Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 65
Example problem on Strong Duality

Hello welcome to another module in this massive open online course. So, we are looking at duality and we have seen the concept of strong duality that is for any optimization problem, standard written in the standard form, one can come up with an equivalent dual optimization problem which is convex.

You can solve that and to obtain the optimal point d star and if strong duality holds then, now usually d star is less than equal to P star where P star is the optimal value of the original primal problem, but when strong duality holds which is usually true for a convex optimization problem we have d star equals P star ok. And now let us understand that through an example alright.

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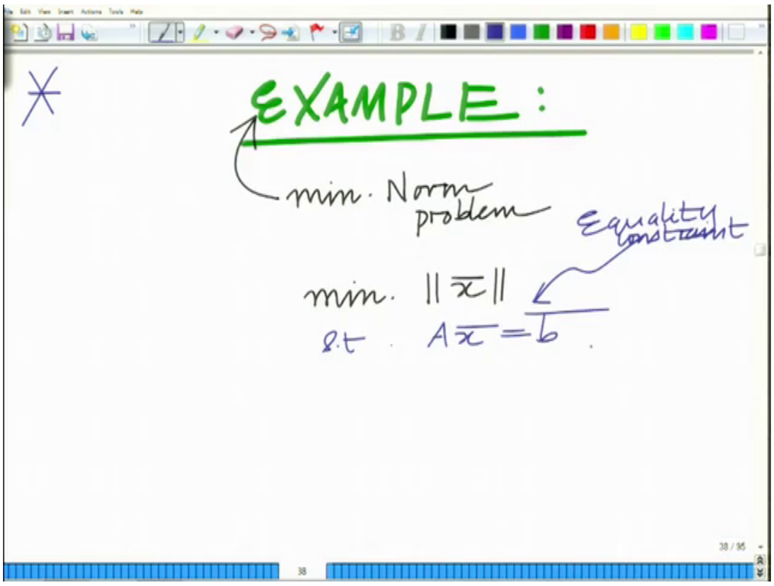

So, let us look at an example to understand the same, this explore duality, in particular let us look at the minimum norm problem that we have seen so far. So, as an example let us look at the minimum norm problem and the problem is the following minimize norm of a vector A bar. So, this is your objective, subject to the constraint A x bar equals b bar ok.

And you can see these are only equality constraints, there is no inequality constraint ok. So, this is only an equality there is only an equality constraint to approach this problem let us form.

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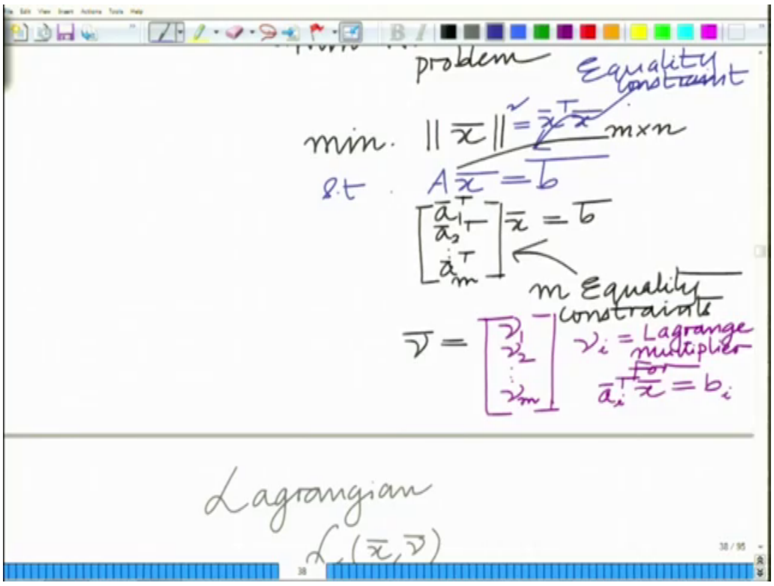

Now, look at this has although this might seem like a single constraint, if A is an m cross n matrix ok, which means it has m rows a 2 bar transpose a m bar transpose correct.

It is an m cross n matrix this is a m bar transpose and x bar equals b bar so, in reality there are m constraints ok. So, there are m equality constraints one for each row of the matrix A. And therefore, the Lagrange multipliers for each equality constraint, you need to have one Lagrange multiplier for each equality constraint so, you have a vector nu 1 nu 2 nu m where nu m is or nu i equals Lagrange multiplier for the constraint, a bar transpose x bar equals bi ok.

So, this is what the ith constraint alright. So, you have m constraint m constraints 1 corresponding to each row of the matrix a the corresponding Lagrange multiplier is nu i alright and therefore, you have the vector, nu bar which comprises of the Lagrange multipliers nu 1 nu 2 up to nu r.

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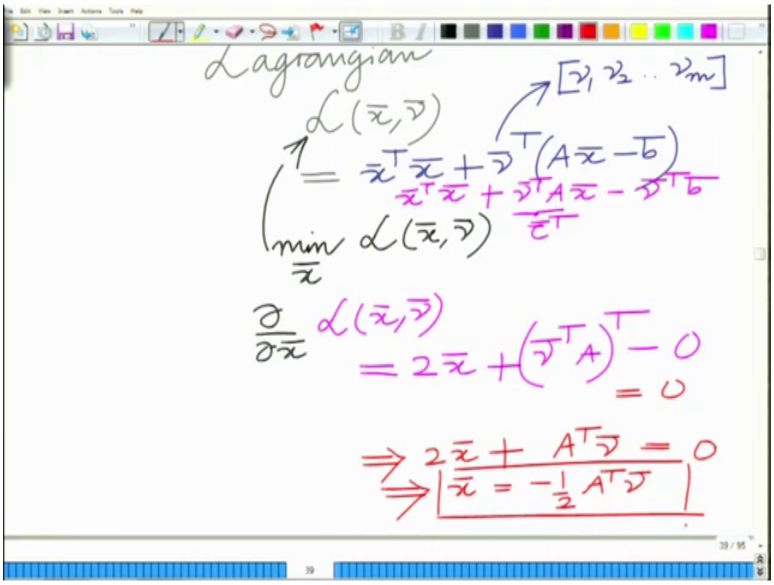

Now, the Lagrangian can be formulated as follows, in the Lagrangian that is a L of x bar and only nu bar is no lambda bar, because there is no there are no inequality constraints.

