Applied Optimization for Wireless, Machine Learing, Big data
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Lecture - 63
Concept of Duality

Hello, welcome to another module in this massive open online course. So, we are looking at different topics and concepts in convex optimization and particularly, particularly from an applied perspective. In this mode, let us start with a new topic and that is Duality.

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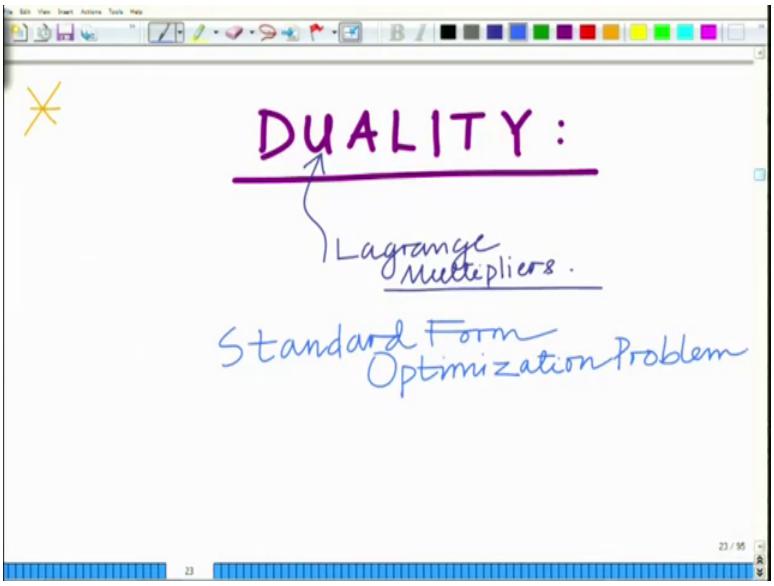

So what we want to start looking at is a very interesting and a very fundamental concept of duality ok. And we have seen this to some extent we have seen this informally, basically what this does is it formalizes the framework of Lagrange multipliers; that we have been sort of informally employing.

So, far so what is a Lagrange multiplier all right, what is the significance of a Lagrange multiplier, what does it indicate all right. And what is the formal, I mean how do you formally define the Lagrange multipliers associated with this problem.

So, that is basically what we are going to see or consider cover now in this when we look at the concepts of duality. So now, let us go back to the standard form optimization problem, recall a standard form a standard form optimization problem is given as follows.

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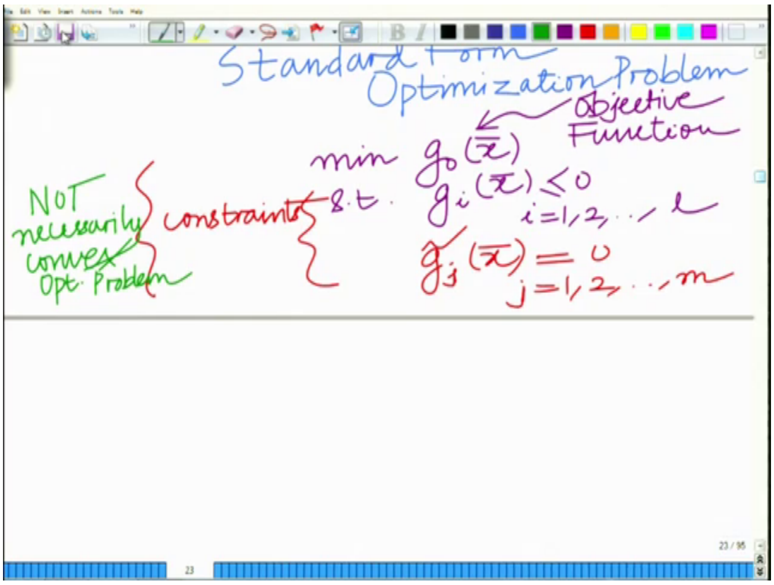

That is, you have the objective function, let us say this is minimize g naught of x bar, this is your objective function ok. Subject to then you have the constraints you have the inequality constraints g i of x bar less than equal to 0 i equals 1 to up to l. And the equality constraints g j tilde of x bar equal to 0 j equals 1 to up to m we have seen this definition before so these are your constraints ok.

