Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture - 58
Example Problem: Orthogonal Matching Pursuit (OMP) algorithm

Hello, welcome to another module in this massive open online course. So, we are looking at schemes or techniques compare compresses using, and we have seen that orthogonal matching pursuit for compressive sensing alright. Or 2 basically for sparse signal recovery, that is to estimate a sparse signal x bar all right. So, we have seen this algorithm in the previous module; let us now look at an example to understand this better.

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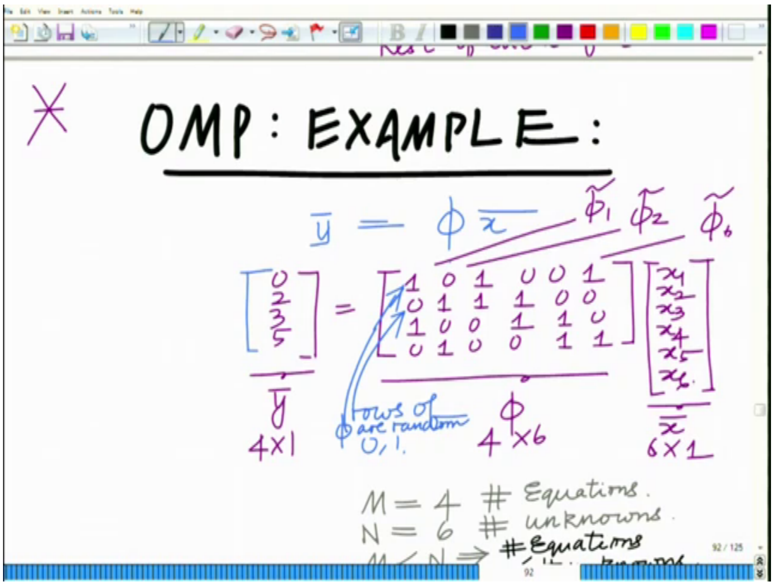

So, what we want to look at is the orthogonal matching pursuit. We have already seen the orthogonal matching pursuit. So now, what we want to see is an example a simple example of paper and pen kind of example for the orthogonal matching pursuit. And let us consider the following example we have y bar equals phi x bar we have to estimate the vector x bar.

So, let us consider the example that is given as follows. The vector y is 0 2 3 and 5 and the matrix phi is the following this is 1 0 1, 0 0 1, this is 0 triple 1 0 0 1. This is the third row 1 0 0 1 1 0 and the 4th row is 0 1 0 0 1 1 times the vector x bar which is basically x 1, x 2, x 3, x 4, x 5.

So, this is your y bar, this is your dictionary of sensing matrix phi this is your matrix x bar, and so, the various parameters of this are as follows. This y is a 4 cross on vector, the matrix phi is 4 cross 6 and x bar is 6 cross 1.

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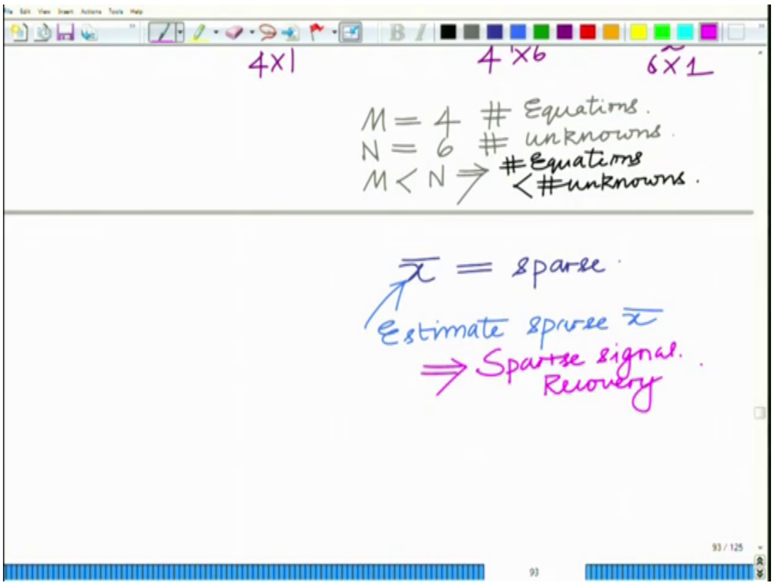

And therefore, in this problem we have M equals 4 which is basically the number of equations. And N equals 6 which is the number of unknowns. And we can see M less than N, which implies number of equations less than number of unknowns. And therefore, to estimate x bar or basically to recover x bar alright, where is x bar.

Remember you cannot use conventional linear algebra, will basically because in linear algebra you need at least number of equations at least or number of equations equal to the number of unknowns or the number of equations at least equal to the number of unknowns to uniquely determine the unknown vector x bar. And therefore, one as to enforce some other condition on x bar to uniquely recover it and the condition that we have seen so far; that is to enforce sparsity.

That is, to determine a sparse vector x bar that satisfies this system of equations alright. So, or that fits this model. So, we assume that x bar is sparse, and then we want to estimate this sparse vector. And this is basically what is termed as sparse signal recovery. The system does sparse signal recovery.

