Applied Optimisation for Wireless, Machine Learning, Big Data
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Lecture – 73
Examples on Duality: Dual Norm, Dual of Linear Program (LP)

Hello, welcome to another module, in this massive open online course. Let us continue looking at examples and in this module let us looking, let us start looking at examples pertaining to duality ok. So, we want to start looking at examples related to concepts of Duality ok, and what we have seen is the following.

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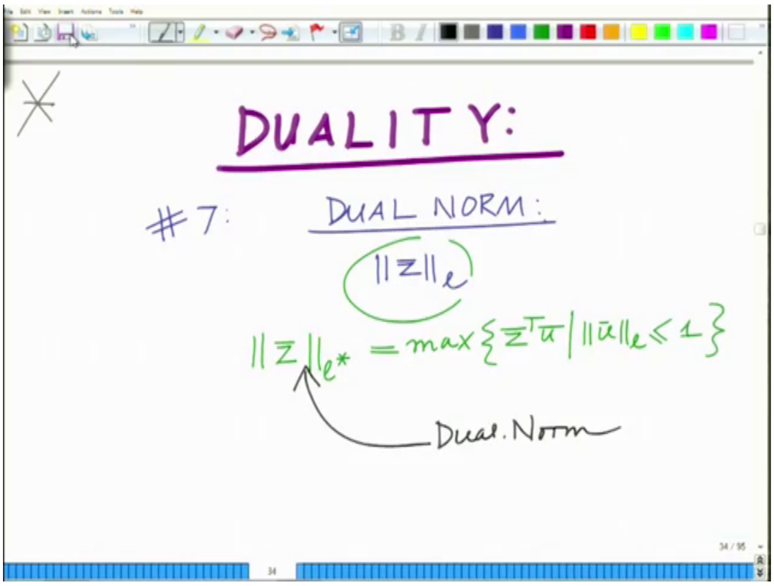

Let us start with the first example that is something interesting pertains to this concept of a dual norm. And now, the dual norm for instance if you have a vector x bar and this is the l norm alright, for instance, l can be 2, that is the l 2 norm or one l 1 norm and so on. Now, the dual norm of this is denoted by norm z l star, which is defined as the maximum of z bar transpose u bar or all elements, such that norm of u bar l is less than or equal to norm of u bar l is less than or equal to 1, ok.

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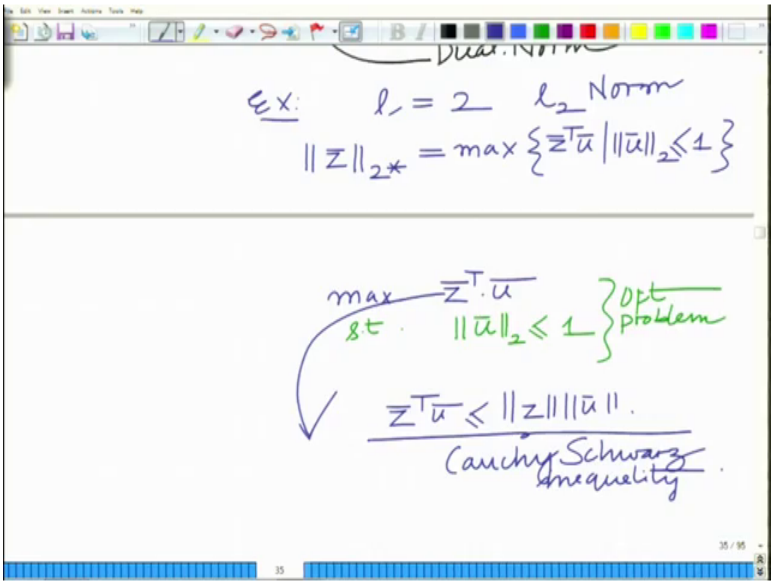

So, this is basically the dual norm. So, this is the l, this is the original norm and this is basically the dual norm, this is the dual norm and now, for instance, let us look at some examples of course, this is simply a definition. Let us look at some examples to understand this. Let us consider l 2, that is we are talking about the l 2 norm. We are talking about the l two norm, therefore, what is the dual norm?

That is the dual norm of the l 2 norm is maximum over all z bar transpose u bar, such that the 2 norm, l 2 norm of u r is less than equal to 1 for instance. Now, let us look at this, let us look at, so this is basically, now if you look at this, you have this maximum of z bar transpose u bar. Over all such vectors norm u bar 2 is less than equal to 1. This is the pertinent optimization problem and of course, you can see, this is convex in nature because, this is a, linear objective convex constraint ok, and now this is easy to solve.

