Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture - 29
Properties of Convex Functions: Operations that preserve Convexity

Hello. Welcome to another module in this massive open online course. In this module, let us start looking at the various properties of convex functions or the operations on convex functions that preserve convexity, ok.

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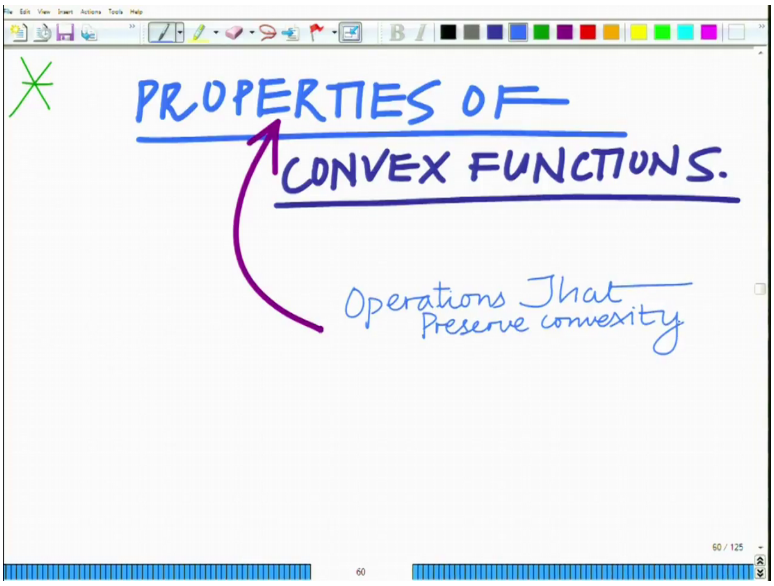

So, what we want to look at are properties or you can also say operations that preserve convexity. We want to look at the operations that preserve convexity.

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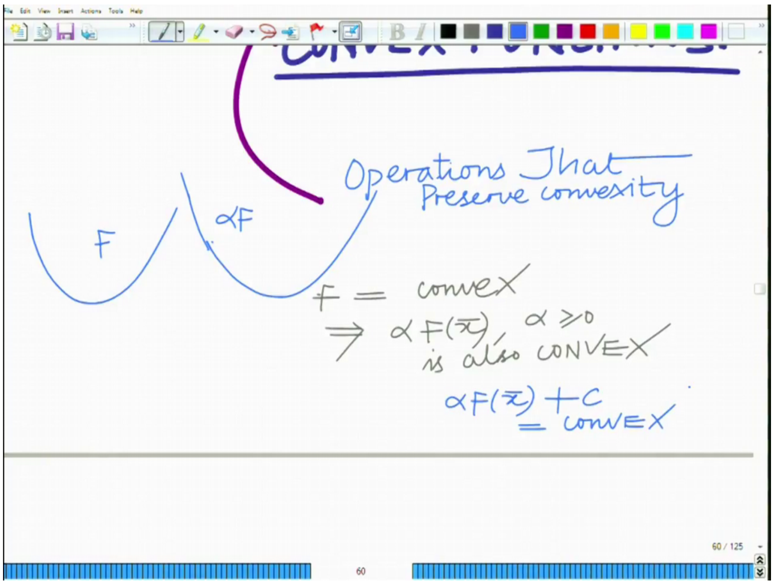

Now, F equals convex. If F is convex so, consider a function. First one is simple, F is convex, then this implies alpha times F of x bar. If that is provided, alpha is greater than or equal to 0 is also convex, correct. And this is very simple if you have what they say, is that you have a convex function and you are scaling it by a factor alpha, right.

So, this is F, this is alpha times F. So, naturally once you scale a convex, function of convex, I mean, it is important that you scale it by a non negative number, alright. So, once you scale it by a non-negative number, a convex function remains a convex function and same for translation. In fact, alpha F plus any constant is also a convex function, correct. So, if you have alpha times F of x bar plus c. This is also a convex function. So, you can scale and translate a function it remains a convex function.