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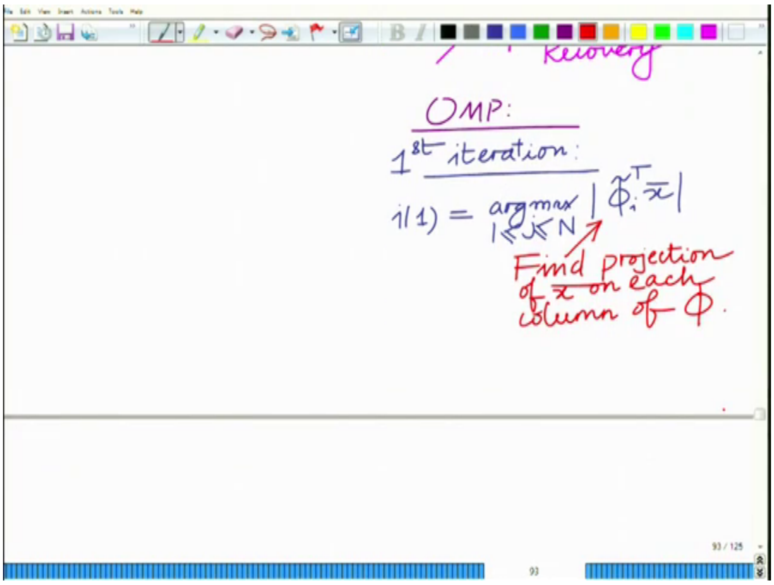

And the algorithm we have seen OMP for sparse signal recovery that proceeds as follows.

First find the projection so, we can see these are the columns, now remember when you talk this y bar, these are the columns for instance this is phi 1 tilde, this is phi 2 tilde and so on and so forth, this is phi N tilde or in this case N equal to 6. So, this is pi 6 tilde. So, what we will do is, we will find so, OMP remember the first iteration.

The first iteration, you find the projection of x bar on each column of the matrix phi which is on each phi i tilde and choose the column which yield the largest projection, all right. So, remember you have i 1 equals argmax j. In fact, one less than equal to j less than equal to N magnitude phi i tilde transpose times x bar. And remember is and what we can and this we can do as a following thing.

So, basically what we are doing is finding projection of fined projection of each or fine projection of fine projection of x bar, find projection of x bar on each column of phi. This can be done as follows, what we are going to do or a fine projection of y bar, I am sorry fine projection of y bar on each column of phi. And what way this can be done as follows.

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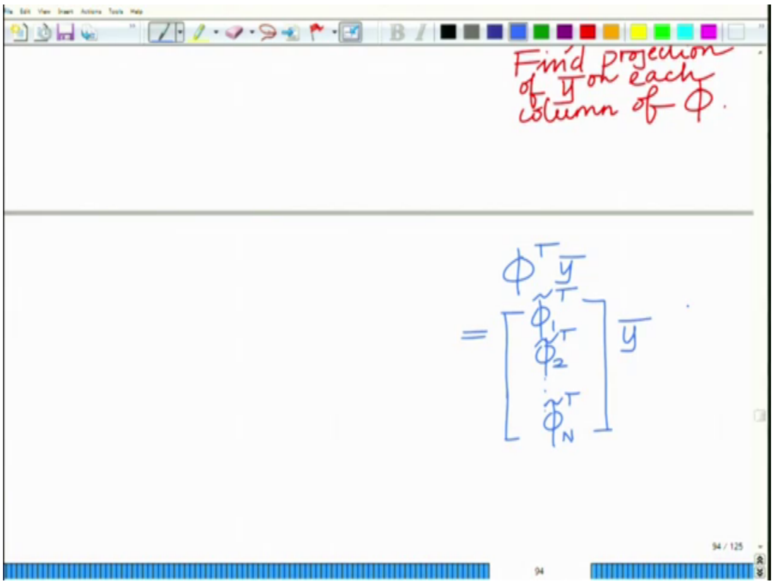

So, what we are going to do is, we are simply going to perform phi transpose y bar. What that gives us is that gives us basically the inner product of each phi 1 tilde phi 1 tilde transpose phi 2 tilde transpose phi N tilde transpose y bar.

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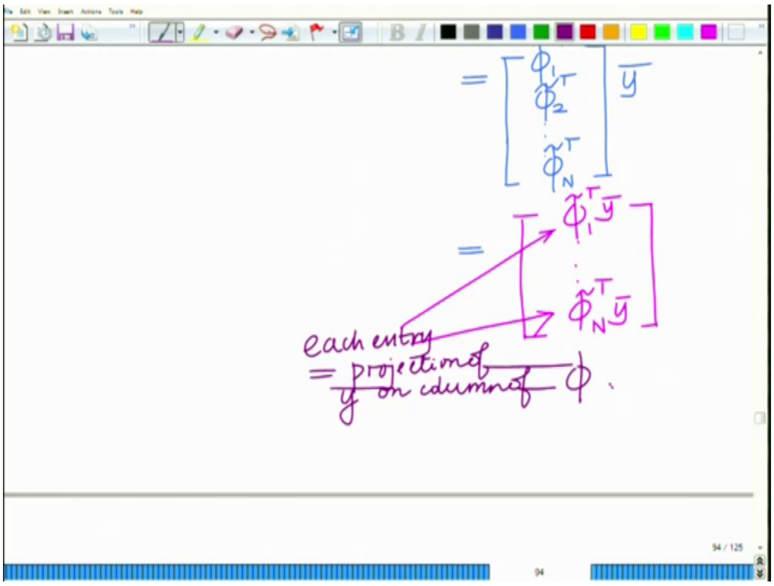

Which is basically nothing but if you look at this, this is basically phi 1 tilde transpose up to phi N tilde transpose into y bar.