And of course, now this is any standard form optimization problem. In addition we have seen that if g naught is convex alright, the objective function is convex correct? Inequality constraints are convex, and the equality constraints are affine it becomes a convex optimization problem. So, right now this need not this is not necessarily a convex optimization problem is simply a standard form this can be any optimization problem, not necessarily a convex optimization problem.

So, this is keep in mind or bear in mind what we are considering now is not necessarily of convex optimization problem. That is a most general framework of duality is applicable even when the problem is non convex ok.

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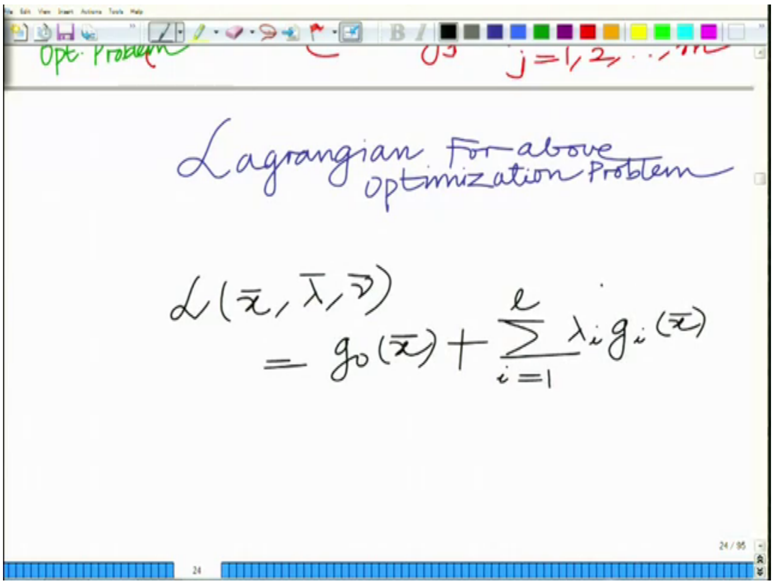

And that is. In fact, the power or that is in fact, the appeal of this framework of duality ok. So, this is a standard form optimization problem. Now for this optimization problem the Lagrangian function for the above optimization problem.

The Lagrangian can be formulated as L of x bar lambda bar nu bar equals the object to g naught x bar plus, summation i equals 1 to l lambda i g i x bar that is each constraint g i there is inequality constraint g i x bar multiplied by this lambda i which is the Lagrange multiplier.

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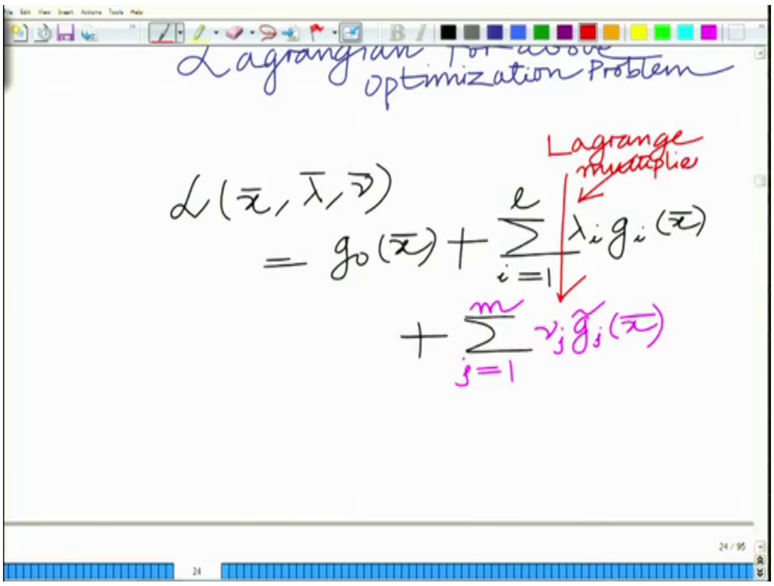