In fact, we know that, magnitude of z bar transpose u bar is less than or equal to, in fact, z bar transpose u bar itself is less than or equal to norm of z bar into norm of u bar. So, we know for two vectors z bar and u bar, z bar transpose u bar is less than equal to norm of z bar times norm of u bar. All right, the dot product is less than equal to the product of the norms. This follows from the Cauchy Schwarz Inequality this is from the.

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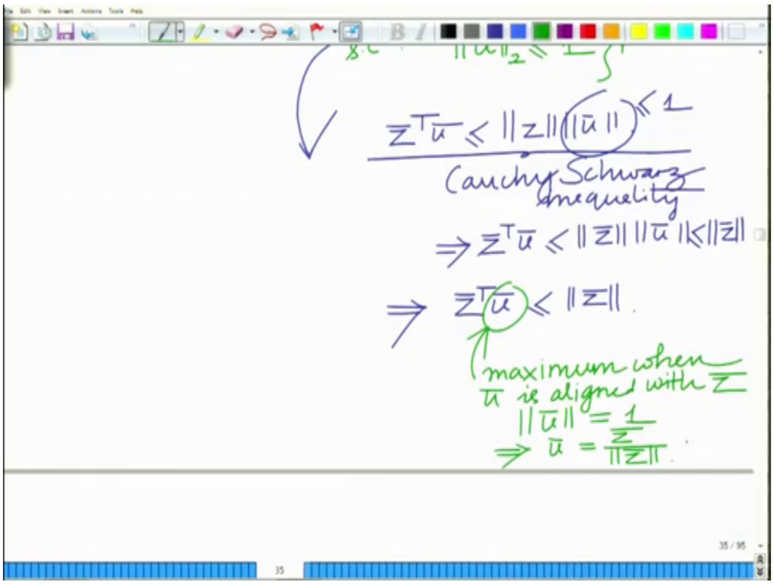

Now, we know that this norm u bar is less than or equal to 1, which basically implies z bar transpose u bar is less than equal to norm z bar times. Norm u bar, which is in turn less than or equal to norm z bar because, norm u bar is less than equal to 1, which implies that basically, z bar transpose u bar less than or equal to norm z bar and when does the maximum error, we know the maximum occurs when u bar is aligned in the direction when maximum for u bar is aligned with z bar, u bar is aligned with z bar and norm u bar equals 1, which implies u bar equals z bar divided by norm z bar.

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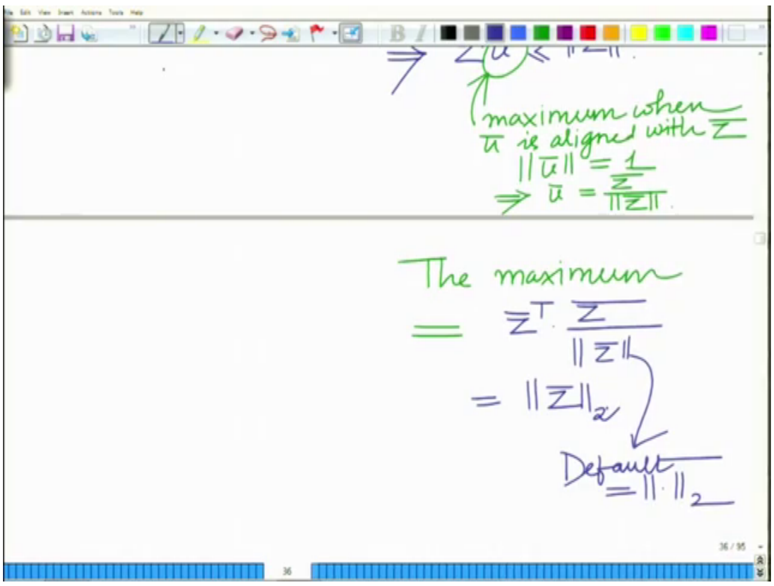

So, the maximum occurs, when u bar z bar divided by norm z bar and the maximum is z bar. The maximum equals well z bar transpose substitute instead of, u bar substitute z bar divided by norm z bar. So, this is norm z bar square by norm z bar. So, its norm z bar square by norm z bar. It is a norm z bar of course, all these are 2 norm because, as I mentioned when there is no norm mentioned by default 2 norm ok.

