Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture –26
Application: MIMO Receiver Design as a Least Squares Problem

Hello, welcome to another module in this massive open online course. So, we are looking at applications of convexity, convexity of a function of a vector, and we are looking at a practical application for a MIMO communication system or a MIMO wireless system. So, let us continue our discussion.

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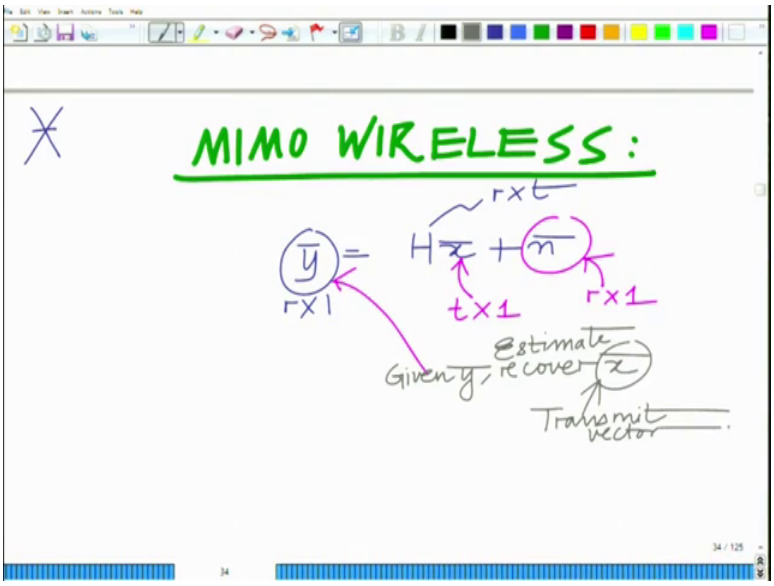

So, we are looking at application of convexity in MIMO wireless system. And what we have said is the following thing I have this model y bar equals H times x bar plus n bar this is my r cross 1 received vector, this is the r cross t channel matrix, this is the t cross 1 transmit vector and this is an r cross 1 noise vector. And the problem is given y bar, we have to recover x bar correct at the receiver all right. One has to estimate recover or basically you can also say estimate, estimate x bar which is basically your transmit vector; given the receive vector, estimate the transmit vector.

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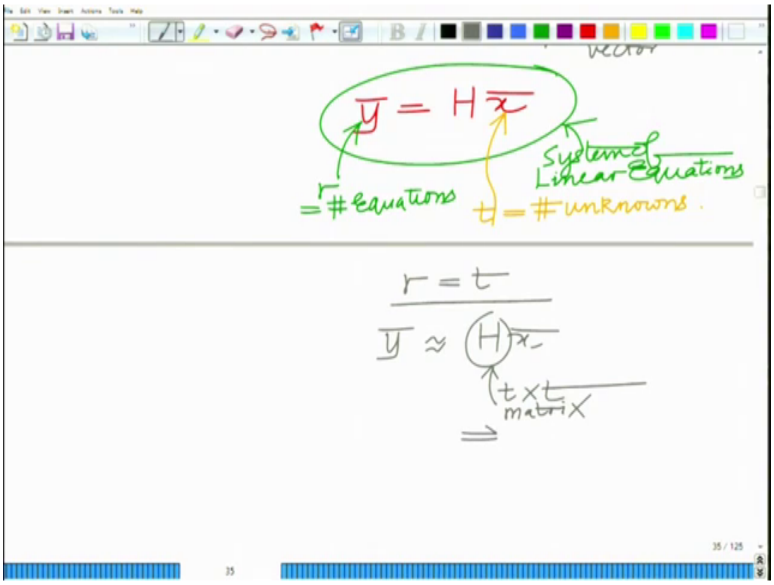

Now, let us go back for a minute look at a simple scenario let us ignore the noise for a little bit. Now, what you can see is if you ignore the noise, this reduces to y bar equals H x bar and this is basically you can see this is a system of linear equations. This is a system or linear system of equations. This is a linear system of linear equations and number of equations is basically you can see r equals number of equations correct.