So, each of these entries corresponds to. So, each entry equals projection of y bar on column of phi. Now the other thing that you must have observed is if you look at these rows, you can see that these rows are random 0's and 1. So, these are noise like waveforms ok. So, that is other important thing. So, rows of phi a random 0 columns. So, these are noiseless remember that is an important criteria remember, we cannot take time domain or special domain measurements. But you have to take the projections of x bar on random noise like waveform alright.

So, each measurement is basically each observation is a projection of y for y bar on this noise like waveform.

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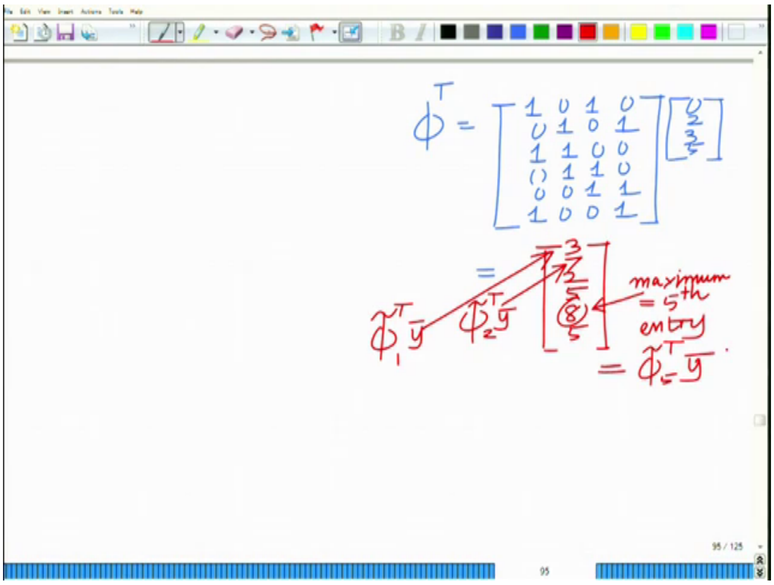

So now let us find phi transpose y bar phi transpose y bar remember, phi transpose is this matrix, in which the rows become columns and the columns become rows. So, first row will be 1 0 1 0 0 1 0 1 1 1 0 0 0 1 1 0 0 0 1 1 1 0 0 1, and you take the projection of y bar.

So, that is 0 2 3 5 and if you compute this, if you evaluate this, you will get the vector 3 7 2 5 8 5. Remember, each of this entries is the projection for instance 3, this is equal to phi 1 tilde transpose y bar 7 equals phi 2 tilde transpose y bar and so on. And if you see the maximum occurs equals 5th entry or 5th component, which is equal to phi 5 tilde transpose y bar. Therefore, the maximum projection of y bar, y bar has a maximum projection along column phi, along column phi; that is, corresponds to phi tilde phi.

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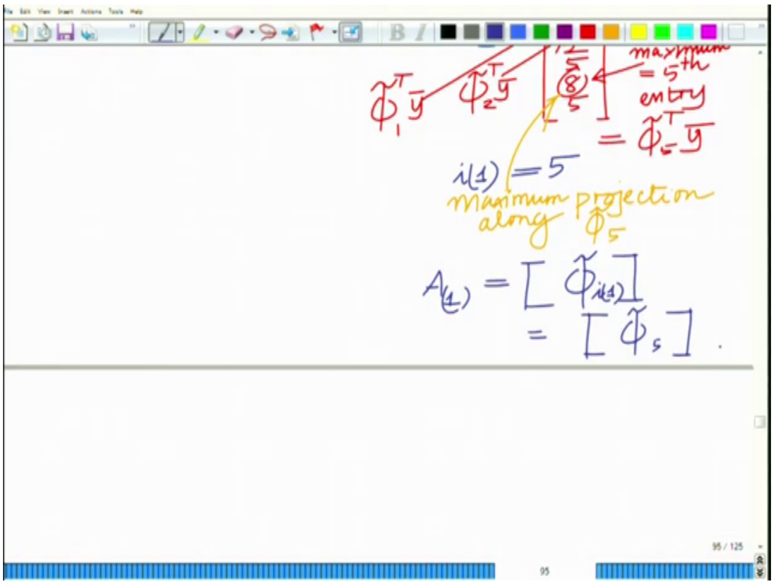

Therefore, now we form the basis matrix using this column phi 5 tilde. Or in other words what we are saying is this quantity i 1; that is, index of the column which has the maximum projection that is phi.