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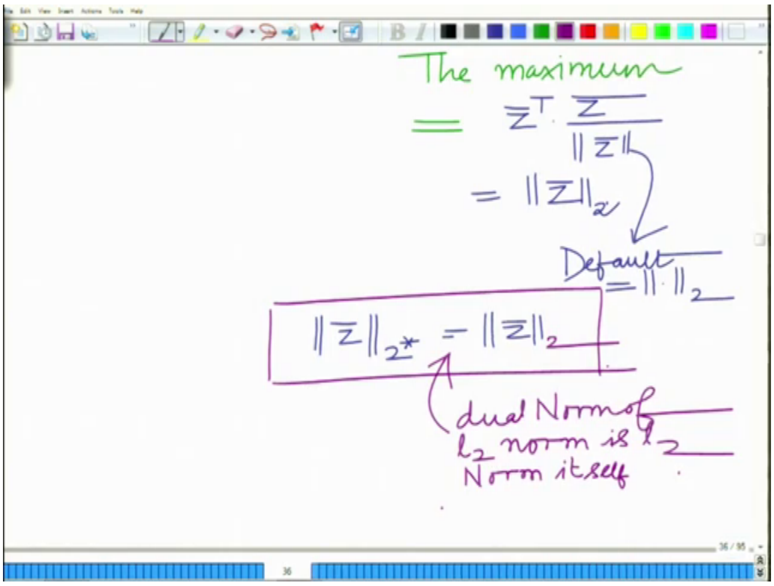

And therefore, you observe something interesting, what you observe is that the dual norm of the 2 norm equals the 2 norm itself ok. So, this is very interesting dual norm of the l 2 norm is the dual norm of the l 2 norm is the l 2 norm itself ok.

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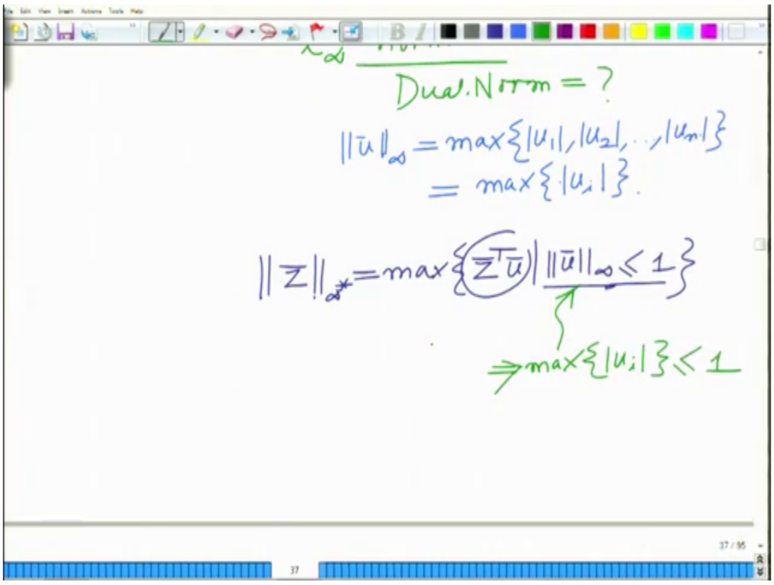

Now, let us look at another interesting one, do you want to consider (Refer Time: 07:01) the l infinity norm? We want to ask the question, what is the dual norm of the l infinity norm? Now remember l infinity norm, that is norm of l norm of u bar l infinity, infinity is simply the maximum of magnitude u 1 magnitude u 2 magnitude u n or simply the, maximum of the magnitudes of all components of this.

Now, what is the dual norm of the infinity norm, that is norm z bar of infinity dual norm is the maximum, that is your maximum of z bar transpose u r such that the infinity norm of u bar is less than or equal to 1, now what is z bar transpose u bar, now norm infinity norm of u bar less than equal to 1. This implies maximum value of magnitude u i is less than equal to 1.

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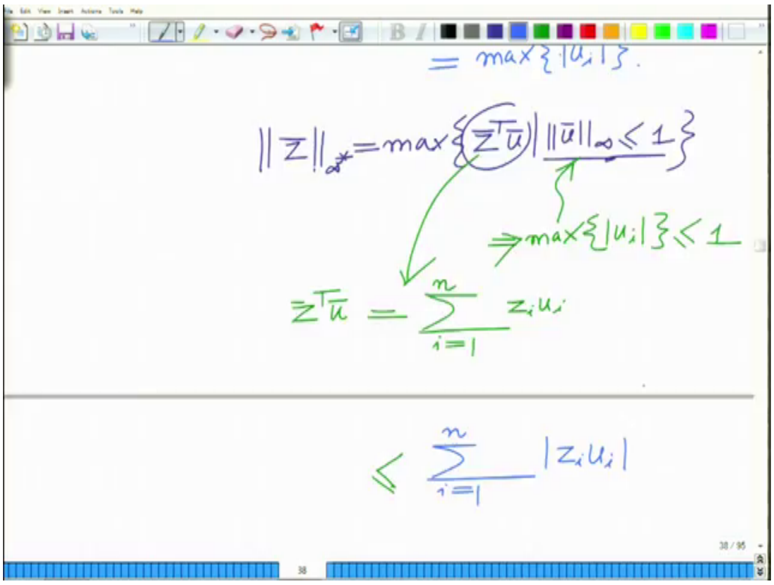

Now, assume z a z bar and u bar both to be n dimensional vectors. Now, what is this? This is simply equal to summation i equal to 1 to n z i times u i, and now if you look at this. Now, therefore, now this is your z bar transpose u bar, which is simply the dot product between two. It is a summation of component y, that summation of component y s product, that is summation of i equal to 1 to n z i times u i, when n is the dimension of each vector. Now, this is less than or equal to; obviously, the summation i equal to 1 to n magnitude z i magnitude z i times u i, which is equal to summation i equal to 1 to n magnitude z i magnitude u i.