And the number of unknowns is basically the elements of this x bar which are the transmitted symbols. So, t equals the number of unknowns. Now, let us consider a simple scenario start by considering a simple scenario with r equals t ok. Now, if r equals t, what happens number of equations equals number of unknowns. Therefore, the matrix H correct that is a square matrix.

And if H is invertible, then I can simply find the receive vector by doing H inverse y x bar equals H inverse y, remember this is an approximation x bar equals H inverse y. Approximation in the sense that we are assuming we are neglecting the impact or the influence of noise. So, if r equals t we have y bar equals H x bar y bar equals H x bar. So, this is t cross t matrix if r equals t because r equals t this is r cross t if r equals t is t cross t implies it is a square matrix. So, implies this is a square matrix.

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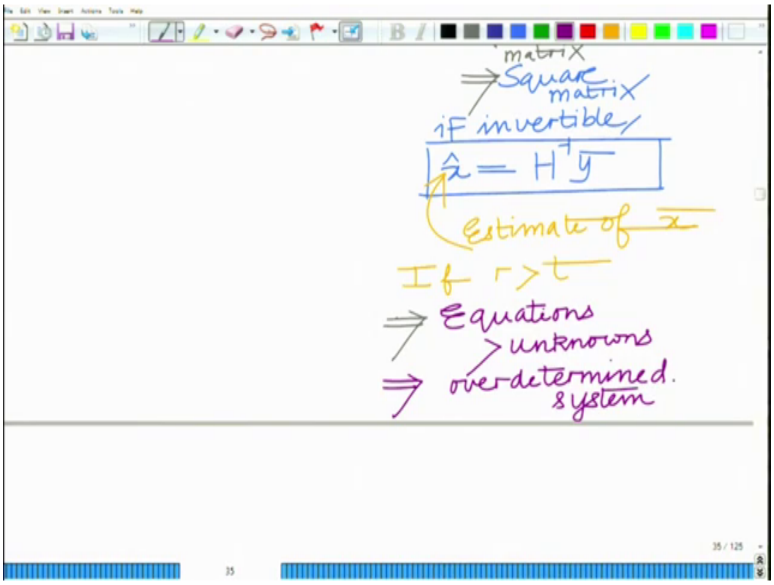

If invertible remember inverse is not guaranteed to exist, if invertible then I can find x hat that is estimate of vector x as H inverse into y bar; x hat equals H inverse y bar this is the estimate of the transmit vector x, estimate of the transmit vector x. Now, on the other hand, if r is strictly greater than t, consider another scenario r strictly greater than. Now, what happens if r is strictly greater than t, the number of equations is much greater than the number of unknowns, it means that the system is over determined.

So, this implies number of equations is greater than the number of unknown, implies it is an over determined system. Now, for an over determined system, typically you cannot solve the system of equations y bar equals H x bar you cannot solve it typically, which means you can only solve it approximately right. You cannot find an x bar such that y bar equals H x bar which means you have to find the best vector x bar such that the error approximation error y bar minus H x bar is minimized.

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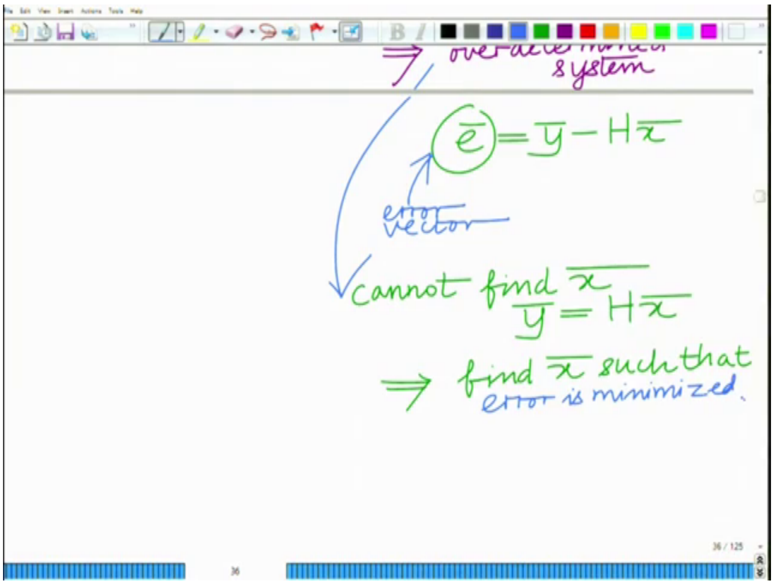

