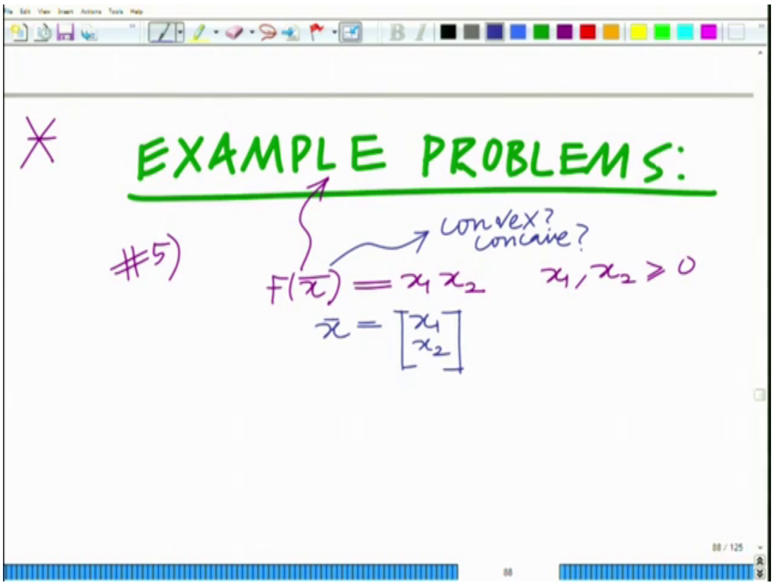
Applied Optimization for Wireless, Machine Learning, Big Data
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture – 32
Example Problems: verify Convexity, Quasi-Convexity and Quasi-Concavity of functions

Hello. Welcome to another module in this massive open online course. So, we are looking at example problems in convex functions and convexity. Let us continue our discussion.

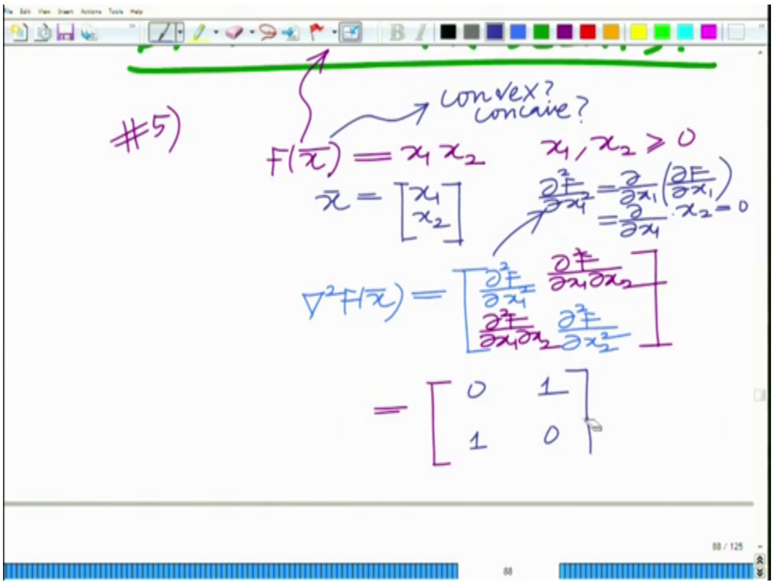
(Refer Slide Time: 00:23)



So, we are looking at example problems in convex functions and well, let us look at this is problem number 5. F of x bar equals x 1 x 2 for the region where both x 1 x 2 are greater than or equal to 0 ah. We want to ask the question is F of x bar of course, x bar we can think of this as the 2 dimensional vector. This is a function of 2 variables x 1 x 2. We want to ask is this convex or concave? Convex concave or neither.

Remember it, function need not need not be only either convex or concave, but can be neither convex nor concave, all right. So, that is an important point to keep in mind.

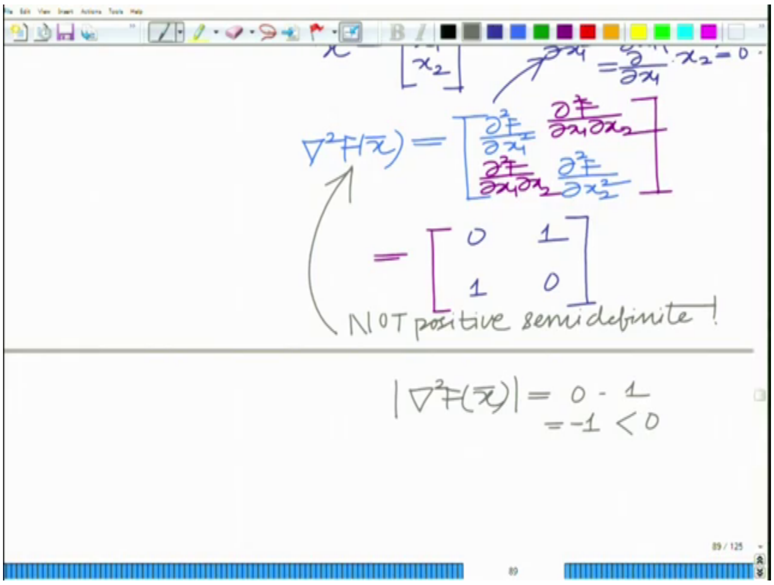
(Refer Slide Time: 01:44)