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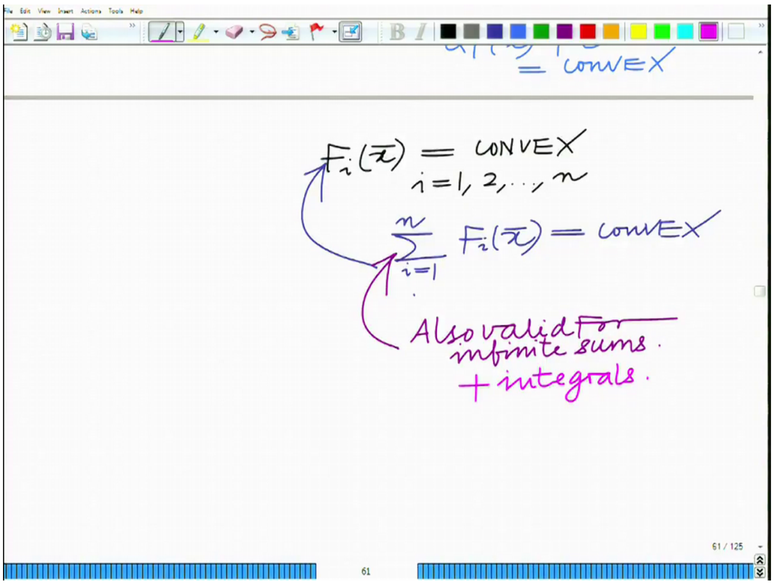

Also, simple to see that, if you have several functions F i of x bar, these are all convex for 1 for i equals let us say, then their sum is convex. So, if you take large number. So, if you take function several functions which are each of which is convex, for instance, here we are considering n functions F 1 F 2 F n, each of which is convex take their sum that is also a convex function.

In fact, this extends the interesting thing about this is this extends even to infinite sum and more importantly, integrals, alright. And, we will see an application of this later, alright. So, this extends to 1 in finite sum. We will try to see applications of this later, you can say also valid for infinite sums plus also integrals which is nothing but a continuous sum with a final function similar to convex sets, correct.

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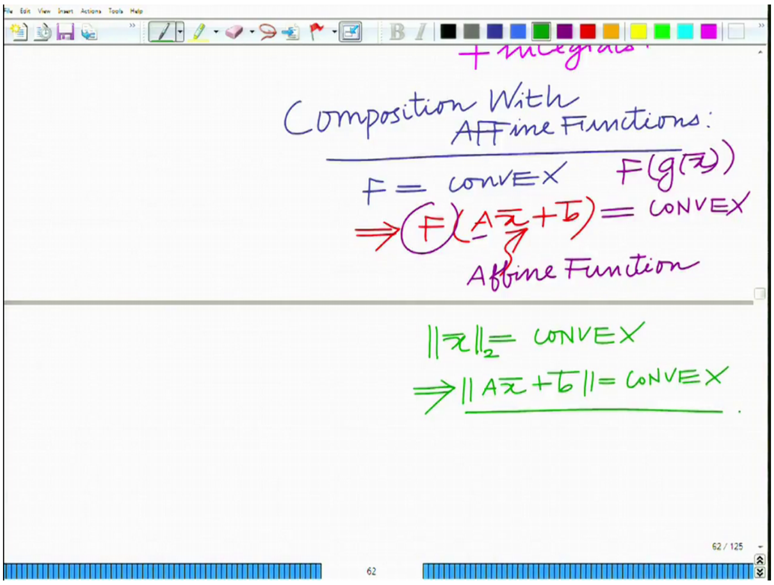

The composition with affine functions F is convex, which implies the composition F x bar plus b bar correct remember this is an affine function, ok. So, F is convex is composition with an affine, that is F of A or x bar x bar plus b bar is also convex that is remember, a composition of a function implies F of g of x bar that is F composition with g. Here, you are taking the composition of F with an affine function A x bar plus b bar. What this is the composition of a convex function like an affine function is also convex.

For instance, a simple example you have known x bar we have seen that this is convex. For instance, if we consider the 2 norm, this implies norm of x bar plus b bar. This is also convex, ok alright.