Now, what this means is since you cannot solve y bar equal to H x bar, you can find the error vector. So, e bar by e bar we denote the error vector over determined. So, you cannot find the vector such that y bar equals H x bar. So, cannot find or cannot find x bar such that y bar equals H x bar implies find x bar such that the error is minimized. The error is minimized implies.

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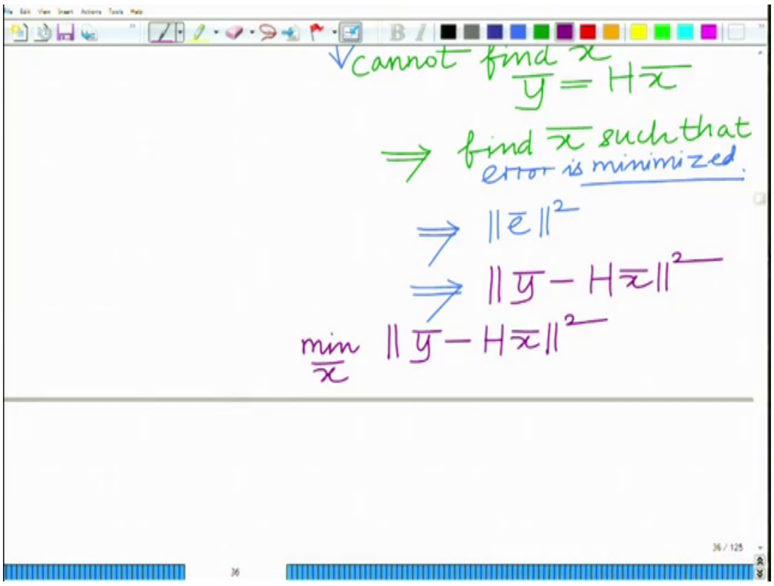

Now, what is the error? Error is we have the error vector e bar error is the norm e bar or norm e bar square ok. You can think of this as the energy of the error vector the total energy of the error that is minimized which basically implies that we want to minimize norm of y bar minus H x bar square. So, we want to find x bar. So, find x bar such that y bar minus H x bar square the error is minimized.

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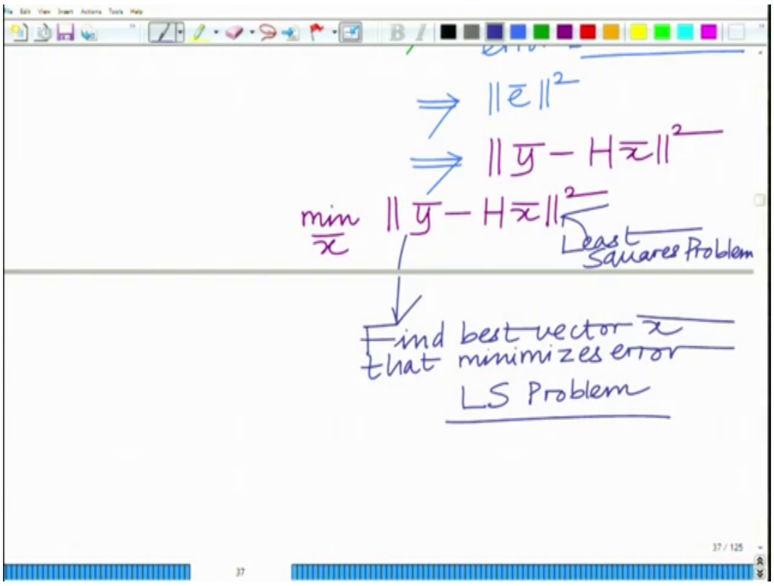