Let us start by looking at once again, remember we have a simple test for convexity, that is the hessian which in this case is simply dou square F by dou x 1 square dou square F by dou x 2 square. These are the 2 diagonal elements and the off diagonal elements are dou square F by dou x 1 partial with respect to x 1 x 2 partial with respect to x 1 and x 2. And this is equal to if you look at this partial with respect to the first partial with respect to x 1 is x 2 partial with respect to x x 1 of x 2.

So, you can evaluate this as follows do partial second order partial with respect to x 1 is partial with respect to x 1 of partial with respect to x 1, but partial with respect x 1 is x 2. So, this is partial with respect to x 1 of x 2 which is 0. So, you can see this is 0 partial with respect to x 1 x 2 you can see this is one partial with respect to x 1 x 2 x 1 the second order partial with respect to x 2 is also 0. Now, first thing you can see is this is a symmetric matrix correct;

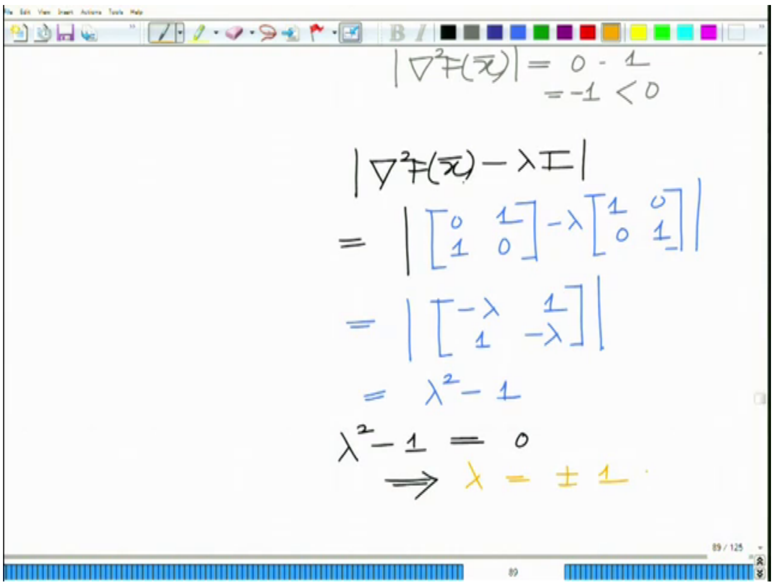
(Refer Slide Time: 03:23)



So, this matrix here, the hessian is a symmetric matrix, but this is not positive semi definite, this is not a positive semi definite. In fact, if you look at the determinant of this is 0 minus 1 equals minus 1. This is negative, ok. So, the determinant is negative.

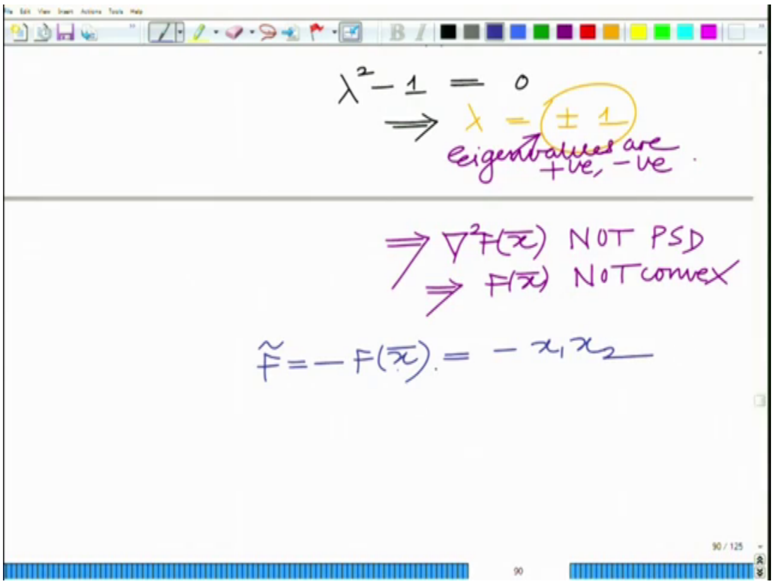
Remember, the determinant of a positive semi definite matrix has to be a positive quantity. Because, the determinant is a product of the eigenvalues, all of these eigenvalues are either are non-negative. Therefore, the determinant has to be greater than or equal to 0 for a positive semi definite matrix, all right.