So, this is phi tilde of i 1 which is basically phi tilde of 5, that is your basis matrix.

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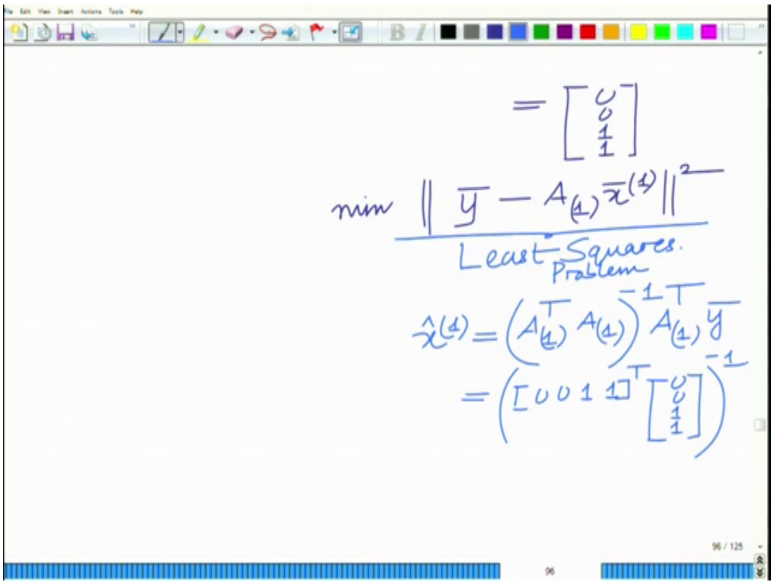

Which is basically that is nothing but this you take the 5th column of the matrix phi. And that will be 5th column of matrix, 5th column of matrix y that will be 0 0 1 1. So, this is the 5th column of, and now you solve the least squares problem. Y bar minus A 1 x bar, remember this is the first iteration.

So, you solve the least squares problem. And once you solve this least squares problem, remember the solution to this is x hat 1 equals A 1 transpose A 1 inverse A 1 transpose y bar. which is A 1 transpose, remember A 1 0 0 1 1. So, this is 0 0 1 1 transpose, this is very simple, simply row vector transpose.

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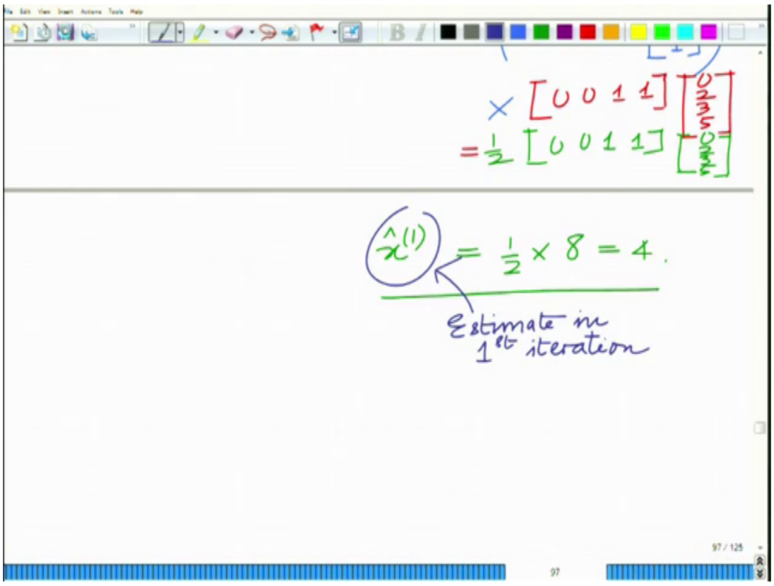

The column vector 0 0 1 1 inverse of this times A 1 transpose 0 0 1 1 times 0 2 3 5, ok. And this will be half because A 1 transpose A 1, this is row vector times of column vector will be 2. So, inverse of that is half times 0 0 1 1 times 0 2 3 5. So, this will be half into 8 equal to 4.

So, this is basically your x hat 1, ok. So, that is basically your estimate of the sparse vector in the first. Remember this x hat 1 corresponds to the index of the column that is chosen in the first iteration that is column number 5. So, your sparse vector so, this entry corresponds to the 5th column or the 5th entry of the vector x bar. Now what we will do is, we find the residue after the first iteration ok. So now, find the residue so, this is the estimate and now what we will do is, we will find the residue for the first iteration.