So, we want to find the best vector x bar that minimizes the error that is norm of y bar minus x bar square. This is known as the least squares, the least squares problem or simply the LS the least squares that is you want to find the x bar which gives you the least squared error or in this case the squared norm of the error simply known as the squared error. So, we want to minimize, we want to find the vector x bar the estimate which minimizes which gives you the least squared error so that is known as the least squares estimate. And this is a very important problem that arises frequently in both communications as well as single processing.

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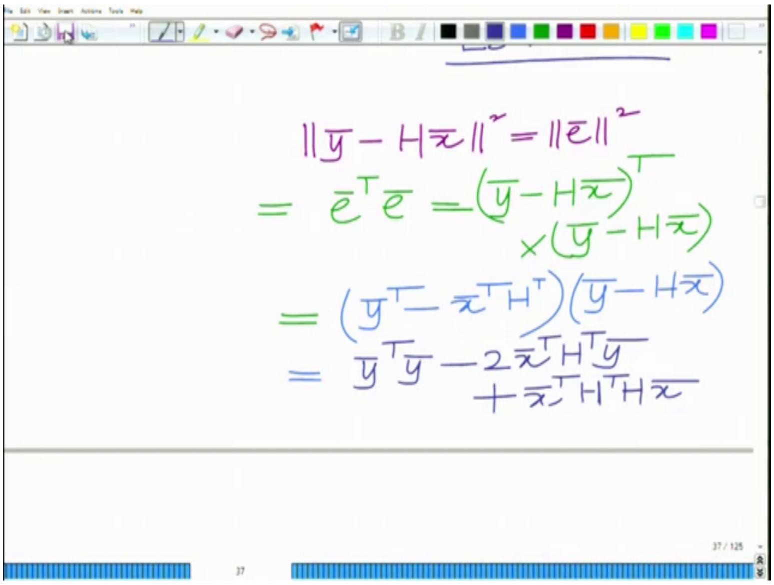

Now, we want to simplify this, let us start by simplifying this cost function norm of y minus H x bar square remember we said this is norm of error vector square. Norm of error vector square is e bar transpose e bar vector transpose itself that is a norm square of the vector which we can now write as y bar minus H x bar transpose into y bar minus H x bar which is equal to y bar transpose minus x bar transpose x transpose into y bar minus H x bar. Which is equal to now multiply this out y bar transpose y bar minus x bar transpose H transpose y bar y bar transpose H x bar these two terms are the transpose of each other remember quantity real number which is transpose of itself is equal to itself.

So, these two quantities you can see they are real numbers simply scalar quantities and they are transpose of each other, and therefore they are equal. So, I am going to simply write this as twice x bar transpose H transpose y bar plus x bar transpose H transpose H into x bar. Now, this is what we get.

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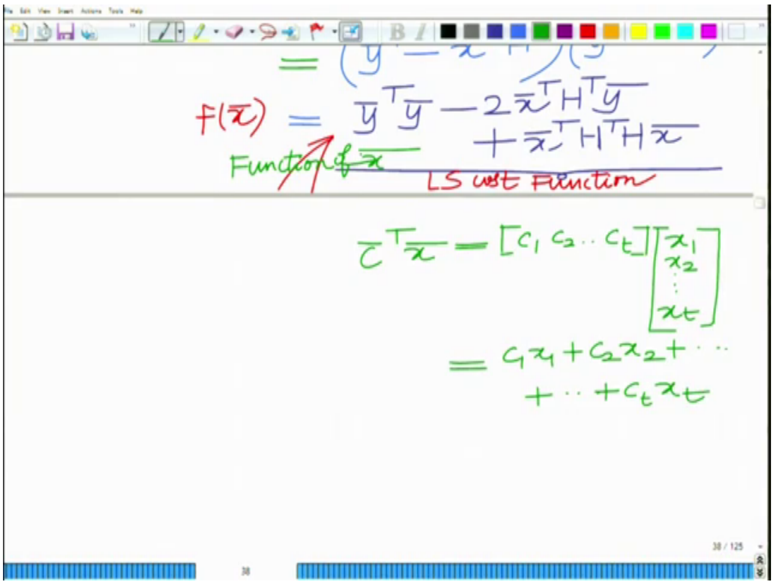