And the summation plus summation j equals 1 to m nu j g j tilde x bar multiplied each equality constraint by this quantity learn nu j and take their sum ok. Now these quantities are the Lagrange multipliers the lambda i s. And we have already seen this to some extent. So, these are the Lagrange now these quantities of the Lagrange multipliers.

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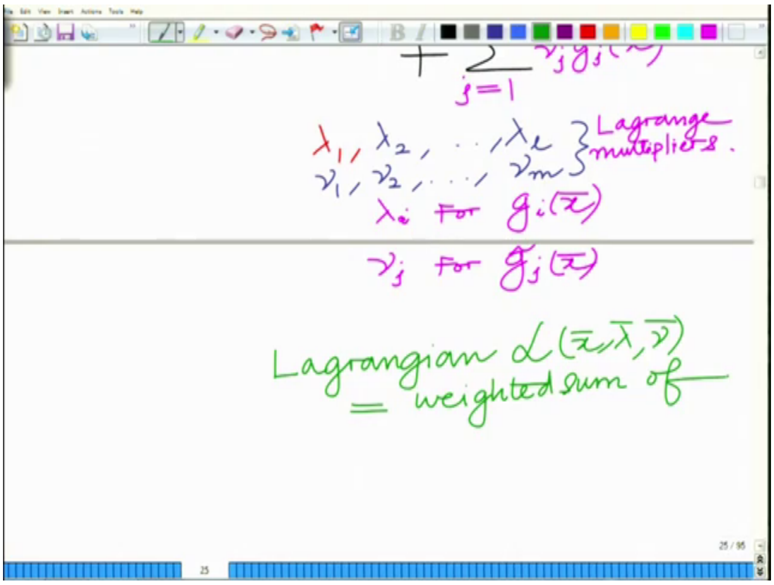

What are the Lagrange multipliers? These are the lambda i s that is lambda 1 lambda 2 up to lambda l nu 1 nu 2 up to nu m, these are your Lagrange multipliers. And of course, these are the Lagrange each lambda i remember lambda i is a Lagrange multiplier for your g i of x bar all right we are multiplying each g of x bar with lambda i. And each nu j is the Lagrange multiplier for g j tilde of x bar right.

So, what we are doing is we are taking g naught x bar objective function plus each inequality constraint g i x bar multiplied by the Lagrange multiplier lambda i sum plus each equality constraint g j tilde x bar multiplied by the Lagrange multiplier nu j.

So, this is a weighted sum all right of the objective function and the constraints, and the weights are basically the Lagrange multipliers ok. So, what the Lagrangian is so the Lagrangian, if you realize it and it is not very difficult Lagrangian that is your L of x bar comma lambda bar comma u bar this is the weighted sum.

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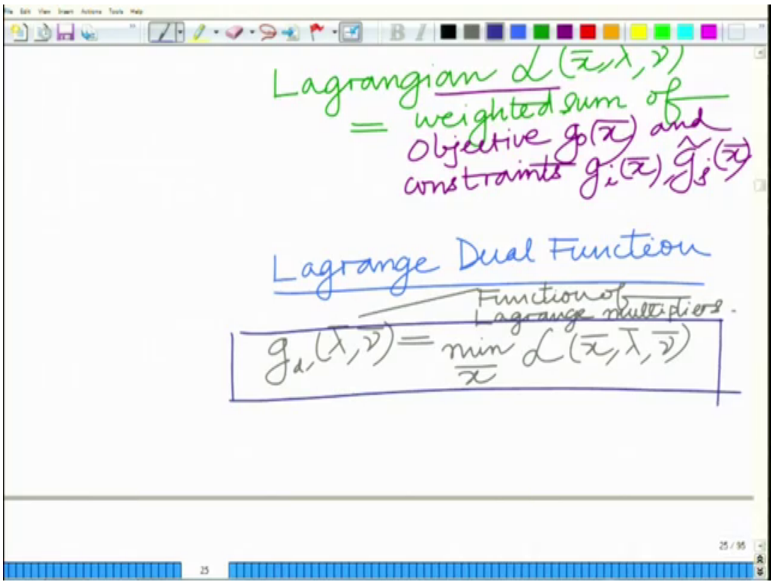