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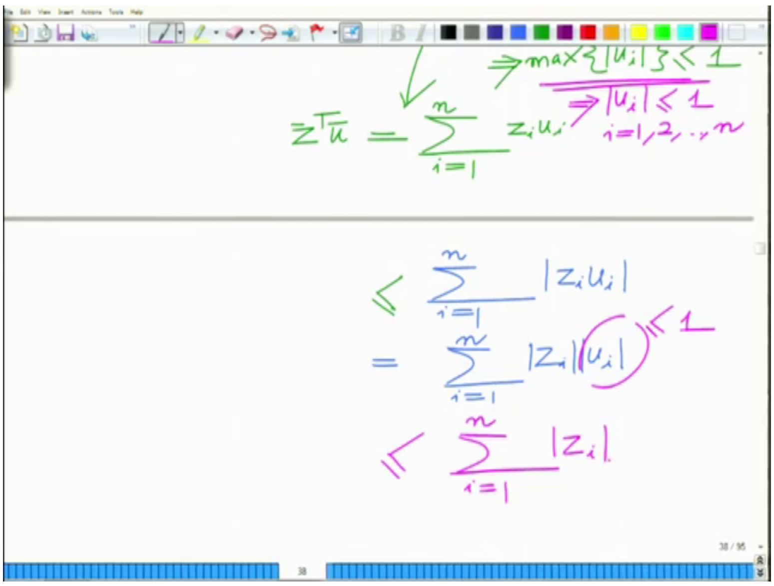

Now, we know magnitude u i less than each magnitude, u i is less than equal to 1. Now, look at this maximum value of magnitude u i less than or equal to 1, this implies magnitude u i less than equal to 1 for all i, i equals 1 2. Since, the maximum itself is less than equal to 1, it means that the magnitude of each component of the vector u bar has to be less than equal to 1, naturally ok. Since, the infinity norm l infinity norm is less than equal to 1 and that gives us a very interesting expression.

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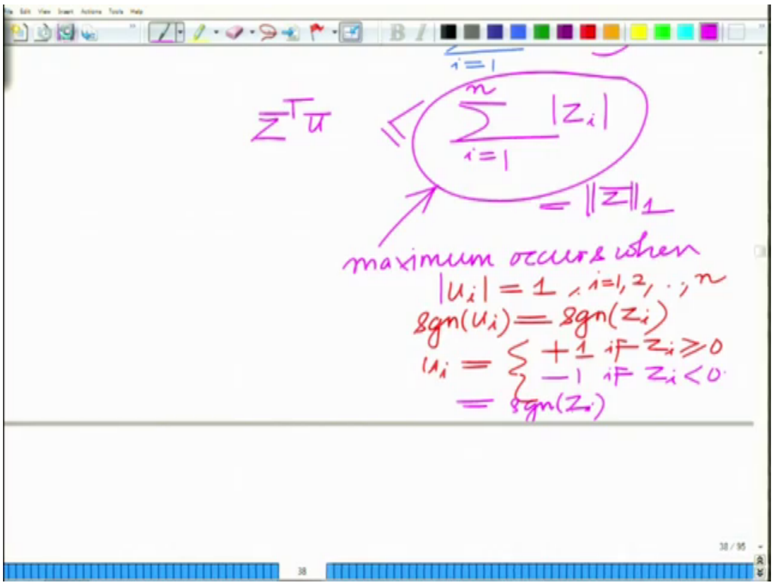

So, this is less than or equal to summation i equal to 1 to n magnitude z i and in fact. So, that gives us the expression, that z bar transpose u bar is less or equal to summation magnitude z i. Now, does the maximum occurs; yes, occur yes maximum occurs. If you think about it, when magnitude u i equal to 1 for each i, that is for all i magnitude u i equal to 1 and the sine of u i equals sine of z i.

That is what you are doing is, if u i z is positive, you are setting u i to be plus 1. If z i is negative, we are setting u i to be minus 1, that is u i equal to plus 1. If z i is greater than equal to 0 minus 1, if z i is less than that is u i is basically equal to, you can say in some sense, sine of z i all right and in that case, what is this? You can see, the maximum values achieved and what is the maximum value?