(Refer Slide Time: 04:24)



In fact, if you compute the eigenvalues, that is find the characteristic polynomial minus lambda times I and take the determinant. This is equal to determinant of 0 1 1 0 minus lambda times 1 0 0 1 and you take the determinant of this that is equal to minus lambda minus lambda 1 1 minus lambda. And if you take the determinant of this, that is basically lambda square minus 1 and lambda square minus 1 equals 0 implies basically lambda equals plus or minus 1. And you can see the eigenvalues are both positive and negative. It has a positive eigenvalue and a negative eigenvalue and negative.

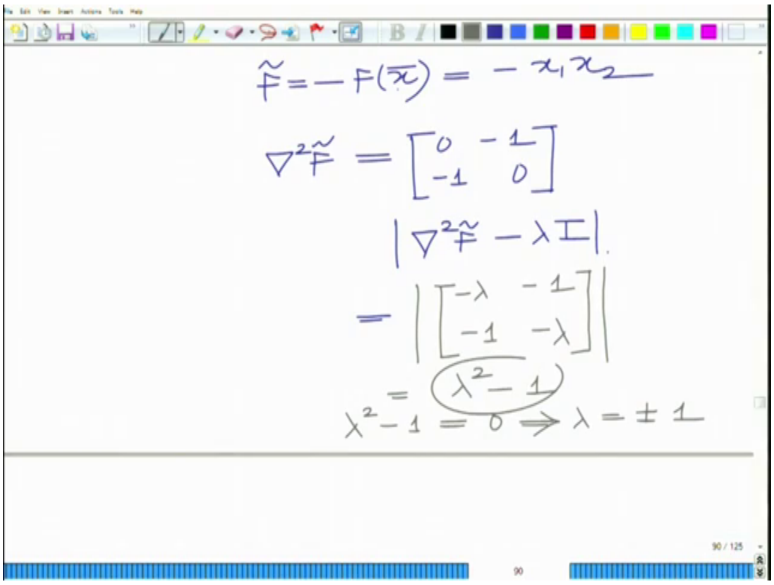
(Refer Slide Time: 05:27)



So, implies matrix remember positive semi definite matrix has only positive semi definite matrix has only non-negative, that is eigenvalues are greater than equal to 0. Here you have a negative eigenvalue that is minus 1 which implies the hessian is not positive semi definite.

And therefore, F of x bar is not convex implies delta square F of x bar is not positive semi definite implies F of x bar is not convex. Now, what about concavity? For that, consider minus x F tilde equals minus F x bar equals remember F of x is concave if minus F of x is convex. So, we consider minus x 1 x 2.

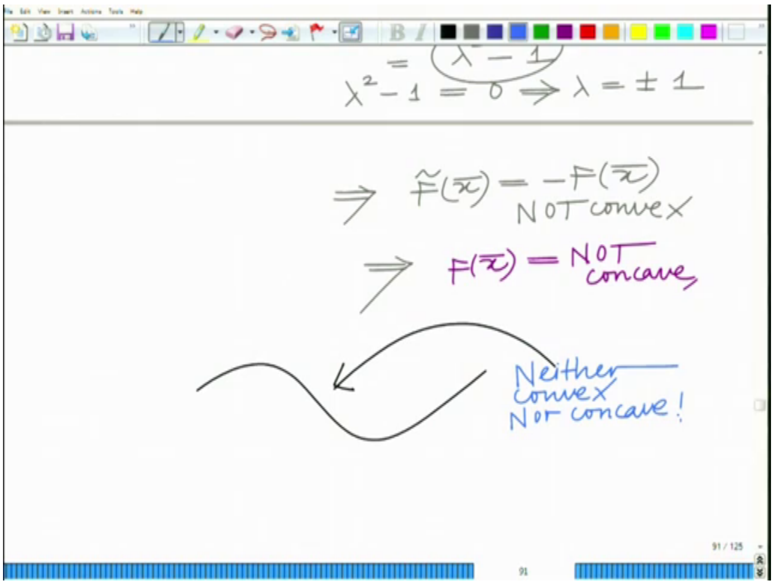
(Refer Slide Time: 06:39)



Now, consider the hessian of F tilde. The hessian of F tilde is simply the hessian minus of the hessian of x bar. So, this will be 0 minus 1 minus 1 0. Now, again check the eigenvalues delta square F tilde; the hessian minus lambda I determinant. Look at the determinant of the hessian, this will be the determinant of well minus lambda minus 1 minus 1 minus lambda equals once again lambda square minus 1 and once again lambda square minus 1 equals 0 implies lambda equals plus or minus 1.

Again, the hessian of F tilde is not positive semi definite which means, the hessian of F which means F tilde is not convex. And therefore, F is not concave. Remember, F is concave only if F tilde, that is minus of is convex. So, this implies F tilde x bar equals minus of F of x bar is not convex implies F of x bar is not concave, x bar is not concave now.