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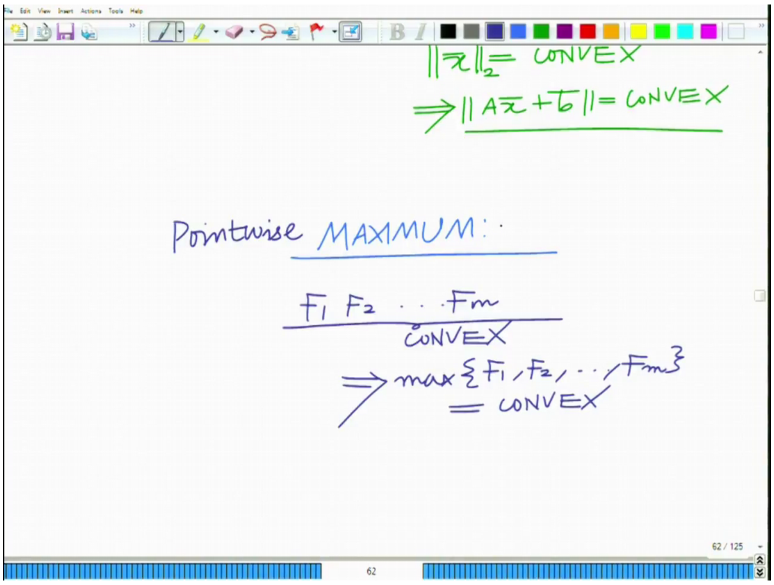

Another interesting property, the point wise maximum or you can simply call this as the maximum. This has a lot of interesting applications. If you take functions F 1 F 2 F of m which are all convex this implies the maximum of these that is F 1 F 2 each point wise, ok. So, this is also you can think of this as the point wise point wise maximum.

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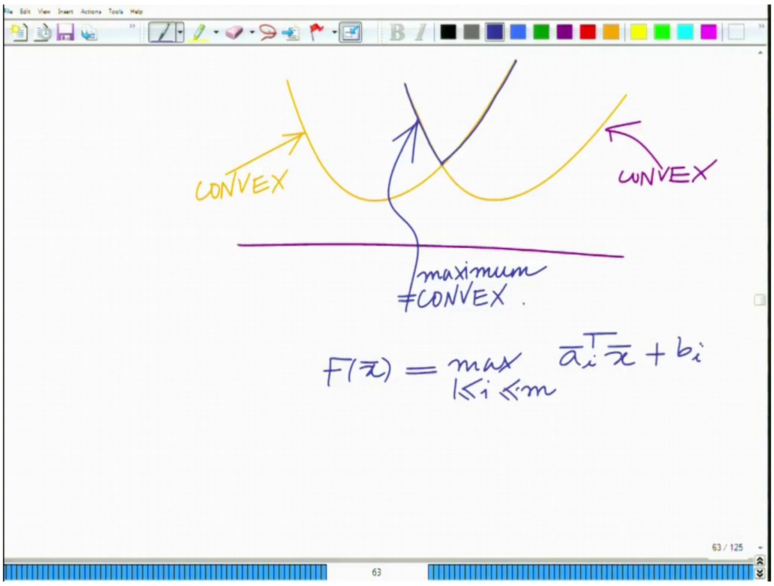

For instance, you have 2 functions convex. So, this you can see, this is convex, correct this is convex. Now, if you take the maximum of these 2, you can see the maximum is this which you can see is also a convex function. So, the maximum of 2 in fact, several convex functions; this is the maximum which is also you can see, this is also convex. For instance, you can take the maximum of a piecewise of a set of piecewise of a set of linear functions, that is maximum of i or 1 less than equal to i less than equal to m a i bar transpose plus b i. And, this is known as a piecewise linear function, correct.