So, this is basically the objective function now, instead of minimizing norm x 1 can also minimize norm x bar square which is basically equal to x bar transpose x bar ok. And therefore, this you can write the objective function was x bar transpose x bar plus nu bar transpose ok. So, basically each Lagrange multiplier is multiplying the corresponding law A x bar minus b bar ok.

So, this nu bar transpose is basically your row vector nu 1 nu 2 up to so, you have each Lagrange multiplier multiplied in a corresponding row. And then you are taking the sum that is what this nu bar transpose is doing. Now, we find the minimum all right in the first type find minimum of the Lagrange multiplier of the Lagrangian of the Lagrangian with respect to x bar.

So, which means we have to compute the partial derivative with respect to x bar and set it equal to 0 ok. So, compute the partial derivative with respect to x bar set it equal to 0, we know how to differentiate this vector, this function of a vector x bar transpose x bar derivative with respect to x bar is twice x bar plus nu bar transpose A into x bar.

So, this is basically your x bar transpose x bar plus nu bar transpose A x bar minus nu bar transpose b bar. Now of course, the derivative of nu bar transpose b bar with respect x bar is 0, derivative of nu bar transpose that is c transpose x bar with respect to x bar is c bar, which is basically this is your c bar transpose. So, derivative is c bar which means it is the transpose that is nu bar transpose A, this is c transpose. So, transpose of this plus or minus derivative of nu bar transpose b bar with respect to x bar is 0 ok.

And this we are setting equal to 0 to find the optimal point, this implies 2 x bar plus A transpose nu bar equal to 0 which implies that x bar equals minus half A transpose nu bar ok. So, x bar equals minus half A transpose nu ok. So, that is basically the x bar for which the minimum is achieved for the Lagrangian corresponding to the original optimization problem ok. Now, to get the dual optimization problem is substitute this.

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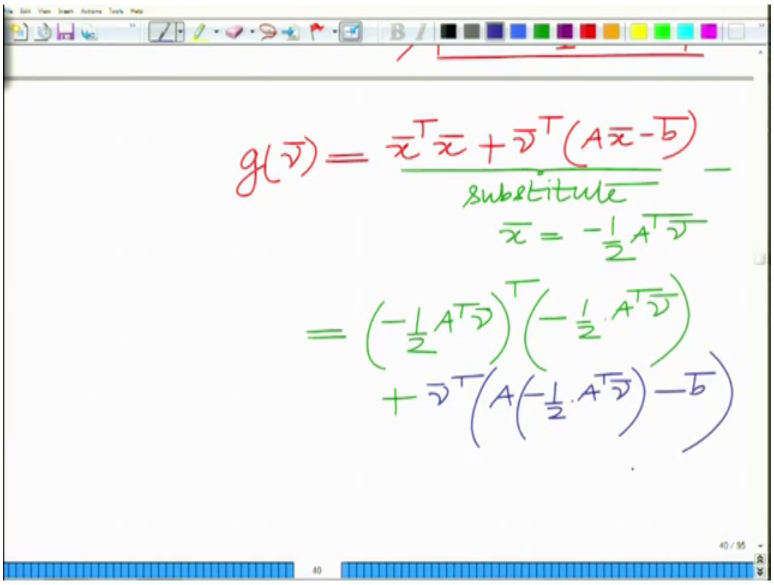

So, now, what we do is g of nu is basically nothing, but you substitute, this that is the minimum value of the Lagrangian ok, for that you substitute the x bar so, that is so in this remember this is your original optimization problem x bar transpose x bar plus nu bar transpose A x bar minus b bar in this what we do is we substitute x bar equals minus half A transpose nu bar.

So, that will give you g of nu bar equals well minus half A transpose nu bar transpose minus half A transpose nu bar plus nu bar transpose A minus half A transpose nu bar minus b bar. So, wherever there is x bar i am substituting minus half A transpose nu bar.

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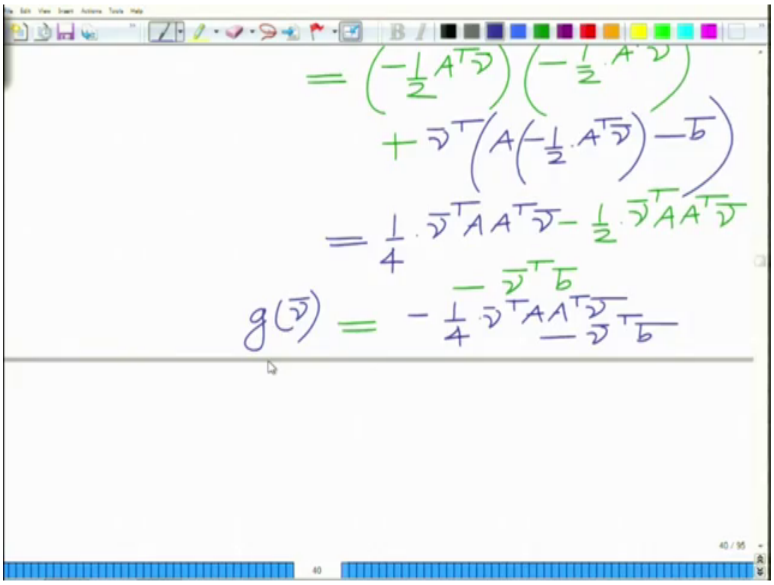