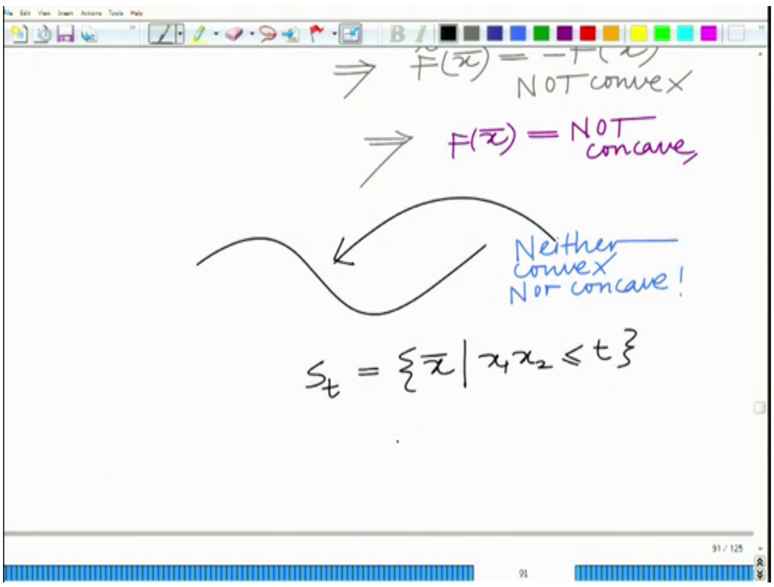
(Refer Slide Time: 04:56)



So, F of; so, x 1 x 2, you can see it is very interesting. It is neither convex nor concave, alright. That shows that any function does not always it is not convex does not automatically mean that it is concave, all right. They can be functions which are neither convex and concave and that is easy to see.

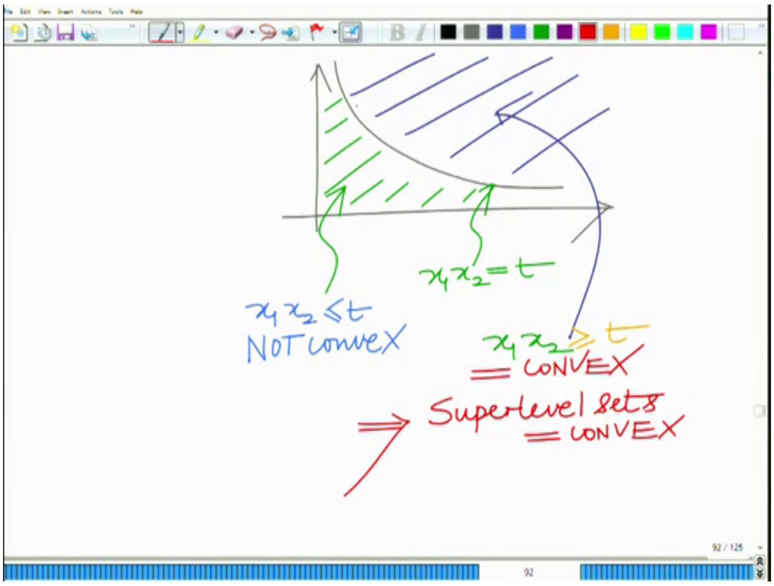
Because, if you have a function that looks something like this; so, neither convex nor concave, ok. And so, these kind of functions these are neither convex nor concave; something to keep in mind.

(Refer Slide Time: 09:14)



How about quasi convexity? Now, for quasi convexity, remember you have to look at the level sets S of t equals x bar such that x 1 x 2 less than equal to t. Now, if you look at the set x 1 x 2 less than equal to t, what you will observe is that, if you plot this if you look at the set x 1 x 2 less than equal to t. That will be this set. This is the curve x 1 x 2 equals t and this is the area that is x 1 x 2 less than equal to t. And you can see this set is not convex, the set is not convex.

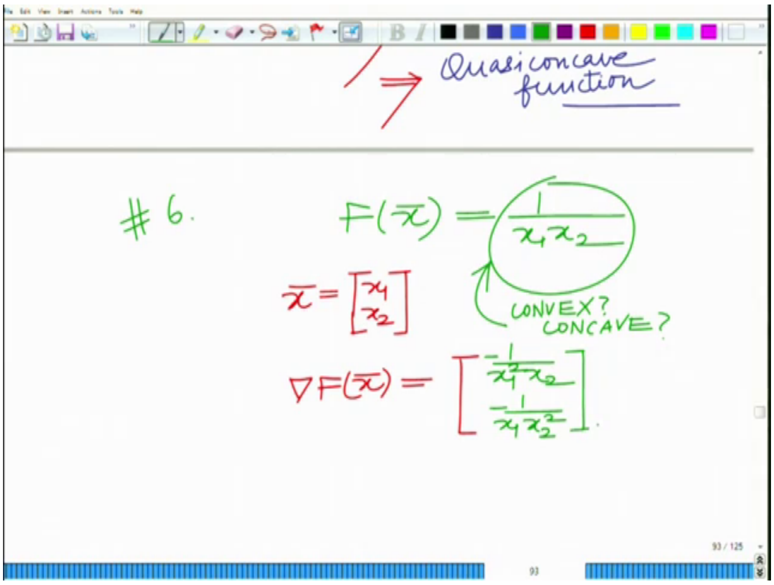
(Refer Slide Time: 09:34)