Weighted sum of objective g naught x bar and the constraints g j x bar or g i x bar g j tilde x bar. And now we can define that so this is a Lagrangian. Now we can define the Lagrange dual function.

What is the Lagrange the Lagrange? Dual function is g d of lambda bar nu bar all right. Now remember now this is a function of the Lagrange multipliers that you can observe, this is a function of the function of the Lagrange multipliers.

This is the minimum over x bar of the Lagrangian L of x bar lambda bar. This dual the Lagrange what we call the Lagrangian dual is you take the Lagrangian alright for a given lambda bar nu bar the Lagrange multipliers. And take the minimum over x bar alright that is the Lagrangian function.

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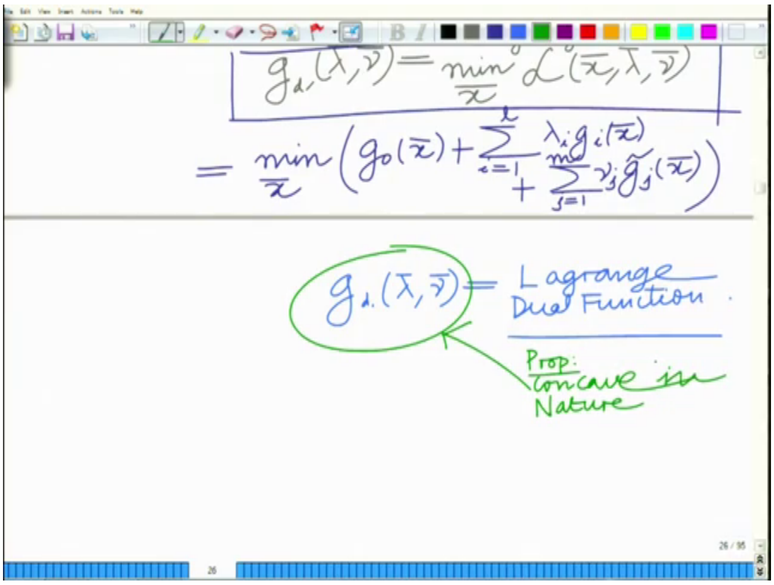

And this is basically you take write try to write it explicitly, the minimum over x bar g naught x bar plus summation i equals one to m lambda i g i x bar plus summation j equal to or i am sorry i equal to 1 to L, in fact i equal to 1 to l, equal to 1 to m nu j g j tilde into x bar and you take the minimum over plus, this is the Lagrange dual function ok.

So, gd lambda bar nu bar equals the; this is the Lagrange dual function corresponding to the positively non convex remember all right, it is important again, I am repeating this again it is important to remember that we have started with the standard form optimization problem which is not necessarily convex.

And this Lagrangian dual function as a very interesting property, the Lagrange the Lagrange dual function can be shown to be concave in nature, irrespective of the original optimization problem which need not be convex. So, this Lagrange the Lagrange dual function this can be shown to be concave in nature. That this is your property this is an important property this g d the dual function.