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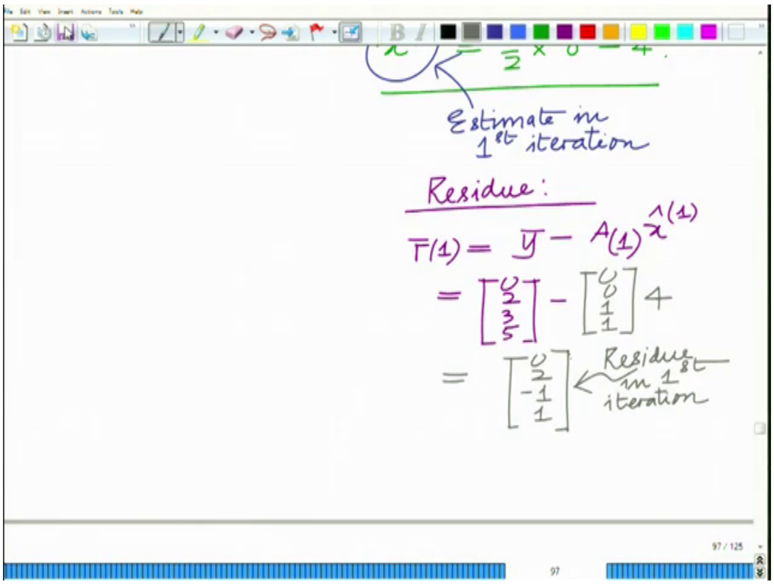

And the residue is r 1 or rather r bar 1, y bar minus A 1 x hat 1; which is 0 3 5 minus A 1 is simply the column 0 0 into 4.

So, this will be basically 0 2 minus 1 comma 0 2 minus 1, this is the residue in first itration, ok. There is a residue after the first iteration. And this is in fact, what we carry over to the second iteration. Remember, subsequently find the projections of the columns of phi on this residue choose the one that has the maximum repeat the process least square solution, alright find the residue repeat the process. Now let us go to the second iteration.

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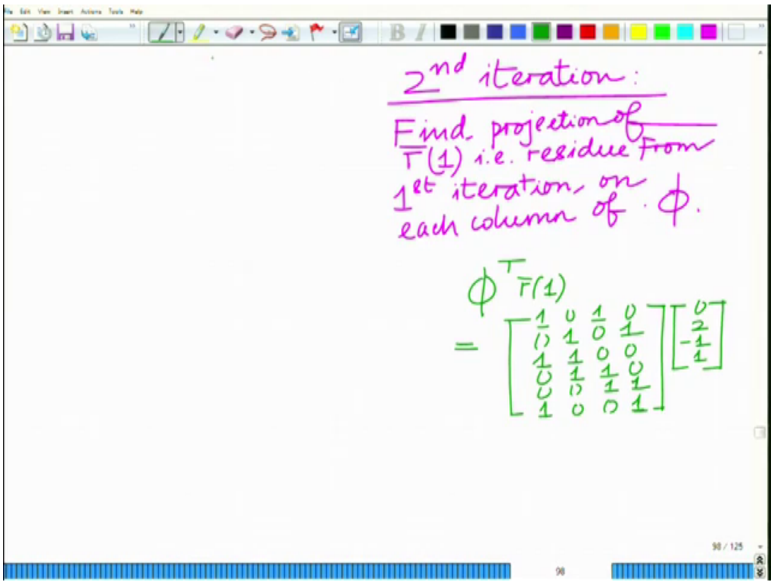

So, let us look at now second iteration. In second iteration, we find projection of r bar 1, that is residue from first iteration on each column of phi. And therefore, again similarly what we will do we will do? Phi transpose r bar 1; which will basically give the projection of the residue on each column of phi ok. So, this will be 1 0 1 0 0 1 0 1 1 1 0 0 0 1 1 0 0 0 1 1 1 0 0 1, on the residue, 0 2 minus 1 1.

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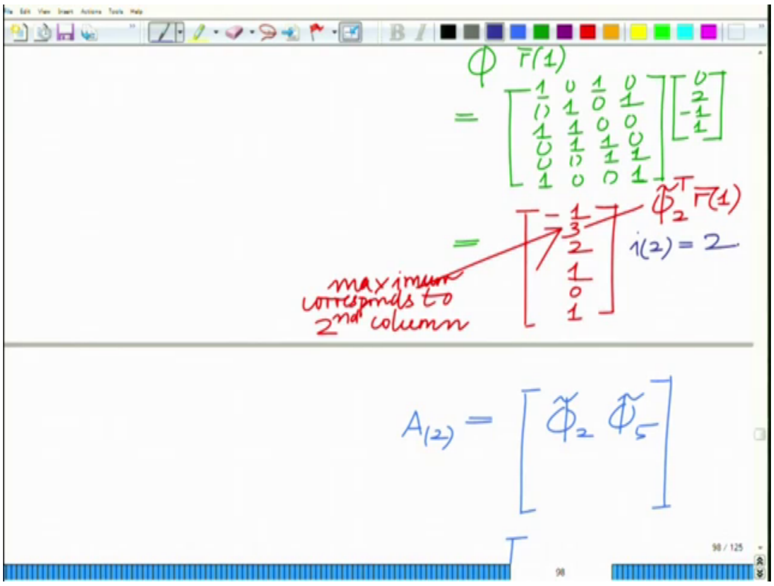

Now, you take this projection, you evaluate this you will see that this comes down to minus 1 3 2 1 0 1. And what you observe that now the maximum is 3, maximum corresponds to second column. This is basically phi 2 tilde transpose r bar 1. So, the maximum entry corresponds to the projection of the residue r bar 1 on the second column that is phi 2 tilde transpose r power. Therefore, you now you choose the second column. You make the augmented matrix. So, the augmented matrix becomes, previously we have phi 2 tilde now we are identify.