On the other hand, if you look at the super level set, that is x 1 x 2 greater than or equal to t, this is a convex set because if you take any 2 points, join them by a line, it lies in the set. This set the super level set x 1 x 2 greater than or equal to this is a convex set, ok.

So, implies thus not the sublevel set super level sets equals a or convex, that is super level set sublevel set. Remember, is the set such that x 1 x 2 is less than equal to t super level set is the set x 1 x 2 greater than or equal to t for any parameter value of the parameter t. So, the super level sets are convex which means, this is a quasi-concave function utilize this is a quasi-concave, this is a quasi-concave.

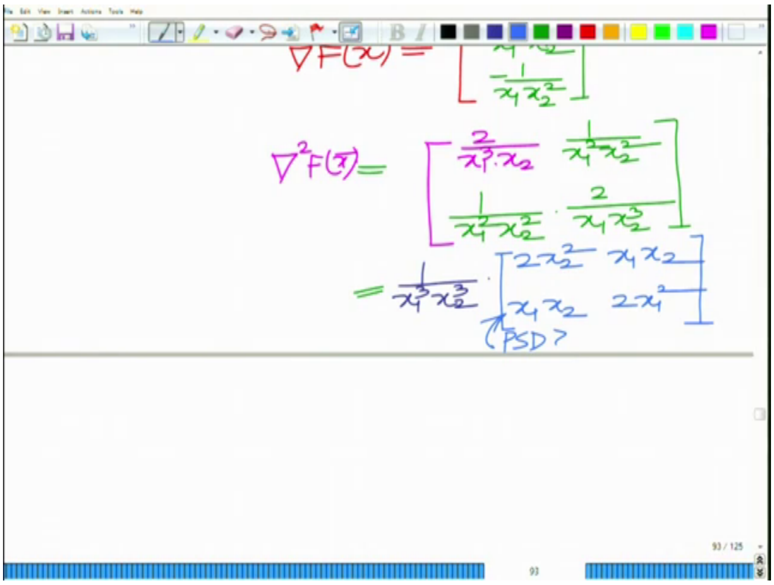
(Refer Slide Time: 11:16)



So, x 1 x 2 neither convex nor concave; it is a quasi-concave option ok. Similarly, let us now move on to the next example. Let us now consider the reciprocal of the previous 1 F of x bar equals 1 over x 1 x 2. Again, we want to ask the same question is this function 1 over x 1 x 2 is it convex or is it concave and you can once again find the hessian of this.

The hessian of this will be well, let us first start with the gradient x bar is again the 2 dimensional vector. This is a function of 2 variables x 1 x 2. So, the gradient with respect to x 1 x 2 will be well this will be derivative with respect to x 1 that will be minus 1 over x 1 square x 2 derivative with respect to x 2 will be minus 1 over x 1 x 2 square.

(Refer Slide Time: 12:52)

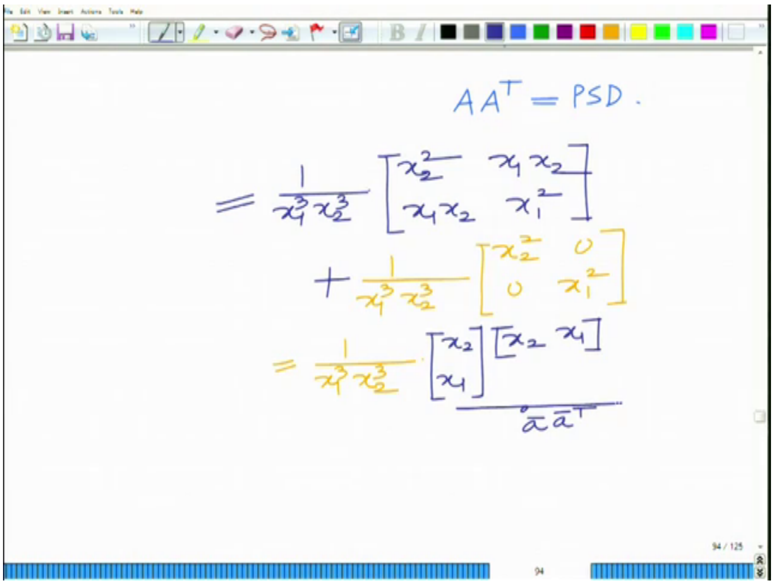


And ah, now the hessian will be take the first element differentiated with respect to x 1. So, that will be 2 over x 1 cube x 2 and now differentiate the first element with respect to x 2. So, this will be 1 over x 1 square x 2 square. This will be 1 over symmetric.