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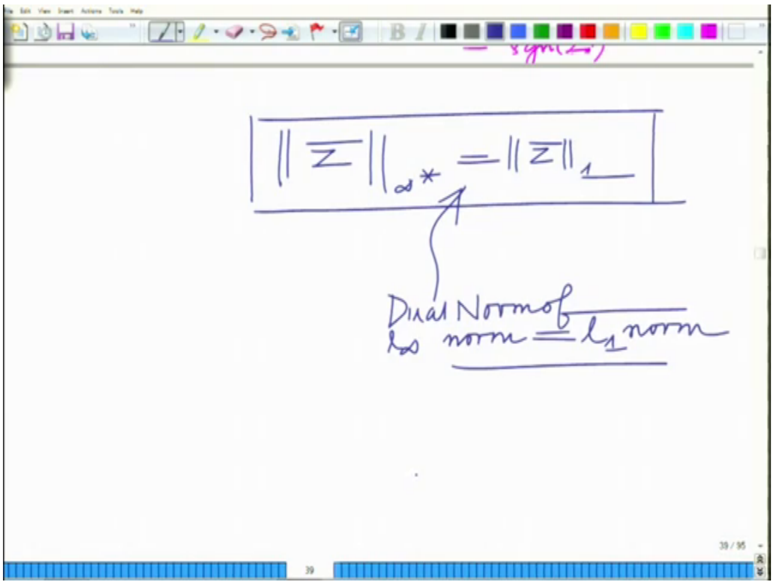

Maximum value is nothing but the l 1 norm, that is the l 1 norm and therefore, what you observe is something very interesting, what you observe is the dual norm of the infinity norm equals the l 1 norm. So, the dual norm of infinity norm, dual norm of the l infinity norm is the l 1 norm all right. In similarly, you can work out the dual norm of other norms. For instance, what is the dual norm of the l 1 norm and you should be able to convince yourself, that it is indeed the l infinity norm. These are the duals of each alright.

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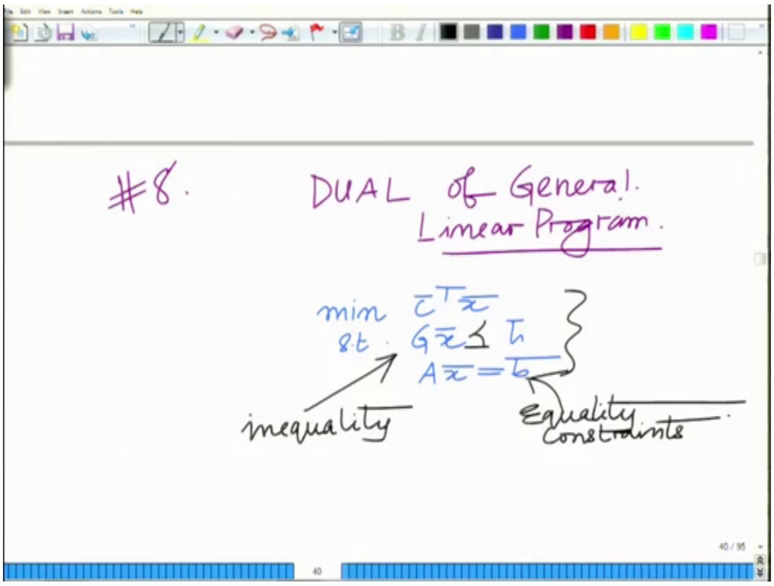

Let us look at another problem, problem number 8 or example number 8, we want to derive the dual optimal problem corresponding to general LP. So, we want to do dual of a general LP or that is your general linear program dual of a general linear program and this can be found. Now, consider your general linear program, that is your minimum c bar transpose x bar subject to the constraint that G x bar is less than or equal to h bar and A x bar equals b bar. Now, the dual problem; now what we want to do is this is a general LP; general LP means, it has of course, these are component wise inequality constraints ok.

So, each element on the left, each element of the vector on the left is less than equal to each element on the right that is of h bar. So, this is, so general LP means it has inequality constraints and it has, does inequality constraints and it has equality constraints.

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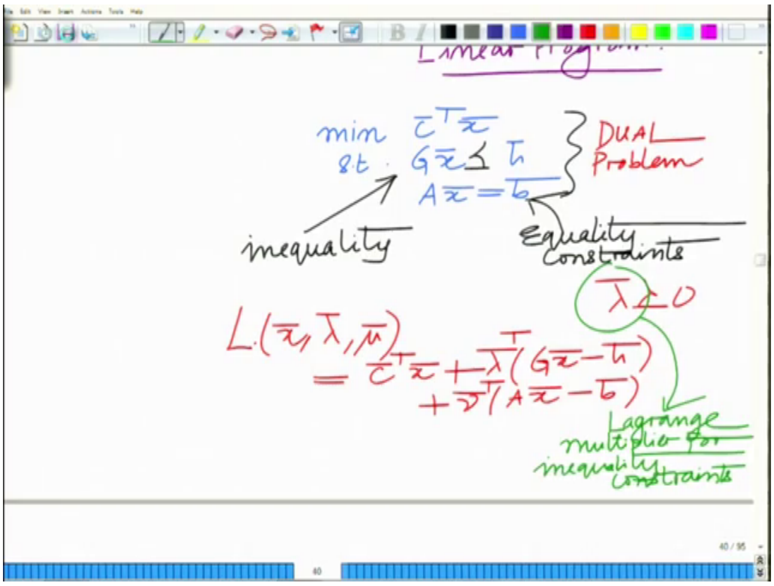

