many cases is not a very good idea, because the it will, not the energy and splitted according to different modes in, in certain manner and the propagation also, will be a little bit of a complicated thing, so we don’t want that, but, sometimes you can't help it. Okay? So, most of the times you design your waveguide in such a way that, for a given frequency range, it should hopefully operate only as a single mode thing. and you can, design such waveguides in the exercises that we will give you. Okay?

Having said that, whatever we have said so far, in terms of beta MN. Right? Interims of the cutoff frequency, M and N and even the waveguide lambda G, which you can find out, note by writing down. So, beta was actually equal to 2 pi by lambda G and since beta MN you know, you can also find out, what is the corresponding guide wavelengths. These are all characteristics of the waveguide geometry; they do not depend on whether you are dealing with a TE or a TM mode. Okay? Of course, once you fix a value of M and n, you have to go back and evaluate, whether that particular T E field components are nonzero or not. for example, the lowest order mode in this particular waveguide is, what is called as, 'T E 1 0', which we will study in detail later on. Okay? There is no corresponding TM 1 0. Because, when you look at, what would be the expressions for transverse magnetic1 & 0, you will find that electric field component is it will be 0. Okay? And because of that, all the other components will also be equal to 0. So this condition, does not really occur, in the sense that, although the cutoff frequencies for TE 1 0 and TM 1 0 are the same, the fact that there are no field components, which are nonzero for, the TM 1 0, means that, that more would not exist.

However, this p 1 0 mode would exist and in fact, it has the lowest, cutoff frequency that is, this would be the mode, that would propagate, right After the frequency F is greater than, F c10. Okay? So, this is the mode that would always begin propagating first, before the other modes can begin to propagate. and as such, this mode is called as, 'Dominant Mode. Okay? We will see what the field expressions for this dominant mode are, but what I wanted to point out is that, all the other characteristics that we saw, but all independent, of whether you are dealing with the TE mode or a TM mode. Okay? Now, without too much of a derivation, I would simply like to give you, the field expressions in the transverse magnetic mode. Okay? When I say, transverse magnetic mode. Okay? I am going to assume that, H Z will be equal to zero. Okay? causing only E Z to be nonzero and because of that, you can actually still see, what would be e Y, EX and H Y and H X, E Z we have already returned down. E Z will be sine and sine and then what would be e Y, well I had asked you to look at the expressions for this, you know, E Y, E X and other things. Right? So, EY will be equal to or will be proportional to, del e Z by Del Y, similarly e X will be proportional to, del Y Z by Del X. Okay?

So, the field components for E Y will be sine K X X but, it would be cosine K Y Y, whereas for the X component it would be, cos K X X and sine k YY. Okay? Please verify this, this is a big exercise. Okay? which follows, the results of which follows from earlier solutions, of the earlier exercises, in the assignment or in the handout, we will actually give you the complete derivation of this equation, so that you don't have to, you know, runaround finding they are revelations? But, I strongly suggest that, you carry out exercises as I have suggested, then you will be able to find these expressions for yourself. Okay? what I want to tell you, at this point, is that well H Y + HX, I will again leave it as, an exercise for you. what I want to tell you here is that, are these solutions sine KX x and cos KYY, plausible from a boundary condition point of view. Right? To do that, let us write down our cross section again, so I have X equal to 0 wall, X equal to A wall. Right? now look at E Y, E Y is a tangential component at X equal to 0 , it is also a tangential component at X equal to A. Meaning that ,it has to go to zero at these two points, the function that would make it go to zero at these two points, would be a sine KX X, that is as a function of X, the E Y component must go to zero, at X equal to zero and X equal to A, consequently it should behave as a sine thing.

However at y equal to 0 and at y equal to a, EY is not tangential, E Y is normal. Okay? And this normal means, they need not necessarily go to 0, which is captured by this cos K y y, well the argument is very strong for the sine part, not so much for the cosine thing, but anyway we'll take it to the case that, when the boundary conditions are not, asking specifically the fields to go to zero, then the possible solution is a cosine wave, not a sine wave. Right? And you can show again, by following this kind of a logic, that EX should go to zero, at both y equal to 0, as well as, a Y equal to B. and therefore, its solution in terms of sine k YY is valid. Right? Because, e x must also go to 0, at the two, walls bottom and the top ones. Okay? So, these are the field equations for TM mode, we will have to write down the field equations for TE mode. But, for TE mode the solution is slightly, complicated or longer, it's not complicated I'm sorry, it is basically just longer. Because, I have to first find out H Z, set the derivatives of H Z with appropriate x and y derivatives to 0, at the appropriate walls. and then, obtain the other field components and then go ahead and find out the corresponding field solutions or the form of the fields. Okay? You can do that, in the next model, I will give you the expressions and you can verify that, you have actually done it correctly, by looking at your expressions and verifying it with the expressions that, we will give. But, what is important in these two aspects is that, most of the times, you can actually write down the solution, by looking at the boundary conditions or at least after you obtain the solutions, you can verify that, they actually satisfy boundary conditions and therefore, the solutions are at least, correct in terms of boundary conditions. Okay? So, we will continue the discussion of T E modes and as well as, point out couple of things, in terms of TN and TM modes, in the next module. Thank you very much.