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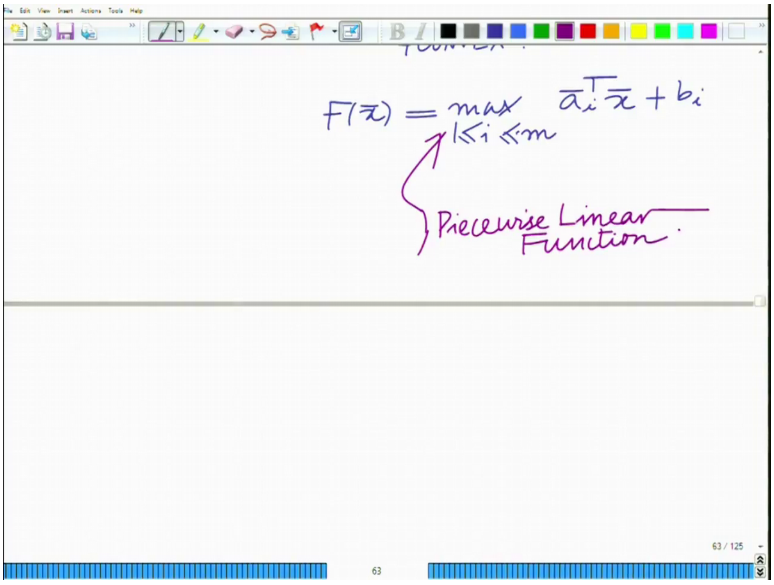

So, if you take the maximum of several linear functions, you know what you get is a piecewise linear function.

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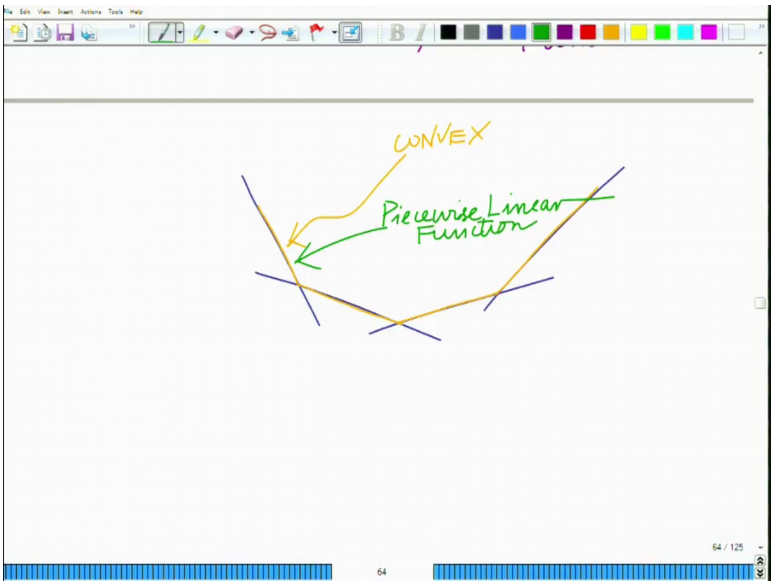

For instance, you have here you take several linear functions and you take the maximum, you can see you get something like this. So, this is basically first you have this is convex. Further, this is also a piecewise, this is a piecewise linear function.

So, you take several linear functions right which are basically hyper planes. You take their maximum, correct. What you get is a piecewise linear function which is also convex and that follows from the property that we have just seen. Let us now look at another concept that is the composition.

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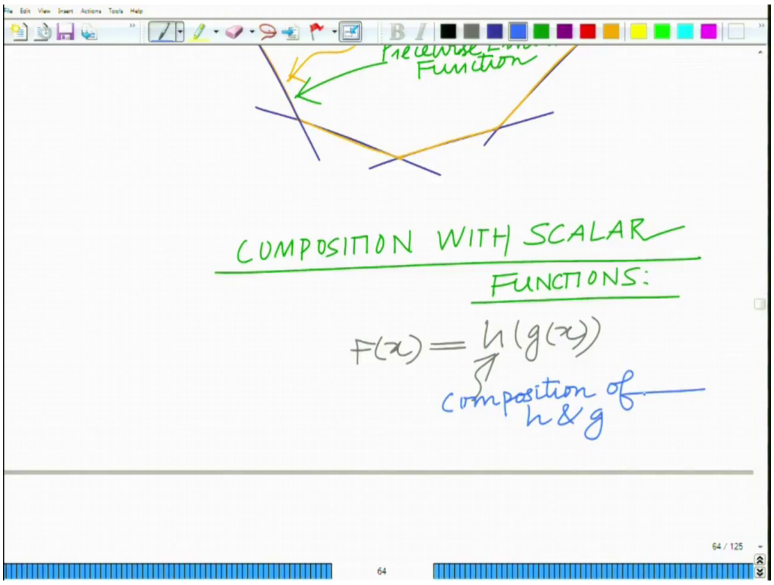

Let us look at a simple case of scalar functions. Let us look at the composition with scalar functions, that is let us say we have a function F of x equals h of g of x that is a composition of h with that is a composition of h with g. So, composition of h and g now, this is convex.