And therefore, if you look at this Lagrangian, you can formulate remember to find the dual problem.

So, we want to find the dual problem for this. This is L bar of x bar lambda bar m u bar and the dual problem of this is a c bar transpose x bar plus lambda bar, G x bar minus h bar plus n u bar lambda bar transpose plus n u bar transpose A x bar minus b bar. Of course, with each, Lagrange multiplier lambda i associated with the inequality constraints greater than equal to 0 ok. These are Lagrange multiplier these are the

Lagrange multipliers for the inequality constraints, and we have seen something similar, before we have seen the linear program with only equality constraint, but not in equality constraint.

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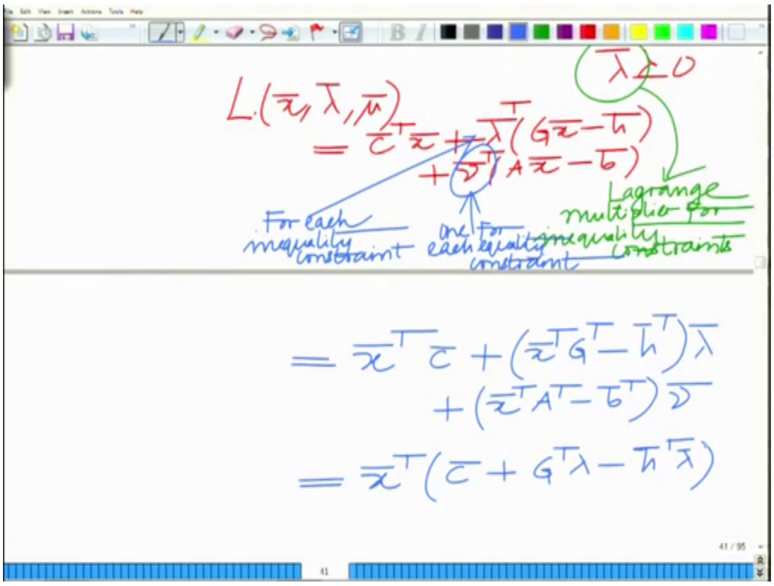

Of course, these are vectors because, you have for each 1 Lagrange multiplier for each lambda bar, one Lagrange multiplier for each inequality constraint, that is if G is m cross n, then you have m inequality constraints all right. So, therefore, you have m Lagrange multipliers alright and nu bar is basically, 1 Lagrange multiplier for each equality constraint, that is if A is m tilde cross n tilde then u bar is; obviously, m tilde. So, ok. So, 1 Lagrange multiplier or each equality 1 for each equality constraint and now I can recast this. I can recently rewrite this just write this as x bar transpose take the transpose of the whole thing because, its scalar quantity.

So, I can simply write the take the transpose of this correct x bar transpose into c bar plus x bar transpose into well, I can write this as x bar transpose c bar plus well x bar transpose G transpose, minus h bar transpose into lambda bar plus x bar transpose A transpose minus b bar transpose into n u bar and collecting all the terms in x bar transpose, this is c bar plus G transpose lambda bar minus h bar transpose lambda bar. I am sorry G transpose lambda bar plus A transpose mu bar minus this will be the constant terms h bar transpose lambda bar plus b bar transpose mu bar.

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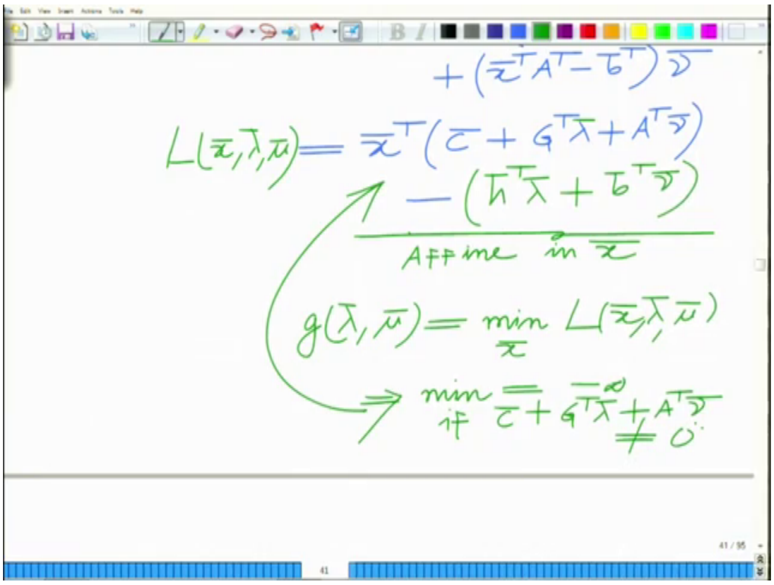