So, this element will also be 1 over x 1 square of x 2 square the 2 cross 2 element will be well that will be 2 over x 1 x 2 q ok. And now, we have to see what s is this matrix positive semi definite and you can simplify this by bringing x 1 cube x 2 cube outside. This will be 1 over x 1 cube x 2 cube times well times. This is twice x 2 square twice x 1 square and this will be x 1 x 2 x 1 x 2.

Now, we want to ask the question is this matrix positive semi definite. This is the hessian and we want to ask the question is this matrix positive semi definite. Now, there are many ways to show this. If the matrix is positive semi definite, one of the methods is of course, to compute the eigenvalues which might be slightly tedious what we are going to do here is, we are going to decompose this in the form of factors which are a times a transpose, alright.

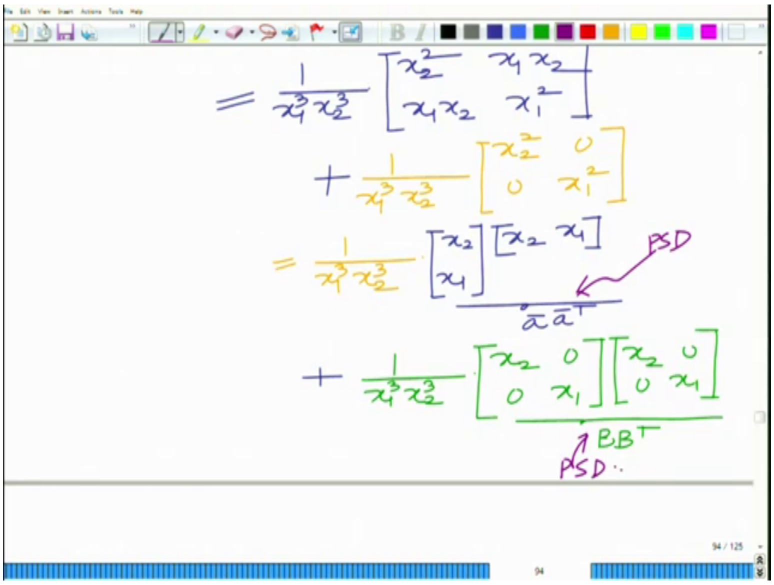
(Refer Slide Time: 15:00)



Now, remember each such factor is positive semi definite. We said that if a matrix can be expressed as A A transpose. This is positive semi definite and the sum of such positive semi definite matrices is positive definite. So, we are going to factorize this into a sum of matrices which can be expressed in this form.

And, you can clearly see it is not very difficult. You can first write this as x 1 cube x 2 cube you can write this as x 1 or x 2 square x 1 x 2 x 1 x 2 x 1 square x 1 square plus this will be 1 over x 1 cube x 2 cube x 2 square 0 0 x 1 and this will be now I can decompose this into factors.

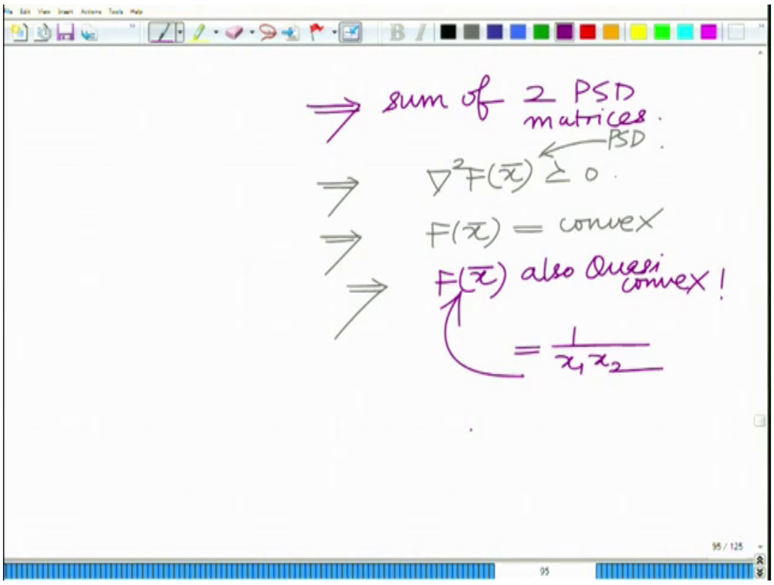
(Refer Slide Time: 16:35)



Now, you can see this will be 1 over x 1 cube x 2 cube times x 2 or x 2 x 1 times x 2 x 1, that is this is of the form vector a bar into a bar transpose plus. Obviously, the next one you can easily see that this is x 1 cube x 2 cube diagonal matrix x 2 0 x 1 0 times x 2 0 and this will be a matrix of the form B B transpose. So, we are decomposed it into the sum of matrices which are factorized as A A transpose.