And this is equal to 1 by 4 nu bar transpose A A transpose nu bar minus half nu bar transpose A A transpose nu bar minus nu bar transpose b bar. And now if you simplify it what you will get is basically minus 1 by 4 nu bar transpose A, A transpose nu bar minus nu bar transpose b bar. So, this is your Lagrangian function.

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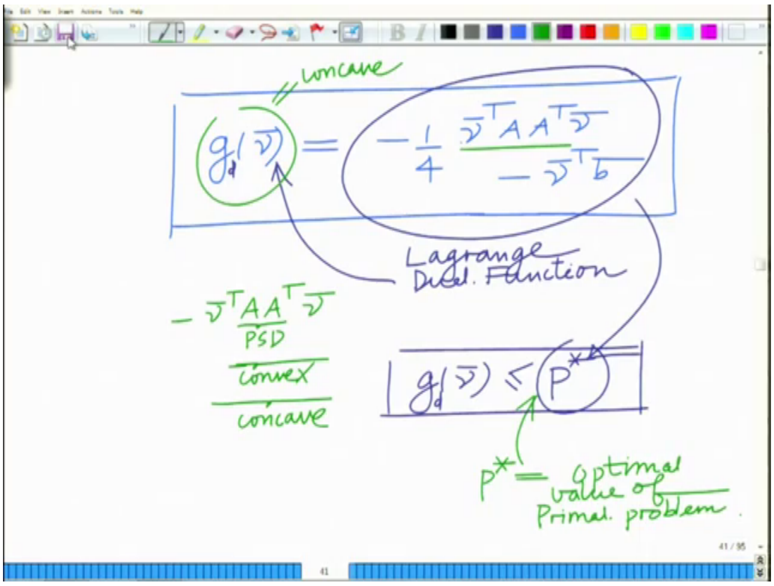

So, let me just write this again when minimizing the Lagrangian, what you are obtaining is a Lagrange dual function, which is minus 1 by 4 nu bar transpose A A transpose nu bar minus nu bar transpose b bar ok. So, this is your Lagrange dual function ok. And this will always give a lower bound, now remember this will always give a lower bound. So, g of nu bar is always less than or equal to P star for any value of nu bar, we will have that g of nu bar that is the value of this Lagrange dual function alright. Remember, there are no inequality constraints so, the there is no Lagrange multiplier or the there is no Lagrange multiplier vector lambda bar.

So, this g of nu bar is a Lagrange dual function alright in fact, this is g d of nu bar so, which is g d of nu bar is always less than equal to P star, where P star equals optimal value of the original or primal problem ok. So, this is always going to be a lower bound. Now, what is the best lower bound and look at this there is also a concave function, because if you look at this you can see here, this is minus of the form minus nu bar A A transpose nu bar. So, this is a PSD matrix positive semi definite so, nu bar transpose A A transpose nu bar is convex minus nu bar transpose A A transpose nu bar equals is concave ok.

So, this is a concave so, you can see that this is clearly and this is of course, concave function nu bar transpose b bar which is basically a linear function so, the base which is also concave so, basically it is a concave. So, it is a g d nu bar so, if you look at this is a concave function ok. And this is a concave function this is always a lower bound for P star.

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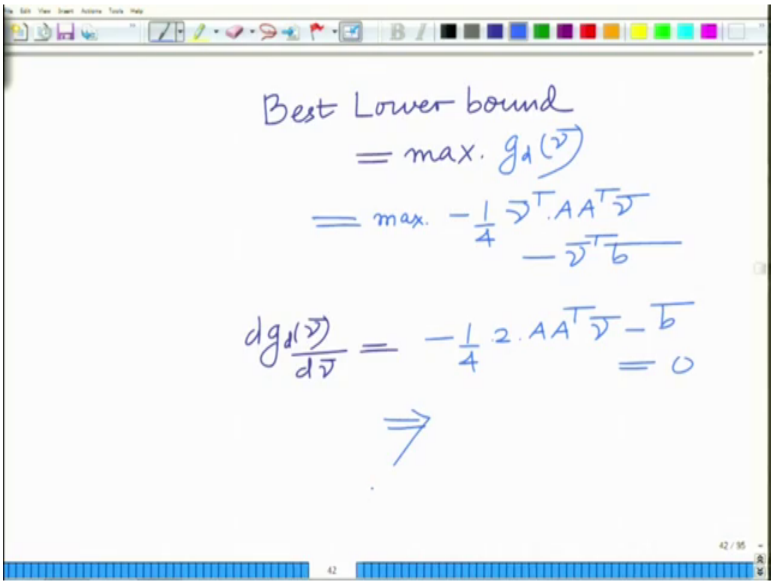

Now, one can ask what is the best lower bound the best lower bound is given by, the best lower bound is given by the maximum value. Once again note that there is no there are no inequality constraints therefore, we do not have the constraint that lambda bar has to be component wise greater than equal to 0 all right. So, I simply have to maximize this Lagrangian dual function which is g d of nu bar.