And now you will observe something interesting, what you will observe is this is a linear function of x or in fact, this is A fine in x bar ok, which means now, now we have our Lagrangian, remember correct if you look at the duality theory.

Now, we have to find G of lambda bar mu bar, which is the minimum of the Lagrangian for each value of lambda bar comma mu bar that is for each Lagrange that is for at every point it corresponds to every lambda bar corresponding to a particular lambda bar u bar. There is Lagrange multiplier vectors, we have to find the minimum with respect to x bar.

Now, you can see this is a fine an x bar, which, implies the minimum equals minus infinity if the linear term, that is a coefficient that is the, that is a vector multiplying x bar is not equal to 0. Then I can take it to minus infinity by choosing appropriate values of x because, it is a hyper plane in x, correct. This is the equation of a hyper plane alright and by choosing if this coefficient vector multiplying x bar is not 0. Then by choosing x is appropriately, I can always take it to any straight line or hyper plane I can always take it to minus infinity ok. So, this is minus infinity if c bar plus G transpose lambda bar plus A transpose n u bar is not equal to 0.

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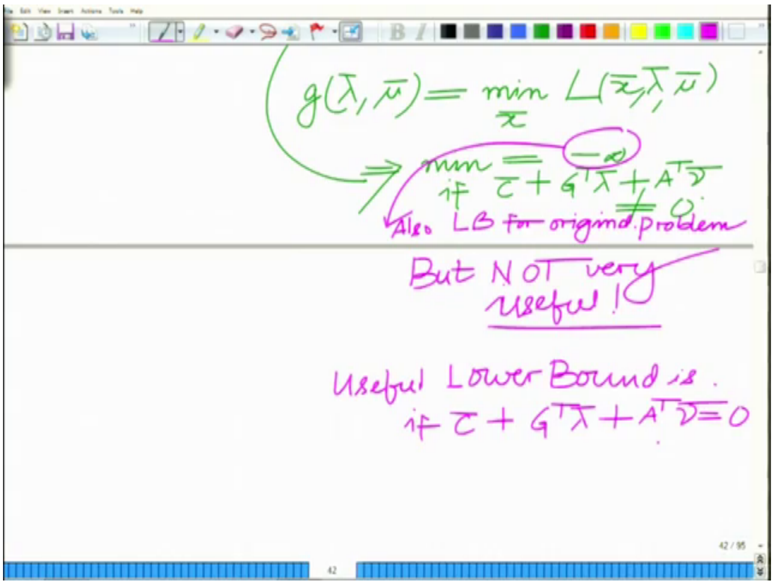

On the other hand something interesting. Now, minus infinity is also a lower bound. Remember G of this thing lambda bar nu bar is always a lower bound. So, minus infinity also lower bound for the original problem, but it is not very interesting because, minus infinity is a lower bound for any optimization problem, but it is not very; let us put it useful it is not very useful. So, instead we want a certain lower bound, which is more useful and that you will get by considering the other case, when c bar plus G transpose lambda bar plus A transpose.

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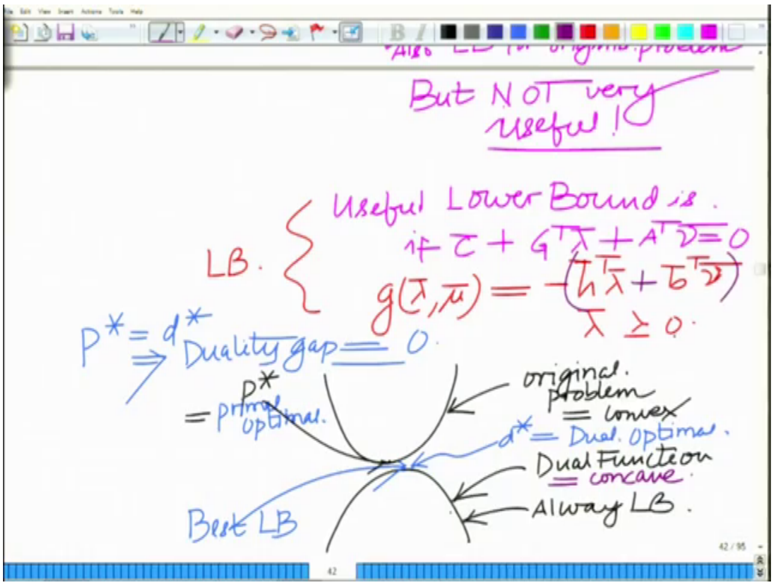