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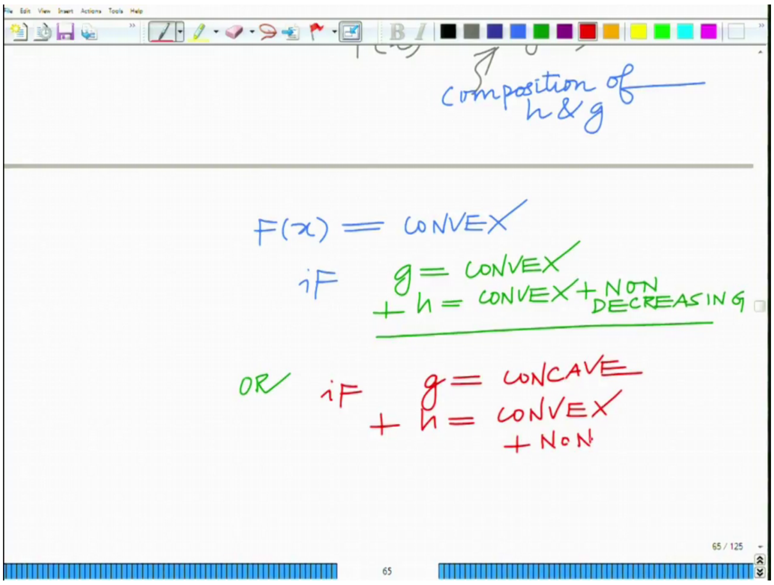

So, F of x is convex if g is convex plus h is convex and non-decreasing, that is basically either increasing or at least non-decreasing. That is, if, so we are looking at h of g of x, if a g of x is g is constant and h is both g is convex and h is both convex and non-decreasing, then F x F of x is convex or if g is concave. If g is concave and if g is concave and h; h is convex and non-increasing, non-increasing that is, it is a decreasing function or at least should not increase that it is a non increasing function.

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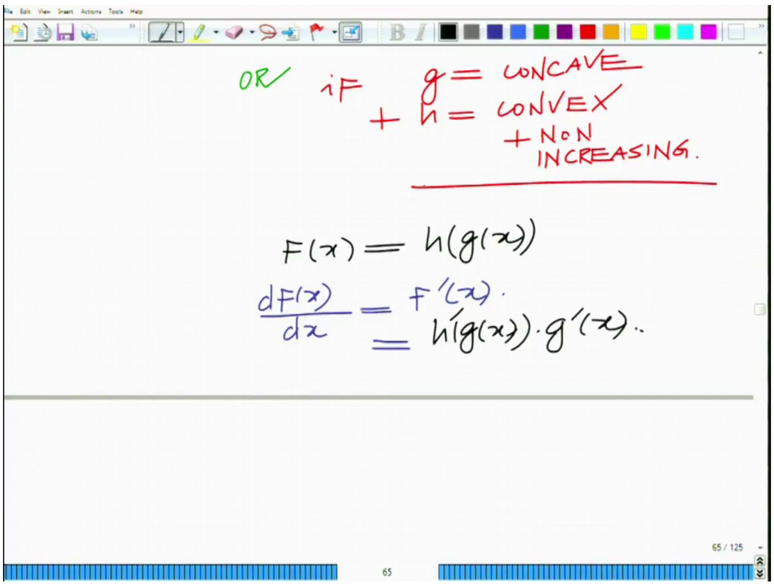

We have this following rule for composition and that is fairly simple to see we are going to use the derivative test to demonstrate this assuming the functions are differentiable, alright. What we could show is that, you have F of x equals h of x equals h of g of x. So, if you take the first order derivative, you can write it as F prime of x represented by F prime of x. This is h prime of g of x. Use the chain rule times g prime of x.