This is maximized minus 1 by 4 nu bar transpose A A transpose nu bar minus nu bar transpose b bar. And now if your different to maximize this, if you differentiate this with respect to nu bar what you get is well this is minus 1 by 4 and nu bar transpose A A transpose nu bar. So, the derivative of that is twice A A transpose nu bar minus nu bar transpose b bar derivative of that with respect to nu bar is simply b bar.

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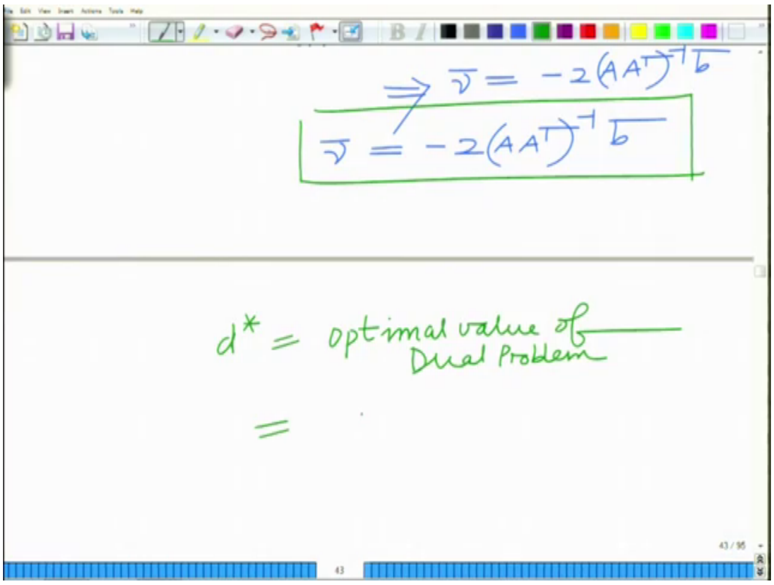

And now you equate it to 0 which implies the optimal value of nu bar for which this Lagrangian, before which the dual function is maximized is minus 2 A A transpose inverse into b bar. So, we have nu bar that is minus 2 A A transpose inverse into b bar ok.

That is the value of nu bar for which the Lagrange dual function is maximized. Now, therefore, the optimal value d star this is the optimal value of the dual problem, simply substitute nu bar in the dual problem, that is basically this nu bar value of nu bar in the dual problem.

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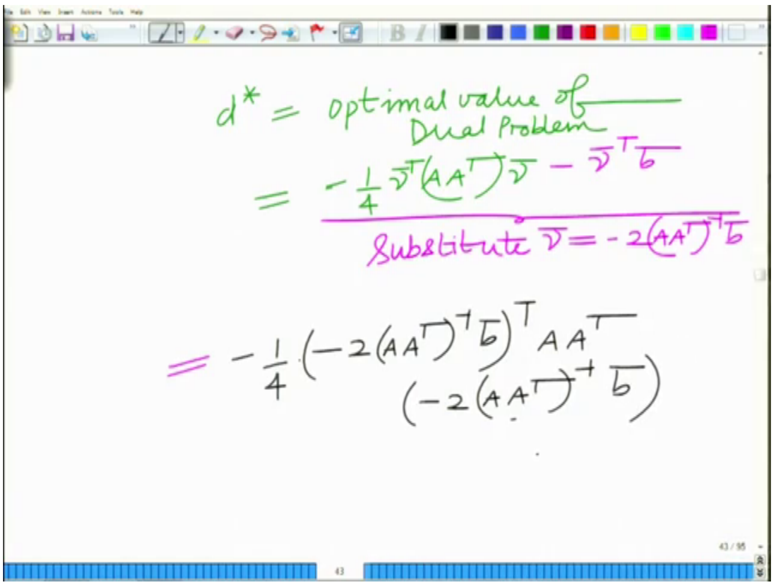

That is minus 1 by 4 nu bar transpose A A transpose nu bar minus nu bar transpose b bar so, what you do here is you substitute nu bar equals minus 2 A A transpose inverse into b bar. So, if you substitute that what you have is it is a little cumbersome, but you can write this so, nu bar transpose which is minus 2 A A transpose inverse b bar transpose into A A transpose into nu bar.

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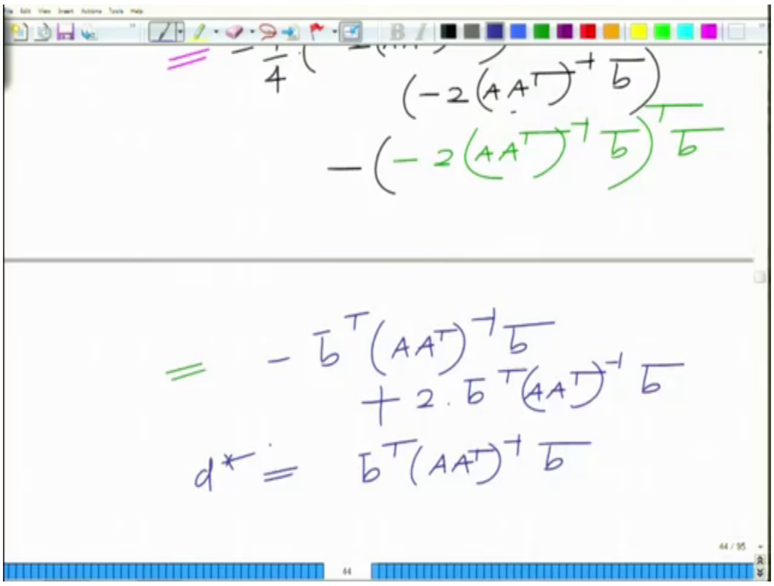

So, that is minus 2 A A transpose inverse into b bar, minus nu bar transpose b bar so that is minus 2 A A transpose inverse b bar transpose of that that is nu bar transpose b bar.