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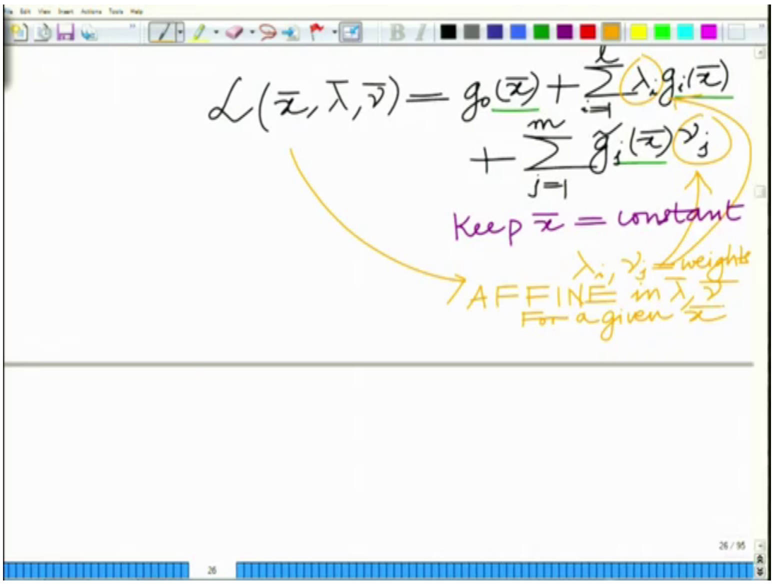

And this is very easy to see this can be seen as follows for instance; if you look at the Lagrangian function let us go back, take a look at the Lagrangian function x bar comma lambda bar comma nu bar. This is equal to g naught x bar plus summation i equal to 1 to l lambda i g i x bar plus summation j equal to 1 to m lambda j tilde, gj tilde x bar into nu j equal to 1 to m ok, and now if you observe this function now, if you closely observe this function.

You can observe that even though this is a complicated function of x bar, this remember this is a linear combination of g naught x bar the g i x bar and the gj tilde x bar. And what are these Lagrange multipliers lambda and nu j Lagrange multipliers lambda i is and the nu j i are nothing but the weights all right. And therefore, it is affine in the Lagrange multipliers and that is important to remember.

If you forget if you keep x bar constant now here if you keep x bar constant ok, for a moment keep x bar equal to constant. Now you observe that if you look at the lambda i's and nu j's these are nothing but the weights ok.

So, lambda I. Coma nu j equals the weights in this linear combination ok. And therefore, this is affine this Lagrangian if you look at this is affine alight. Affine in the sense that it is some constant plus some vector transpose times lambda bar plus some vector transpose times nu bar ok. So, this is affine in lambda bar nu bar because remember we are keeping x bar constant for each x bar, for a given x bar for a given x bar this is affine in lambda bar nu bar ok, that is what you have to say.

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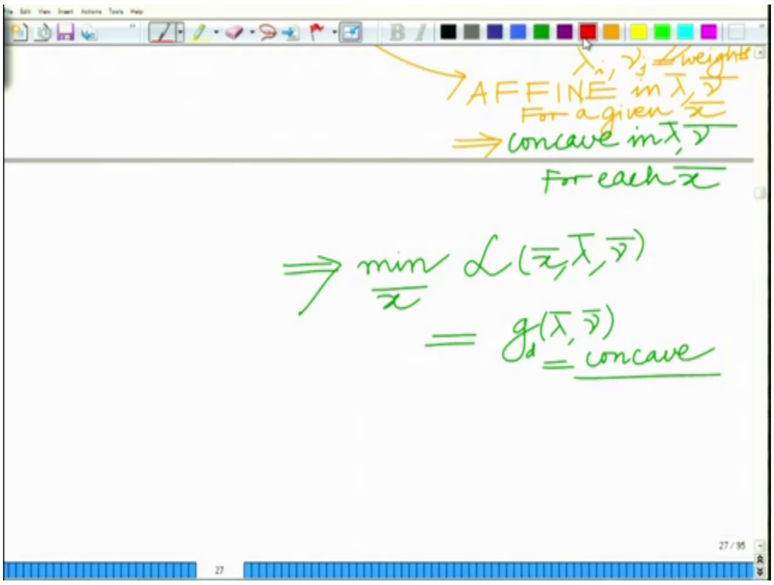