So, a more useful lower bound more useful, let us say lower bound is when, if c bar plus, when c bar plus G transpose lambda bar plus A transpose nu bar equals 0 then the lower bound if this is a constant. Therefore, G lambda bar mu bar in this context, in this case G lambda bar mu bar equals well, what is it? It reduces to the constant, which is minus h bar transpose lambda bar plus b bar transpose m u bar ok, and therefore, now, so this is a lower bound, this is a lower bound. What does this mean?

This means that, for any lambda bar nu bar and of course, lambda bar has to be remembered, that constraint is always their lambda bar has to be comprehensives greater than equal to 0. This is all the Lagrange multipliers associated with the inequality constraint have to be greater than equal to; so, for any such lambda bar m u bar satisfying this constraint alright, G of lambda bar mu bar is a lower bound for the original optimization and therefore, what is the best lower bound, that is the dual problem.

So, the best lower bound, which means something that, is close. So, everything is a lower bound, what is the best lower bound something, that is the maximum value. So, everything; so you can, if you remember the picture, this is the original problem, which is convex. This is the dual function and this is always a lower bond ok, for any the entirely lies below the optimal value.

So, this is the primal optimal and this write here is d star, which is the dual optimal and this is what we call as the best lower bound because, its closest it is the one, that is closest to the optimal optimum value p star of the primal, primal optimization problem and of course, if d star equal to p star that implies, that the duality gap is 0 the primal optimal equals the dual optimal ok, p star equals d star implies duality gap equal to 0.

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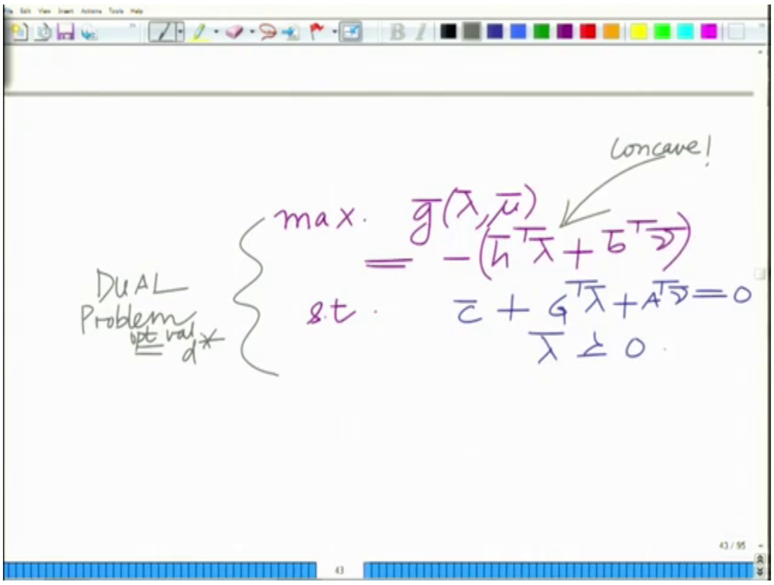

And therefore, the dual problem is basically the best lower bound, which is maximizing G bar lambda bar mu bar, which in this case, is well c bar plus G transpose lambda bar, I am sorry, which is in this case is and observe that the dual function, this is concave ok. So, G of lambda bar mu; so, this is minus h bar transpose lambda bar minus b bar transpose nu bar minus h bar transpose lambda bar. I am sorry or you can write plus, but this is brackets minus h bar transpose lambda bar plus b bar transpose nu bar negative of the whole thing ok. That is the constant.

So, minus h bar transpose lambda bar, I am just going to write it like this minus h bar transpose lambda bar minus of h bar transpose lambda bar plus b bar transpose u bar, but of course, you have constraints subject to the constraints that remember, this is only when c bar plus G transpose lambda bar plus A transpose nu bar equal to 0, and of course, each lambda bar is component each lambda is greater than equal to 0 or lambda bar is component wise greater than equal to 0 and this is the dual problem and you can see this is concave because, it is a linear function linear in lambda bar nu bar.

So, it is both convex and concave or it in particular the dual problem is concave and therefore, you can find d star this gives solution equal to the optimum value equal to d star, which is, in fact, less than equal to p star, but in this case d star will be exactly equal to p star because, this is a linear program which is a convex optimization problem. So, in general for a convex optimization problem strong duality holds, which implies that d star equal to p star ok, all right. So, we will stop here and continue with other examples in the subsequent modules.

Thank you very much.