And if you simplify this what you will get is minus b bar transpose A A transpose inverse into b bar plus twice b bar transpose, you can simplify this A A transpose inverse into b bar and that is basically b bar transpose A A transpose inverse into b bar and that is your d star, that is the optimal value of the dual problem ok. So, this is d star.

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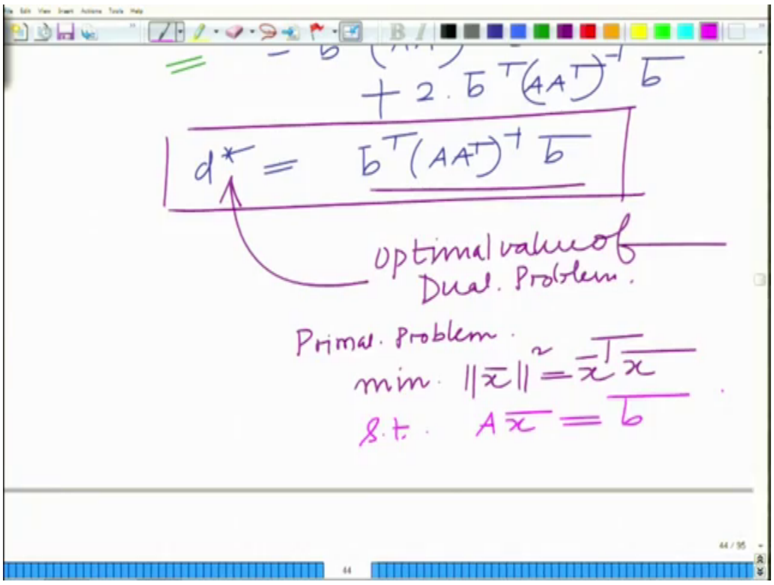

This is the optimal value of the; this is the optimal value of the dual problem ok. So, this is d star which is of course, always less than or equal to P star. Now, let us see what is P star that is optimal value of the primal problem we already know that, because you solve the minimum norm problem alright. So, this is the optimal value of the dual problem. Now, let us go back to the primal problem, remember the primal problem is minimize norm x bar square, that is x bar transpose x bar subject to the constraint A x bar equals b bar.

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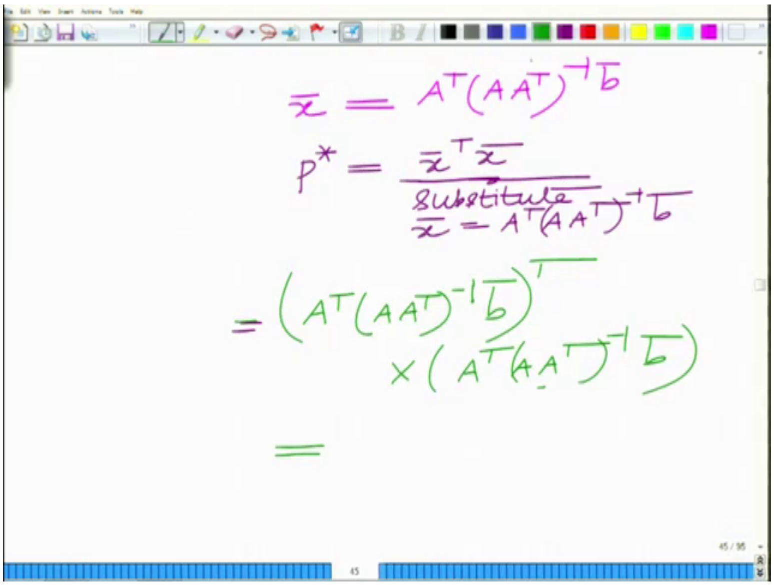

And we know that the optimal solution for this is x bar equals this we know from the previous modules that optimal solution for the minimum norm problem is A transpose A A transpose inverse into b bar.

And now P star which is optimal value of the dual problem that is basically your x bar transpose x bar. And here you substitute x bar equals A transpose A A transpose inverse into b bar. And that gives you what does that give you that use u well that gives you A transpose A A transpose inverse b bar transpose into A transpose A A transpose inverse b bar, which you can simplify and if you simplify it no wonder what you are going to observe is this is B bar transpose A A transpose inverse into b bar.

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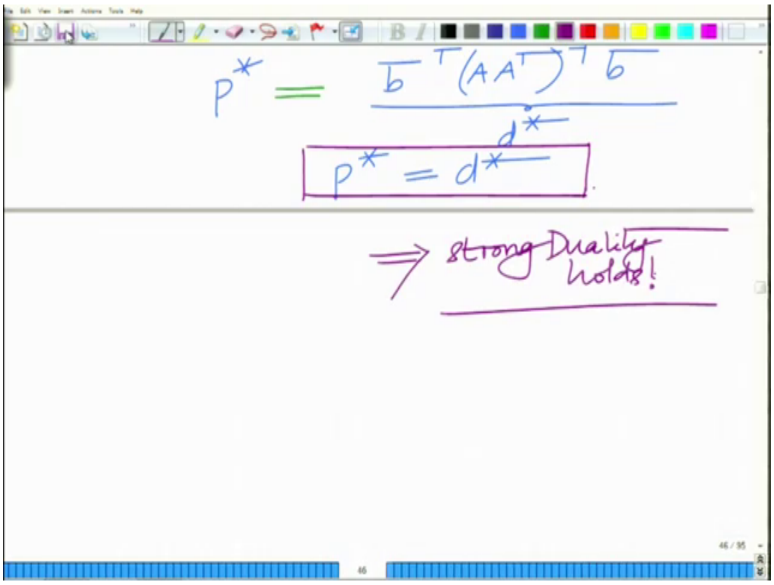

This is P star and from what you can observe above this also exactly equal d star b b bar transpose A A transpose inverse b bar so, this is in fact exactly equal to d star so, we have P star equals d star and therefore, strong duality holds and in fact, one can immediately say that, there is a value of the dual problem for because the dual optimization.