Now remember you can also write it as phi 5 tilde comma phi 2 tilde it does not matter. It does not as long as you are clear that this is the order and therefore, corresponding the entries of x tilde will x hat will correspond to these 2 columns. So, basically your matrix in fact, I should write it like this. These are simply the columns of the matrix ok. So, you have matrix you are picking the columns phi 5 tilde phi 2 tilde phi 5 tilde.

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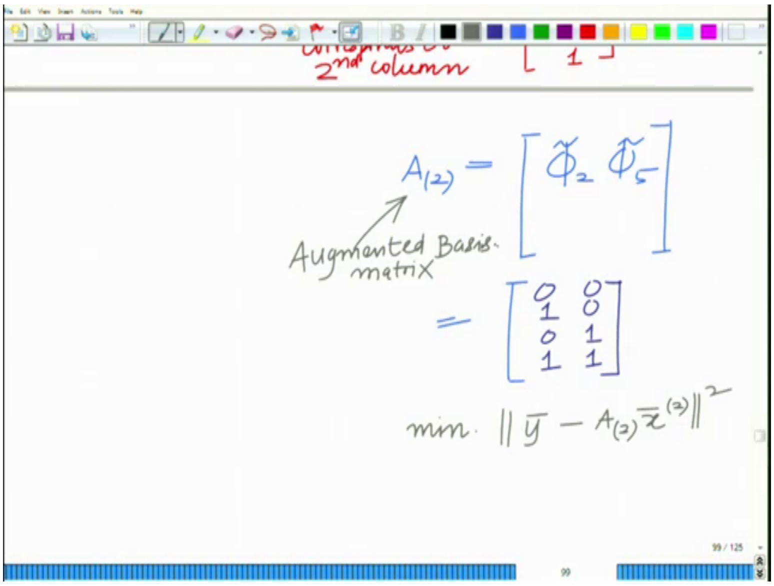

And these will be so, this is basically what it says is i 2, that is the index picked up in the second iteration is basically 2, ok. And the columns corresponding columns are 0 1 0 1 and 0 0 1 1. And now you again solve the least squares problem. Now you have the estimate x hat; which is basically you solve the least squares problem nor y bar minus A 2.

Second iteration x bar 2 whole square it is a least squares problem. Remember this is your, you can call this as the augmented basis matrix.

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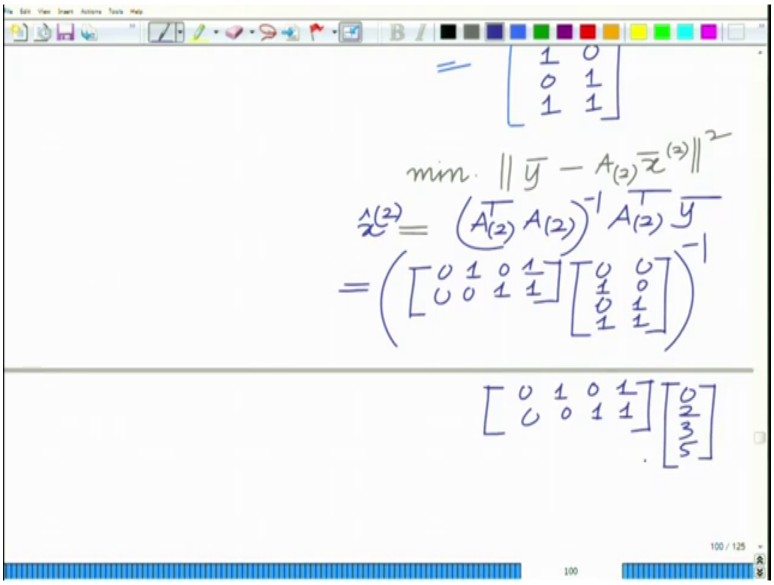

And once you solve this least squares problem what you get is, x hat 2 equals A 2 transpose A 2 inverse A 2 transpose y bar; which is basically if you look at A 2 transpose, that is 0 1 0 1 0 0 1 1 times 0 1 0 1 0 0 1 1, A 2 transpose A 2 into inverse into A 2 transpose.

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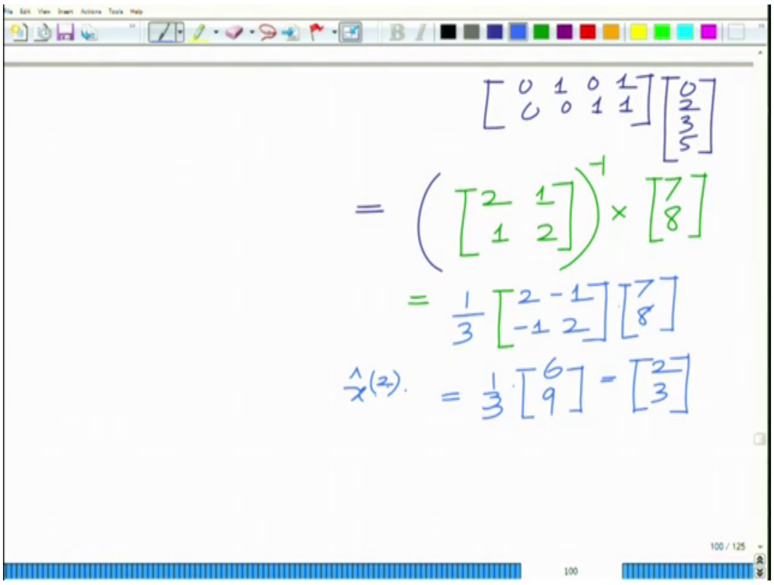