So, for remember affine lambda bar nu bar for a given x bar which means it is concave remember any affine function. Let us affine in the sense that this is a hyper plane. This is a concave function, this is affine lambda bar nu bar and therefore, this is a concave function, this implies this is a concave function for each x bar this is a concave function in lambda bar nu bar.

And therefore, now when you take the minimum remember what is the dual doing this is taking the minimum. So, implies when you take the minimum over x bar. So, this is concave for in lambda bar nu bar for each x bar, when you take the minimum over the x bar what you get is the Lagrange dual function g d lambda bar nu bar, which is naturally concave and that can be seen simply as follows.

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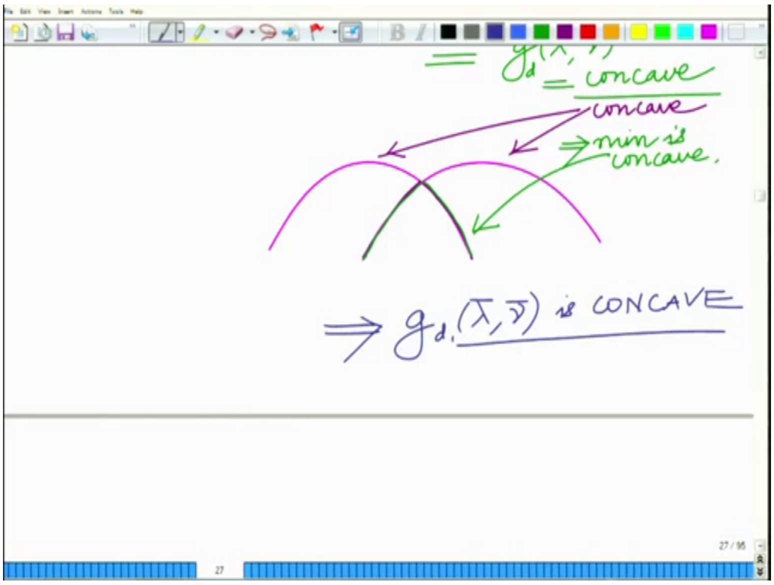

You take 2 concave functions and now you take the minimum the minimum and you take the minimum. So, these are the concave functions. And now if you look at the minimum of this the minimum is concave; so, the Lagrangian is the Lagrangian function is concave it is in fact, affine in each lambda bar nu bar which is basically concave. So, the moment you take the minimum of this over x bar it is going to be a concave function which means the Lagrangian dual function g d is a concave function.

So, that is an important point and again I am belaboring the point. So, g d is concave, but that is the Lagrangian (Refer Time: 17:22) and this is very important and again I am repeating this over and over.

So, that you do not forgetted that is this is not and this holds true even when the original problem is not necessarily convex and that is a big advantage, because you can see we are going to see that we are going to convert a non convex. Because one can convert a standard non a possibly non convex optimization problem into a concave, or an equivalent convex remember concave optimization is the same as convex.

Because if you are maximizing a concave function that can we could take the negative of the objective function you can write it as minimizing a convex objective. So, concave and convex optimization in that sense are equivalent.

So, one can convert so the power of the duality framework is one is that one can convert a possibly non convex original problem into a standard form concave or for that matter convex optimization problem. And then one can use all the tools and techniques all right associated with the framework of convex optimization. And this has this is a very powerful framework or this is a very powerful result, which has a widespread application and simplifies several convex optimization problems.

So in fact, can we used to simplify also obtain simplified forms of several possibly non convex optimization problems, as we are going to see subsequently.

Thank you very much.