This is the cost function also known as the cost least squares cost function. You can also think of this as a the least squares cost function. Now, let us denote this least squares cost function by f of x bar, remember this is our function of the vector x bar ok. So, this least squares cost function is a function of the vector x bar ok. Now, here before we consider the now we have to consider remember the Hessian to demonstrate this is convex. So, first let us look at the gradient the properties of the gradient and then we will look at the Hessian ok.

So, let us consider a simple function C bar transpose x bar. If you look at the gradient of this, the gradient of C bar transpose x bar, now remember C bar transpose x bar you can write this as the vector c 1 c 2 c t row vector c 1 c 2 c t times column vector x 1 x 2 up to x t which is equal to c 1 x 1 plus c 2 x 2 plus so on up to c t x t. And now the gradient of this C bar transpose x bar, we can see derivative with respect to x 1 is c 1 with respect to x 2 is c 2 with respect to x t is c t. So, this is simply C bar.

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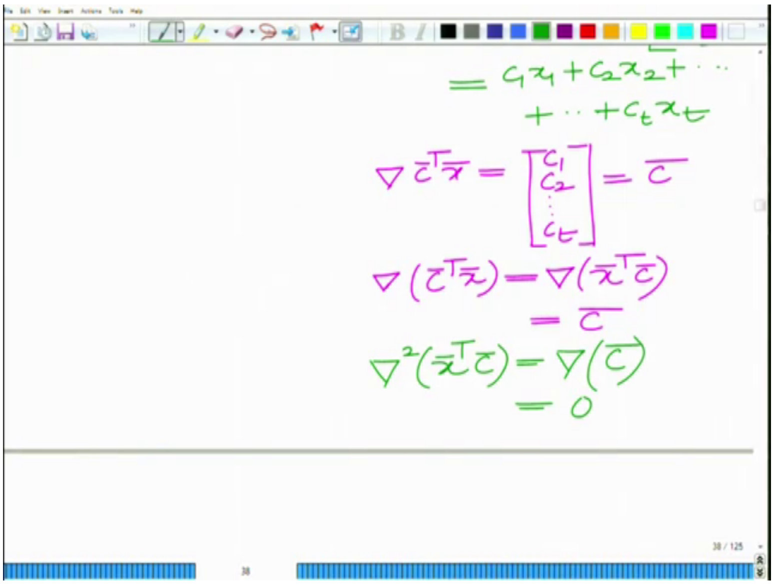

Similarly, now the gradient now remember C bar transpose x bar is equal to x bar transpose C bar therefore, gradient of C bar transpose x bar is gradient of x bar transpose C bar equal to C bar. So, this is what we have over here ok. Now, what about the Hessian? Now, if you look at the Hessian, therefore, second order derivative of x bar transpose C bar or C bar transpose x bar that will be the gradient of the that will be the gradient of C bar which is, but C bar is a constant so gradient of. So, if you look at this Hessian of x bar transpose C bar, x bar transpose any constant vector C bar that is going to because remember this is a linear in x bar all right. If you differentiate it twice, it is going to be 0. So, the gradient of the term x bar transpose Hessian of the term x bar transpose C bar is 0.