So, each of these matrices component matrices is positive semi definite. Therefore, the sum is positive semi definite. You can easily see that if 2 matrices are positive semi definite alright, compatible matrices. If you sum them, you get another positive semi definite matrix ok. So, this is positive semi definite, this is positive semi definite

(Refer Slide Time: 17:38)

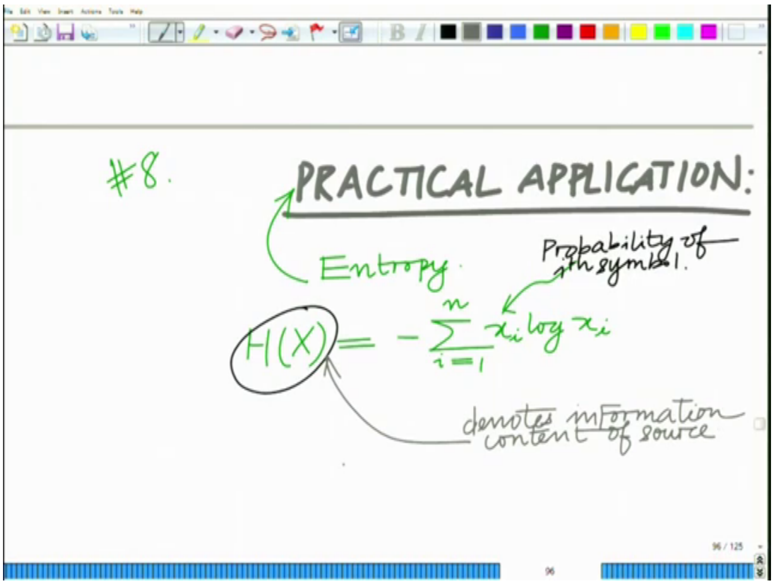


And therefore, this implies that basically your matrix this implies that this is the sum of 2 positive semi definite matrices. Remember, that this is also only for x 1 comma x 2 greater than or equal to 0 which means these factors, that is if you look at these factors, these factors 1 over x 1 cube x 2 cube is or also greater than equal to 0. Therefore, this is positive semi definite matrix. Matrices weighted by positive factors. So, these are positive, these are positive.

So, the resulting matrix is positive semi definite. So, that means, hessian is positive semi definite ok. Hessian is positive semi definite. Remember, this notation which implies that F of x bar equals convex and it can also be seen that since it is convex, it is also quasi convex because any convex function is also quasi convex ok.

So, this implies that F of x bar also quasi convex sorts 1 over x 1 x 2 remember what is F of x bar if x bar this is 1 over x 1 x 2 this is quasi convex alright. So, 1 over x 1 x 2 is basically convex and hence, it is also quasi convex.

(Refer Slide Time: 19:49)

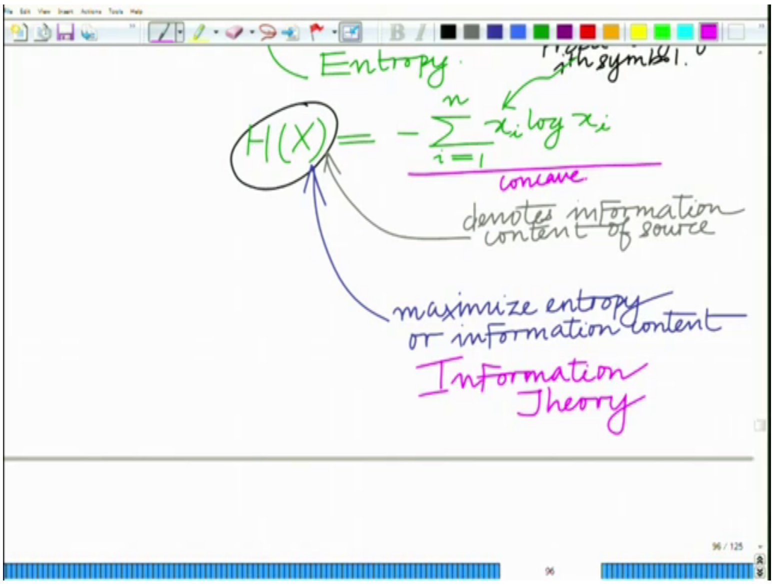


Let us look at another example and this is you can treat this as a practical application. In fact, it is a very interesting we are going to look at the entropy function, that is which can be defined as the entropy of the source H can be defined as minus summation or entropy of a minus summation i equals 1 to n x i log to the base e natural log x i, where x i equals probability of the i'th symbol.