Because the dual objective is always less than or equal to the primal, dual objective always is a lower bound for the primal objective function. So, the dual objective in the primal object were coinciding that implies that, that point is the maximum value of the dual objective function dual optimization problem and is also the corresponding point is the optimal value of the primal objective.

And in this case P star we have P star equal to d star and therefore, strong duality. And in fact, the optimization problem is convex and that is what we have set for convex optimization problem, typically strong duality holds. So, P star equal to d star this implies that strong, strong duality holds and that is what we have already seen that strong duality holds for this optimization problem ok. So, this is one of the simplest and most elegant optimization problems that is the (Refer Time: 19:51). Let us look at another interesting problem and that problem is as follows.

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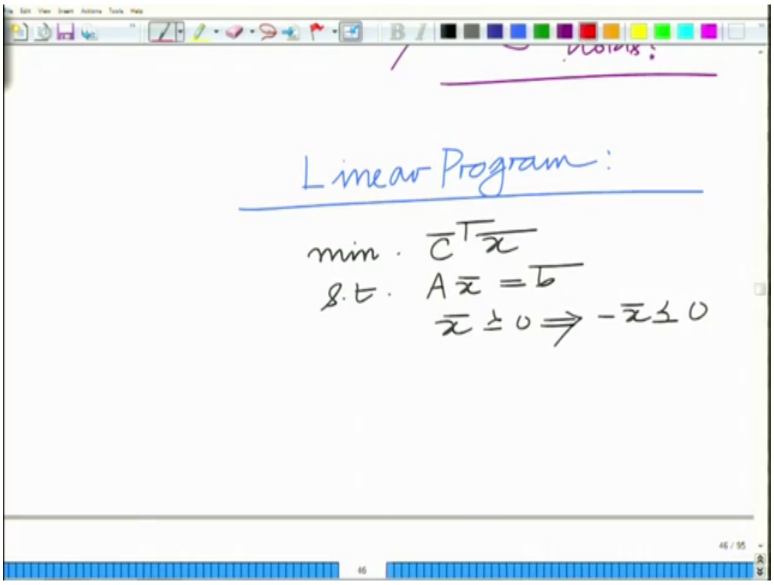

Let us look at another interesting problem and that is a linear program let us see what duality has to tell us. Now, for a linear programs look at the standard linear program or one of the versions, that is minimize the linear objective c bar transpose x bar subject to A x bar equal to b bar these are the equality constraints, and then let us say that x bar is component wise greater than equal to 0 each component of the vector x bar is greater than equal to 0. You can write this as a standard from convex optimization problem, by saying each component of minus x bar is less than or equal to 0 ok.

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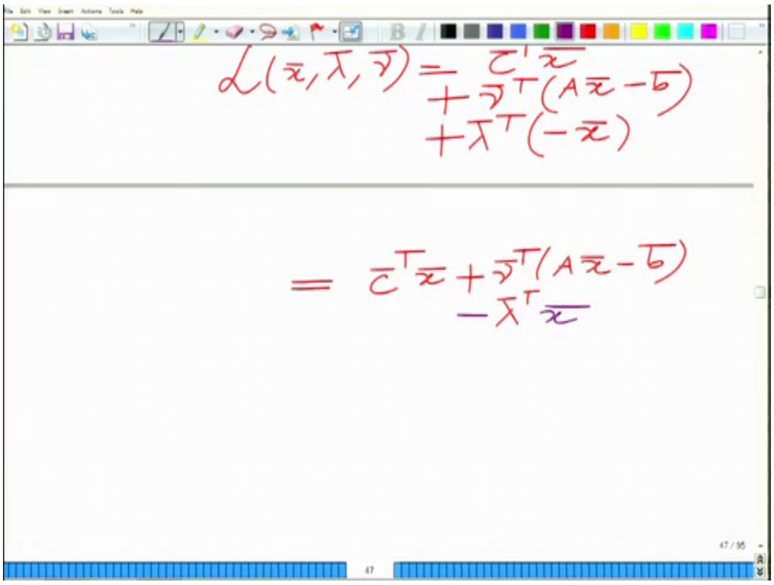

Now, the Lagrangian of this can be formulated as x bar, now you have both inequality and equality constraints. So, the Lagrangian will be objective function a x bar I am sorry c bar transpose x bar plus 1 Lagrange multiplier for each equality constraint, that is your same as before nu bar transpose A x bar minus b bar nu bar is the vector comprising of the Lagrange multiplier for the equality constraint plus lambda bar transpose 1 Lagrange multiplier for each inequality constraint minus x bar ok, equals c bar transpose x bar plus nu bar transpose A x bar minus b bar plus lambda bar transpose minus x bar ok.

Each in fact, the size of the vector lambda bar is equal to x bar, because you have one Lagrange multiplier for each component of x i less than or greater than equal to 0 ok, we can directly write this as minus lambda bar transpose x bar. Now, we have to take the minimum of the Lagrangian right and typically for that we differentiate it with respect to the vector x bar, but since this is a linear this is an affine function we will follow a slightly different approach.