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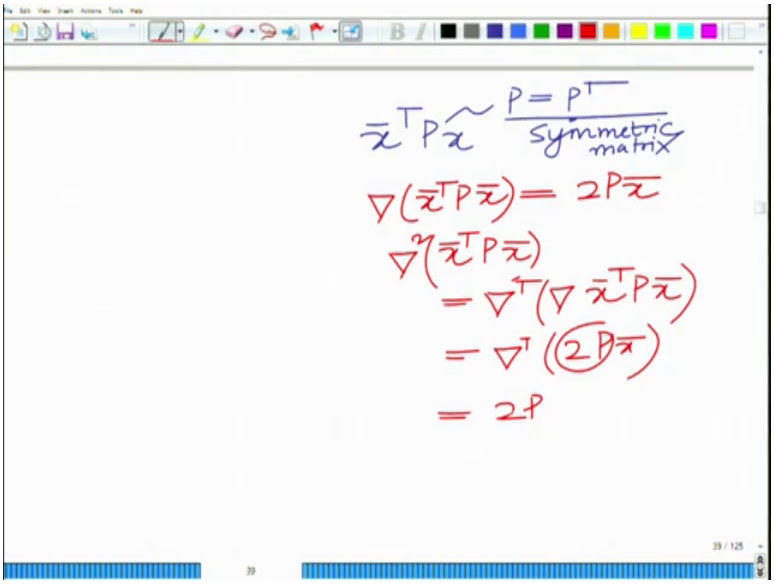

On the other hand, if you look at this, now look at this term of the form that is x bar transpose P x bar with P equal to P transpose ok. So, P is symmetric. You can show that the gradient of this first we will start with the gradient, gradient of this is x bar transpose P x bar, this is you can show this is twice P times x bar. Now, you take the gradient the Hessian of x bar transpose P x bar, you can also write this as the row vector of the gradient of that is you are taking the gradient and so the Hessian first you differentiate by the row, then differentiate by the column and then you differentiate by the row right.

So, transpose gradient transpose gradient of twice of x bar transpose P x bar which you can write as the gradient transpose of now we have seen already the gradient of x bar transpose P x bar that is twice P x bar. Now, this is of the form well this is of the form twice C bar transpose x bar. So, you can see this will be you can check that this Hessian will be twice of P.

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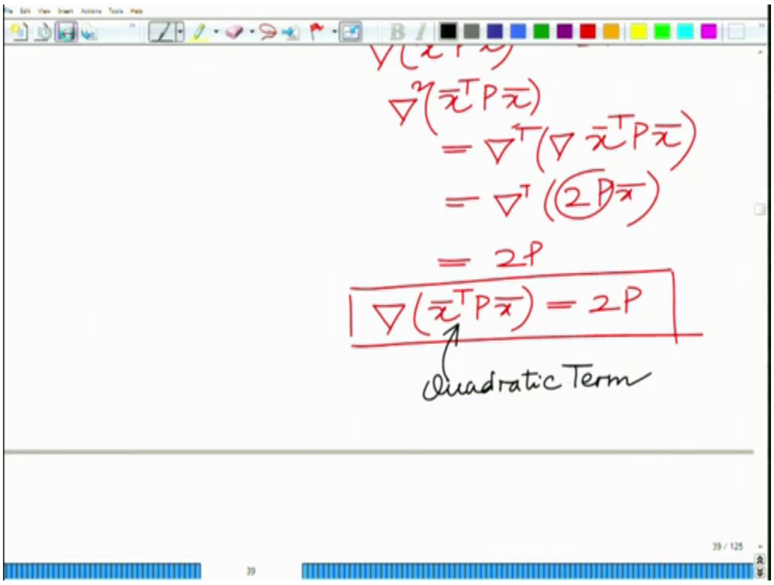

So, the gradient of x bar transpose P x bar will be twice into the matrix P this is your quadratic term this is a quadratic term. Now, we go back to our original least squares cost function that f of x bar and then we compute its Hessian.