This is the probability of the i'th symbol and this entropy denotes the information content of the source. This entropy denotes the information content of the source that is given in source with n symbols which have probabilities x 1 x 2 up to x n. What is the average information content per symbol of this source? That is given by the entropy, that is minus summation x i log log of x i.

And therefore, the higher the entropy, it means the higher the information content of the source. And therefore, we would like to maximize this entropy quantity and that has very important applications in information theory.

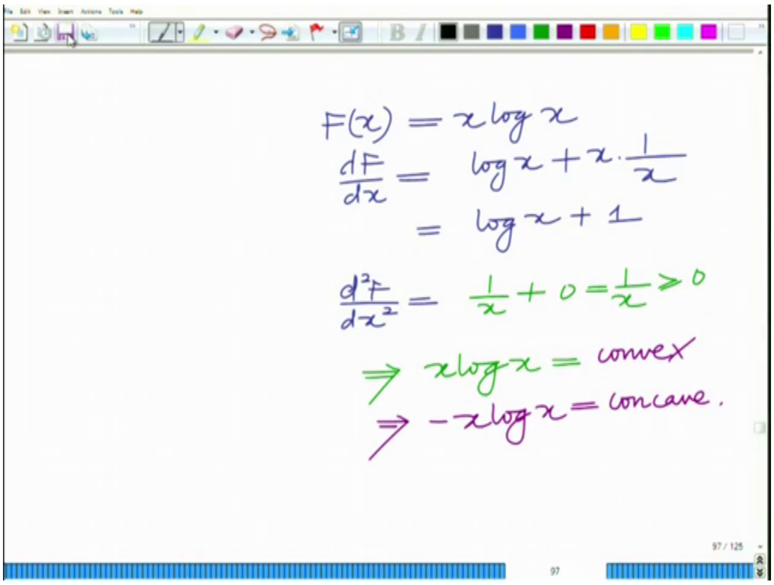
(Refer Slide Time: 22:02)



So, we would like to maximize the entropy or the information content, ok. So, this has a applications in information theory. So, this is a very important quantity in information theory and by extension in also wireless communication and signal processing.

In fact, many fields machine learning also since information theory has widespread applications in several right. As several applications is wide several applications in various fields, alright. So, therefore, this quantity entropy is very important. Now what we want to show is that, this quantity entropy is a concave quantity and that is relatively easy to show.

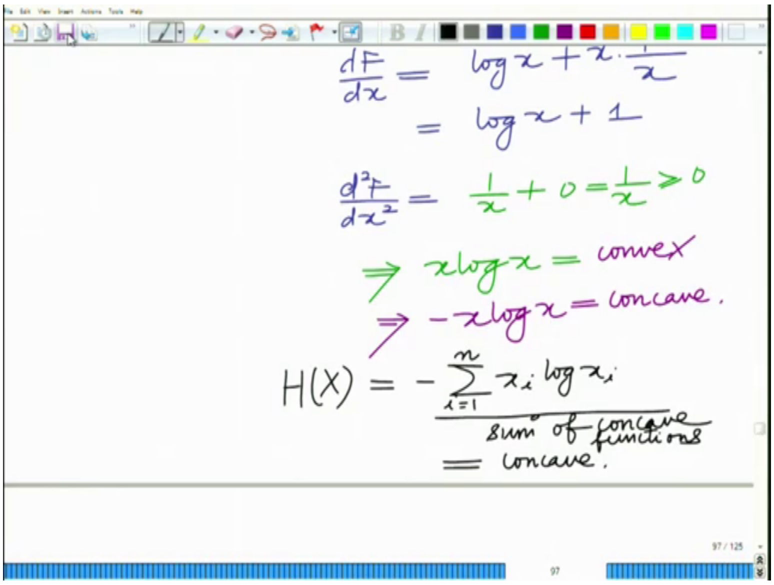
(Refer Slide Time: 23:16)



We start by considering F of x equals x log x, that is the natural logarithm and we demonstrate that this is convex ah. You take the first derivative, this is very simple, this is x. So, we use the product rule. So, first differentiate with respect to x, that is 1 times log x plus x into the derivative of log x which is 1 over x which is basically log x plus 1. And now, if you look at the second derivative, that will be derivative of log x which is 1 or x plus 0 which is 1 over x that is greater than equal to 0.

So, this implies x log x hessian is greater than or equal to 0. This implies x log x or second derivative is greater than equal to 0 implies x log x is convex which implies that minus x log x equals concave.

(Refer Slide Time: 24:42)



And therefore, now, if you look at the entropy log to the base, you can see this is the sum of concave functions implies this is concaves the entropy is the sum of concave functions. And therefore, this is in turn concave. And therefore, one can maximize the entropy; thus maximizing the average information of the given source, alright. So, we will stop here and continue in the subsequent modules.

Thank you very much.