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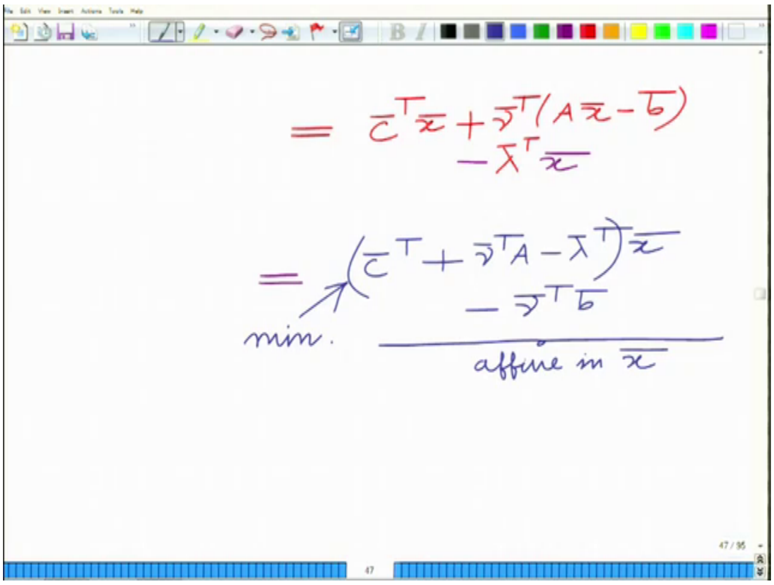

And that is as follows and interestingly, if you separate the terms, if you write this as the terms corresponding to x bar plus nu bar transpose A minus lambda bar transpose x bar minus and the constant term.

You can see this is affine in x bar which is basically it is a, you can see this is the equation of this is the equation of basically a hyperplane, this is the equation of a hyperplane correct. Now, if you see what is now what we have to do is now we have to minimize this ok. And what you will observe is, you will observe something interesting. Now, this is an affine function it is like a line correct.

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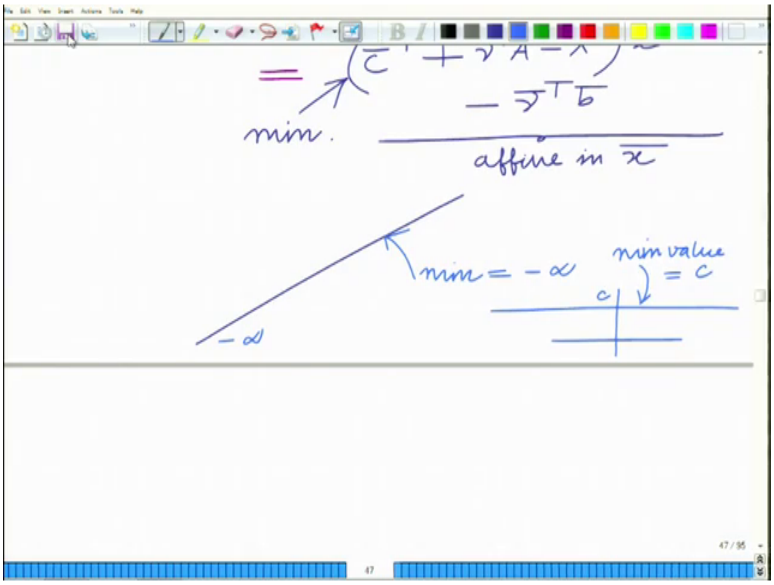

So, let us look at this line if this line has a slope, then the minimum value of this will always be equal to minus infinity, because one of the ends will always be minus infinity. Only if the line is parallel, then the minimum value that is the line is a constant ok.

And then the minimum value equal to c that is a constant ok. So, this is very interesting, because it is affine if the vector multiplying x bar is non zero, then the minimum value is always going to be minus infinity right in that case it has a slope. If that is 0 that is a vector multiplying x bar is equal to 0, then the minimum value is the constant which in this case is minus nu bar transpose b bar ok.

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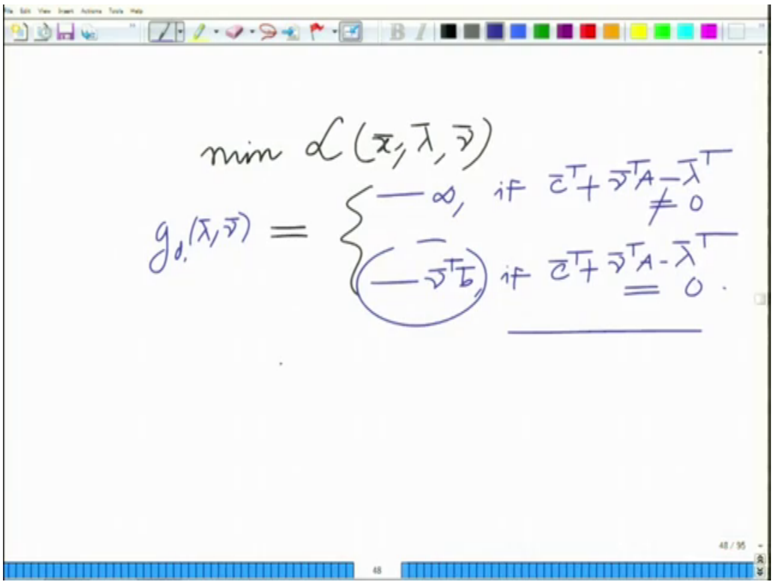

So, with that observation we have the minimum of the Lagrangian, this is L if c bar transpose plus nu bar transpose A minus lambda bar transpose is not equal to so, the minimum is minus infinity, if c bar transpose plus nu bar transpose A minus lambda bar transpose equal to 0, that is this vector which is multiplying x bar. This vector is not equal to 0 then the minimum is minus infinity, on the other hand if that vector is 0 then the minimum is simply the constant that is minus nu bar transpose b bar.