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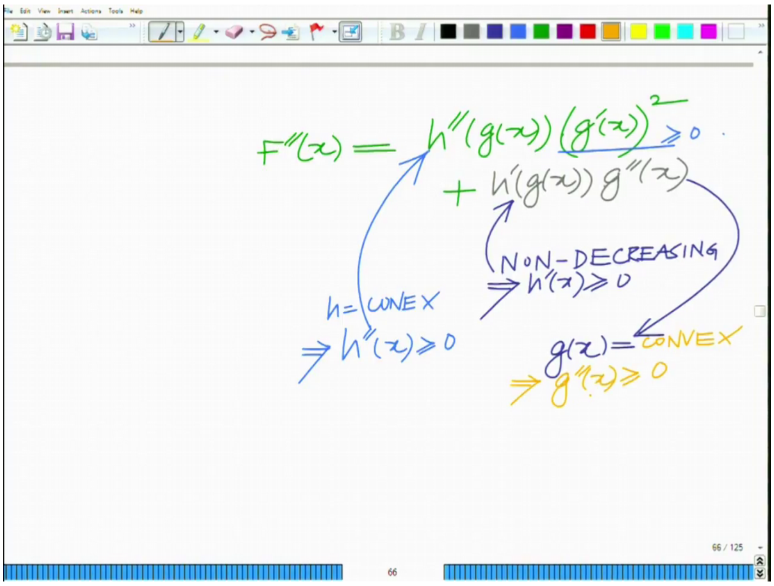

Further, you would have F double prime of x second prime of second derivative of F F of x or the derivative of F prime of x you use the product rule. So, it is h prime second derivative of g of x into derivative of g of x into derivative of g of x. So, this is g prime of x square plus well h prime of g of x into derivative of g of x which is g prime of x,. Now, there are 4 components.

Now, first you can see that g prime of x. This is always greater than equal to 0. Now, h prime of now, since h is convex, this implies h prime of h double prime of x is greater than equal to 0. Remember, the second order derivative test convex. So, the second order derivative is greater than or equal to 0. Now, this is interesting. Now, this is non-decreasing correct implies the first order. This implies h prime of x or h prime of g of x is greater than or equal to 0 and the last condition we have is g of x is convex implies the second order derivative of g of x is greater than or equal to 0.

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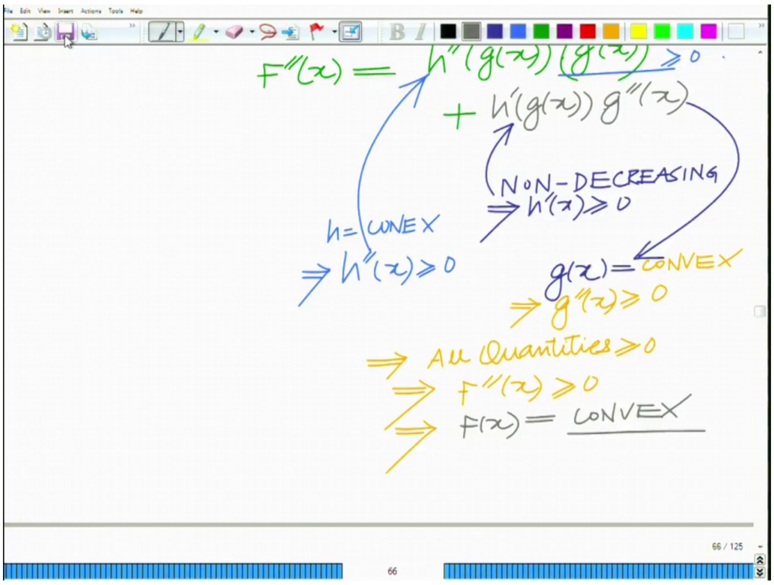

So, this implies also you can see all quantities are positive in plus second order derivative greater than or equal to 0 this implies that F of x equals convex. So, this implies, you can see if g of x is convex, h is convex and non-decreasing, correct. You can see that the composition h of g of x is convex, ok.