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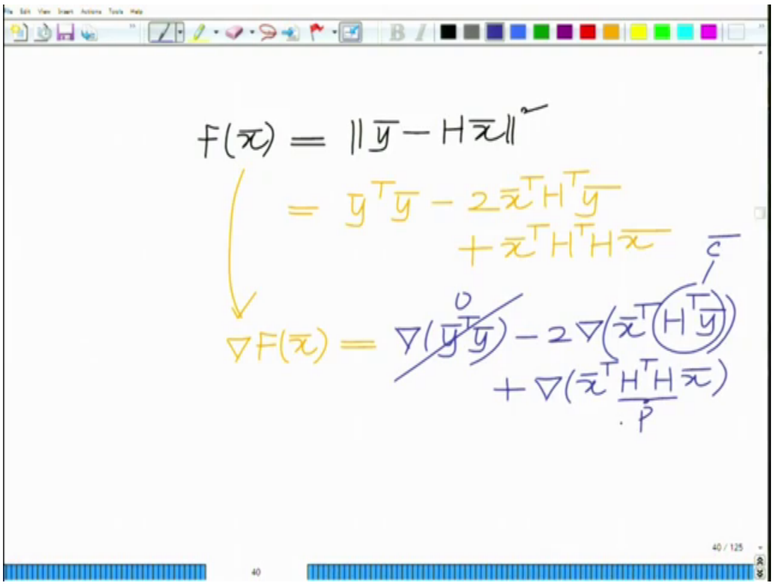

So, now if you go back to the least squares cost function, you will see that the cost function is norm of y bar minus H x bar square which is y bar transpose y bar minus twice x bar transpose H transpose y plus x bar transpose H transpose H into x bar. Now, if you first let us take the gradient of this thing, now observe that the gradient of y bar transpose y bar this is a constant y bar transpose y bar given the vector y bar so this is 0 minus twice gradient of x bar transpose your vector C bar H transpose. So, this gradient is simply going to be remember this is your C bar. So, this is simply going to be C bar plus gradient of x bar transpose H transpose H into x bar. Now, remember this is your matrix P which is we can see symmetric.

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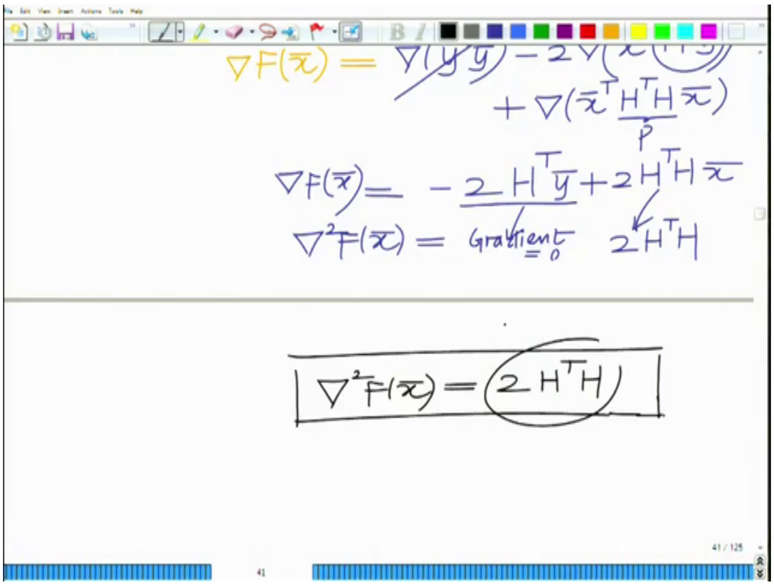

So, this gradient will be simply twice P x bar or twice H transpose H into x bar so that is it. So, now you have minus 2 times C bar which is H transpose y bar plus twice P which is H transpose H into x bar this is the gradient. And the Hessian will now be if you differentiate this again of course this is a constant term gradient of this is 0. And corresponding to this what we will have we have already seen twice P into x bar Hessian that is you take the gradient of that that will be twice P. So, this will be twice H transpose H. So, the Hessian of this reduces to twice H transpose H, this is twice H transpose H.