Once again 0 1 0 1 0 0 1 1 into y bar; which is 0 2 3 5 which basically equals this is 2, this is 1 1 2 inverse of this matrix times 0 1 0 1 0 0 1 1 times y bar 0 2 3 5, that will be basically 7 8. And inverse of this 2 cross 2 matrix is very simple. You interchange the diagonal elements which are the same. Negative of diagonal elements, and you divide by the determinant 1 by 4 minus 1 which is 3 times 7 8. So, this is 1 by 3 times 16 minus 7 is 9. I am sorry, 14 minus 8 is 6 16 minus 7 is 9, and this will given now give us 2 3 very simple.

And this is basically this is nothing but your x hat 2. Estimate of x hat 2 estimate in the second iteration.

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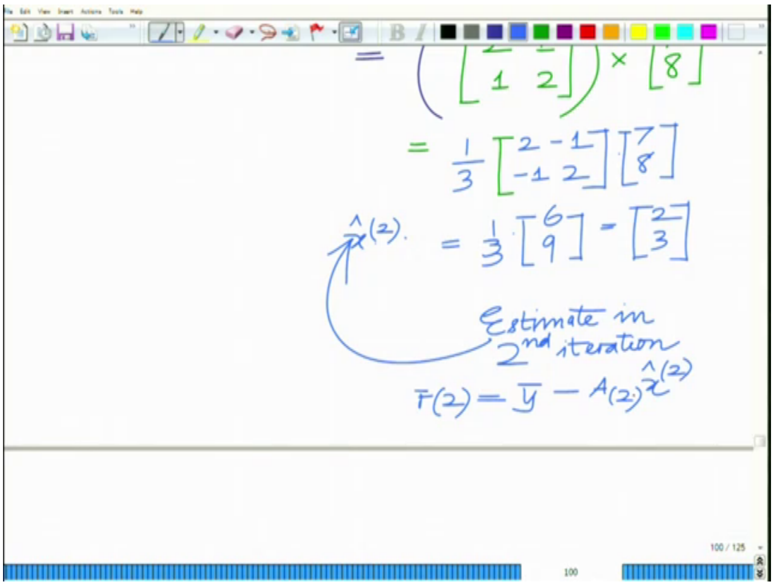

This is estimate in the second iteration. And now of course, we again need to find the residue. That is r bar 2 equals y bar minus A 2 x hat 2.

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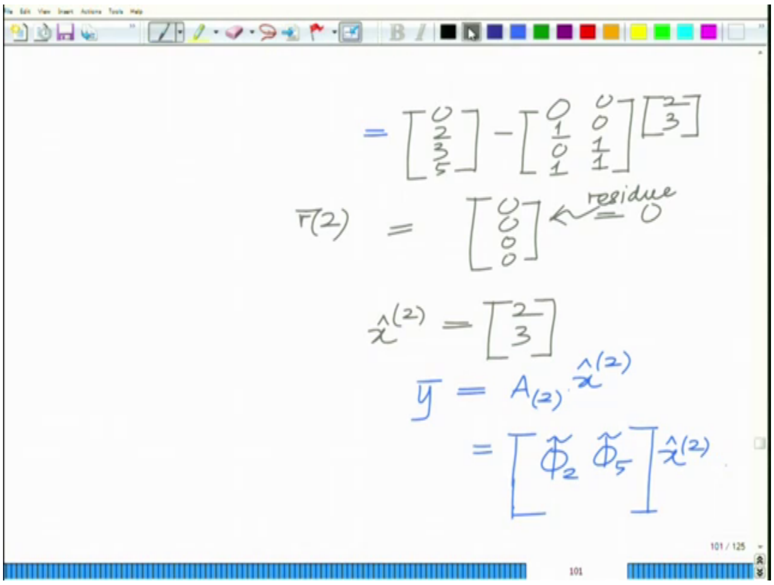

This will be 0 2 3 5 minus A 2; which is 0 1 0 1 0 0 1 1 2 3. And you can calculate this, and what you will see is this residue is exactly 0. Residue equal to 0, this is your r bar 2 r bar 2. So, the residue equals 0 which basically means that you are exactly able to exactly approximate y bar in the second iteration, which means you are using the basis matrix in the second iteration. That is comprising which comprise of the columns phi 2 tilde and phi 5 tilde.