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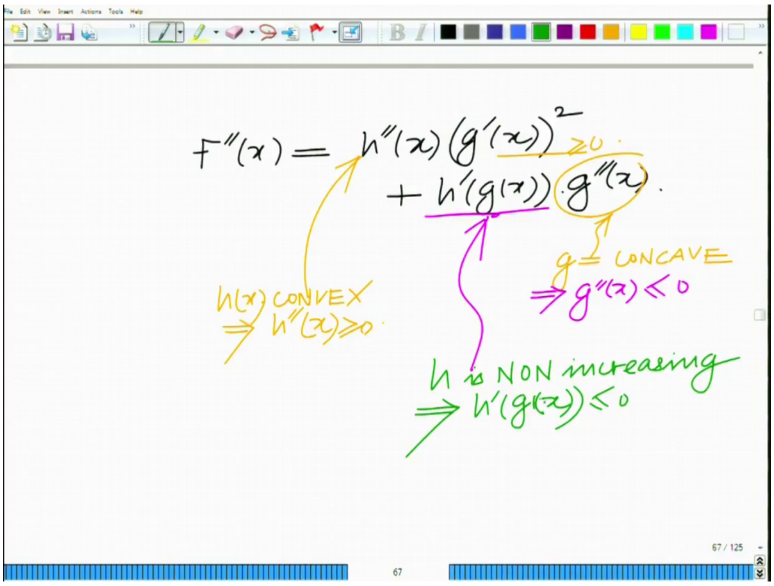

Now, similarly you can show it easily for the other condition; that is, start with again F double prime of a let us write that the second derivative is second derivative of h of x g prime of x square plus h prime of g of x minus g prime of x. Now, you can see this quantity g prime x square is always greater than equal to 0. H of x is convex in the second condition. Also, if you look at that, h of x is convex.

So, this is greater than or equal to 0, convex implies h that is the second derivative greater than equal to 0. Now, coming to this g prime, g is concave. This implies g second order derivative. G double prime is less than or equal to 0. H is non-increasing implies h prime g of x is less than or equal to 0. Now, together g prime x less than g double prime x less than equal to 0, h prime of g of g of x is less than equal to 0.

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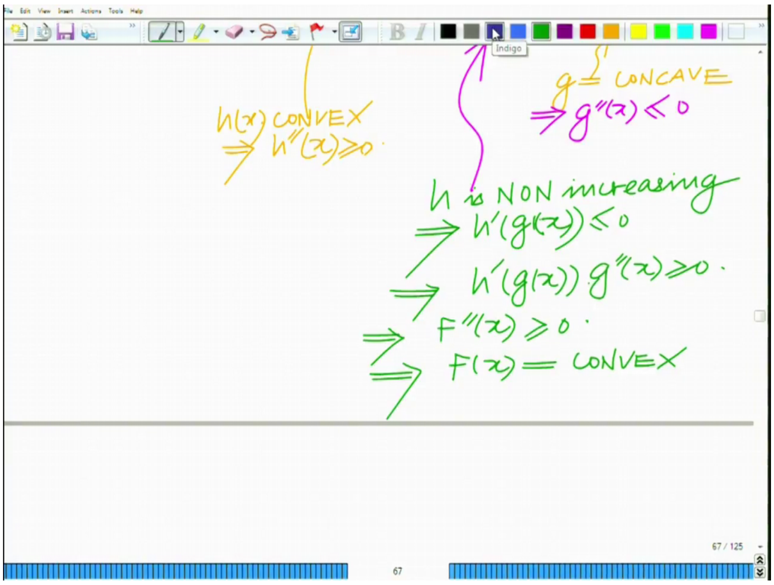

So, this implies since both of these are less than equal to 0, this implies h prime g of x into g double prime x greater than or equal to 0. So now, both the quantities in the sum are positive. So, h double prime x into g prime x square is non negative and h prime of g of x into g double prime that is second order derivative of g of x.