Now, of course, minus infinity is always a lower bound for any optimization problem all right. So, this is your Lagrange dual function minus infinity is always a lower bound correct, it is very uninteresting the interesting lower bound occurs for this that is minus nu bar transpose b bar. And the best lower bound is a where ever is when you maximize this with respect to nu bar lambda bar.

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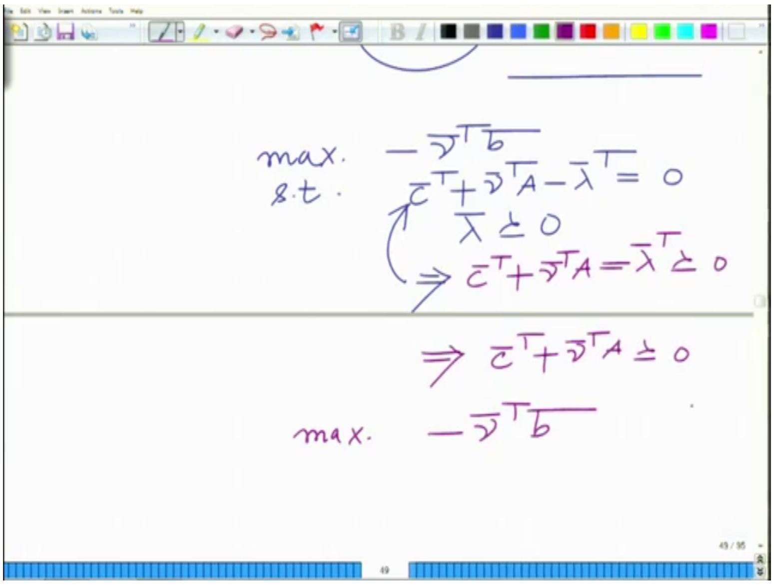

So, the dual optimization problem can be equivalently written as maximize minus nu bar transpose b bar subject to the constraint, c bar transpose plus nu r transpose A minus lambda bar equal to 0. And of course, we always have this constraint as well that is the Lagrange multipliers corresponding to the inequality constraint, or component wise greater than equal to 0 that is each Lagrange multiplier lambda is greater than equal to 0.

Now, you can see that c bar transpose plus nu bar transpose A minus lambda bar transpose equal to 0. So, if you will now we can simplify this further. So, this implies, what does this imply? This implies that c bar transpose plus nu bar transpose A equal to lambda bar transpose which is component wise greater than equal to 0. So, this implies that c bar transpose plus nu bar transpose A is component wise greater than equal to 0 ok.

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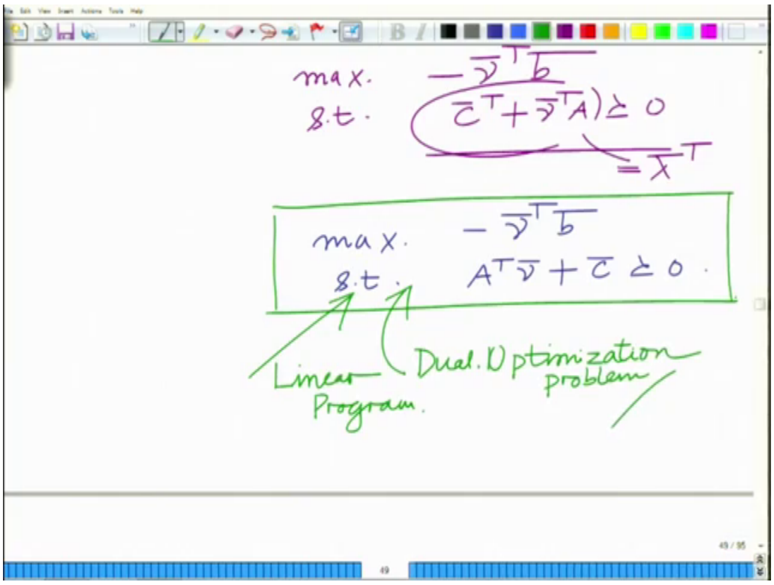

And therefore, now you can simplify this optimization problem as simply this, maximize minus nu bar transpose b bar subject to the constraint that c bar transpose plus nu bar transpose A, component wise greater than equal to 0 that is and whatever is c bar transpose plus nu bar transpose A, that is equal to now this quantity is equal to itself equal to you can set this quantity. Once you obtain this quantity you can set this quantity equal to lambda bar transpose alright.

And in fact, I am just going to take the transpose of this. So, I am just going to take I can also write this as this is a row vector, I can write this as minus nu bar transpose b bar subject to the constraint A transpose nu bar plus c bar, this is component wise greater than equal to 0. And this is the equivalent dual optimization problem yes, this is the dual optimization problem.

And since this is a convex optimization problem, that is the original problem is a linear program the dual optimization problem you can also see is a linear program, that is the dual of a linear program is a linear program is a convex optimization problem. Therefore, strong duality holds P star optimum value of the original dual optimal; original optimization problem, you will see is equal to the d star which is optimal value of the equivalent dual optimization all right. So, we will stop here and continue in the subsequent modules.

Thank you very much.