Therefore, your x hat 2 or the second iteration and what we have is y bar equals A 2, it is basically as 2 columns, phi 2 tilde phi 5 tilde into x 2.

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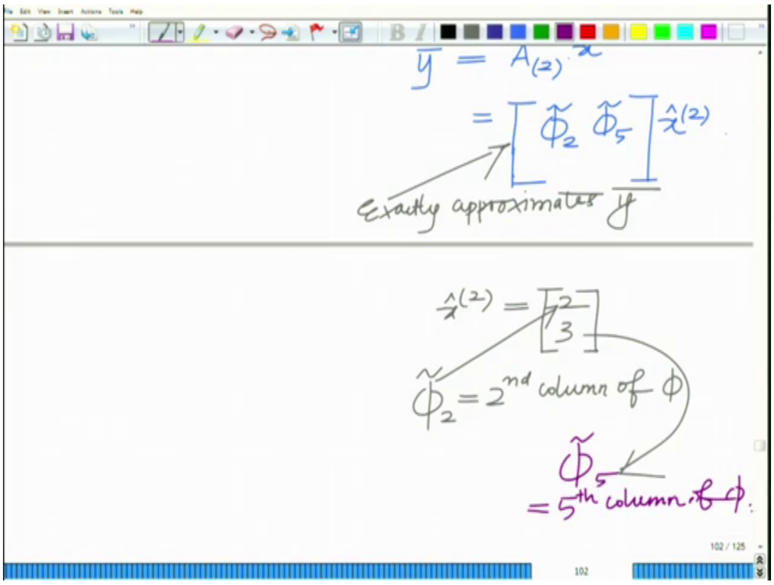

So, this is able exactly approximates y bar. And therefore, residue 0 no further iterations are needed. So, which means if you look at x hat of 2; which is equal to 2 comma 3, 2 corresponds to remember, each correspond 2 corresponds to phi tilde 2; which is basically second column of phi. And 3 corresponds to phi tilde 5 equals 5th column of the matrix phi.

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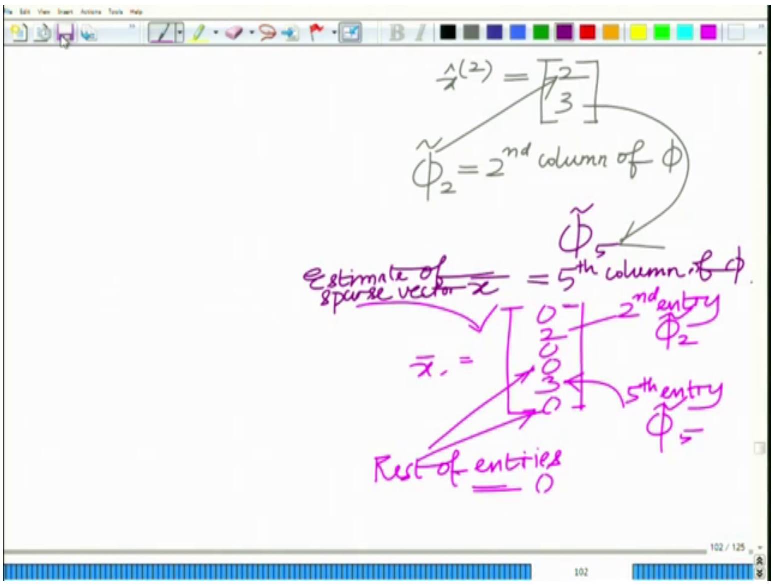

And therefore, now you can reconstruct the sparse vector x bar as follows, only the second entry will be 2, and 5th entry will be 3 and the rest of the entries are 6.

So, this is second entry corresponding to phi 2 tilde, this is 5th entry corresponding to phi 5 tilde, and rest of the entries, rest of the entries are 0. Rest of the entries are 0, and therefore, this is your estimate of the sparse vector x bar. And of course, as we said this is a simple example it is simply a paper and pen example something that you can do on the back of an envelope kind of a calculation.

But of course, problems in practice this is just for the purpose of illustration, with problems in practice are frequently more complex, and they will be quite involved for instance the size of the dictionary matrix 5 can be of the size, 500 cross let us say 10,000. Remember, the characteristic of 5 it will always have many much fewer rows than columns, because the columns represent the unknowns, the rows represent the observations all right.

But you can use this OMP algorithms scale it up and use it for similar scenarios. And as I have already said OMP the interesting thing about OMP is that it is a very simple algorithm, all you are doing at each stage is finding the projection of y bar.

And it is finding the position of y bar or the residue on each column of phi; choosing the one that has a maximum projection all right. Computing the least squares estimate that gives you the estimate x hat corresponding to the sparse vector in that iteration, remove the current estimate the best approximation to y bar all right. Form the residue and then continue in the subsequent iterations right.

So, this is OMP algorithm this example clearly illustrates the process hopefully it clarifies. Some of your doubts or, some of how do I put it points which are points which earlier lacking clarity, because the algorithm was theoretical in nature. So, this algorithm sort of explicitly illustrates the OMP the working of OMP through an exam, all right. So, let us stop here, and continue in the subsequent modules.

Thank you very much.