The product is greater than equal to 0; their sum is greater than equal to 0 implies if the second order derivative of x is greater than equal to 0 which implies F of x is convex implies F of x is convex, all right. So, we have seen these 2 conditions, alright. These are the 2 conditions to demonstrate that is these are the 2 conditions that ensure that the composition F of x, there is obtained by the composition of h of g of x is also convex and let us look at a simple example.

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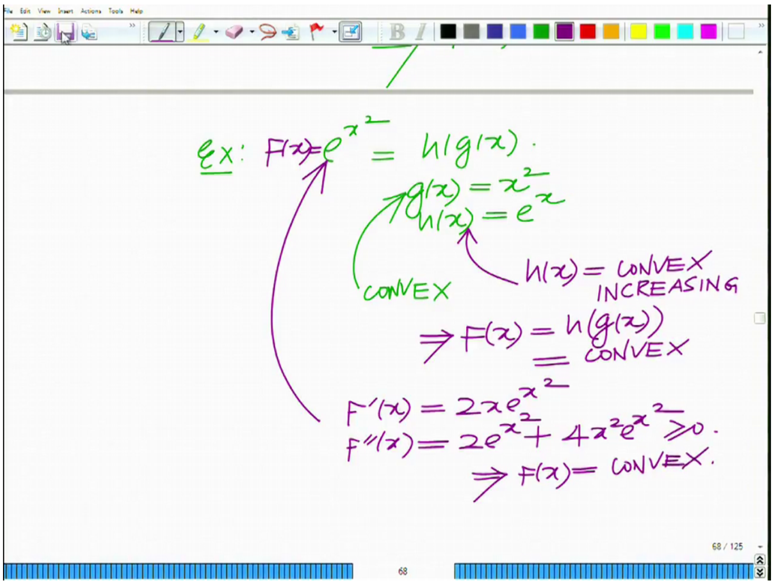

We can look at a simple example to understand this for instance example you take e raise to x square. Now, this is equal to h of g of x g of x equals x square h of x equals e raise to x. Now, you can see g of x x square. This is convex and if you look at h of x, h of x equals convex. And in fact, it is increasing. So, g of x is convex, h of x convex, non-decreasing which implies F of x which is h of g of x is convex.

And, you can just check that if you take F prime of x, you get 2 x e power x square F double prime of x is 2 e power x square plus 4 x square which is greater than equal to 0 implies that F of x equals convex; that means, F of x is a convex. Similarly, one can do derived results for the concavity of that is when is F of x concave, given that it is a composition of h of x which is one can derive the corresponding conditions for concavity, ok.

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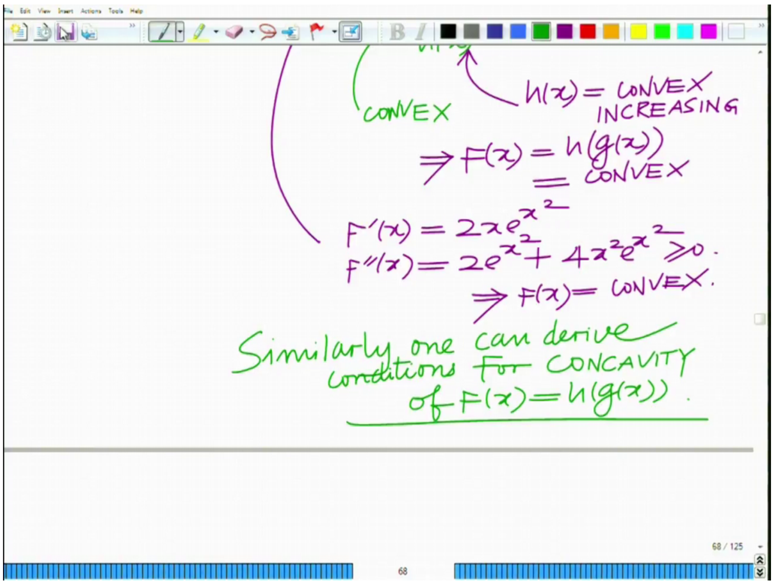

Similarly, one can derive conditions for the concavity of F of x, that is the composition of h of h n g h of g of x. So, we will stop. So, we looked at several properties of convex functions. We will stop here and continue in the subsequent modules.

Thank you very much.