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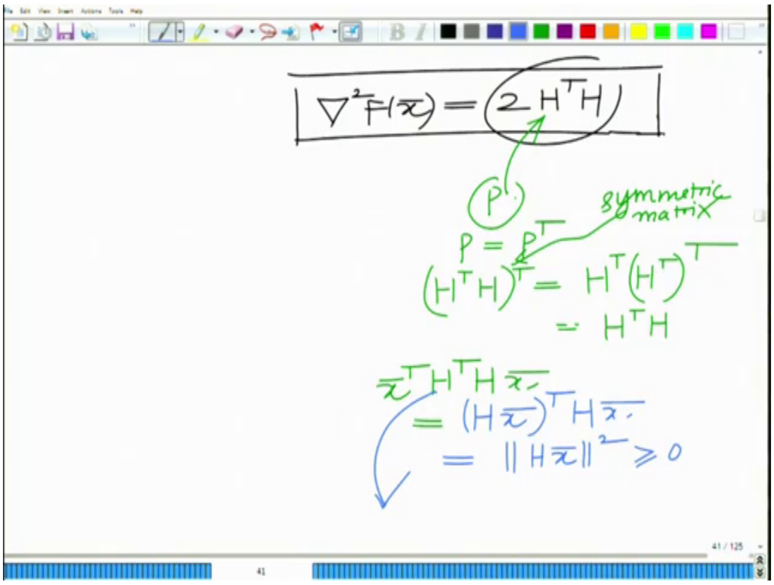

And now you can see if you call this matrix as P, first you can see P equals P transpose because H transpose H transpose A B transpose is B transpose A transpose. So, this is H transpose into H transpose transpose, but H transpose transpose is itself. So, first this is a symmetric matrix. And further if you look at x bar transpose H transpose H into x bar, well what is this, this will be equal to H x bar transpose times H x bar which is equal to norm of vector transpose itself that is norm H x bar square which is greater than equal to 0 which means x bar transpose P x bar our x bar transpose H transpose H x bar is always greater than equal to 0 for any vector x bar. Which means that this matrix P H transpose H is always positive semi definite which implies twice H transpose H because you are multiplying it by a positive constant is also going to be positive semi definite.

Therefore, the Hessian is positive semi definite which implies that the least squares cost function is convex that is a very important property which helps us design the receiver in this MIMO system. In fact, that is what we are going to do when we solve the convex optimization problem.

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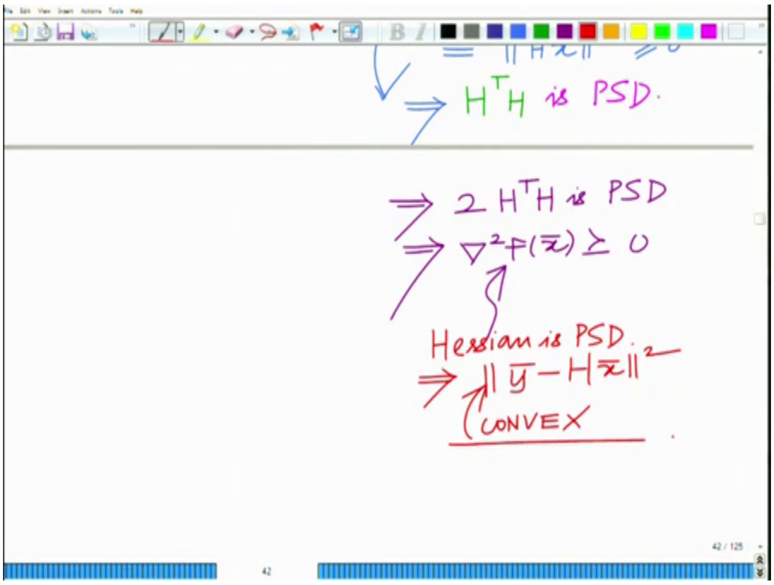

So, what you will realize here is that this implies H transpose H is a positive semi definite matrix which implies that you are multiplying it by a positive constant 2, which implies twice H transpose H is positive semi definite which implies that delta square f of x bar is that is this Hessian is positive semi definite implies y bar minus H x bar square. This is a convex, this is a convex function of x bar. So, this f of x bar this is convex. And solving this convex optimization this convex problem this optimization problem one obtains the estimate of the transmit vector x bar. And this is going to be important when we talk about the receiver design for a MIMO system ok.

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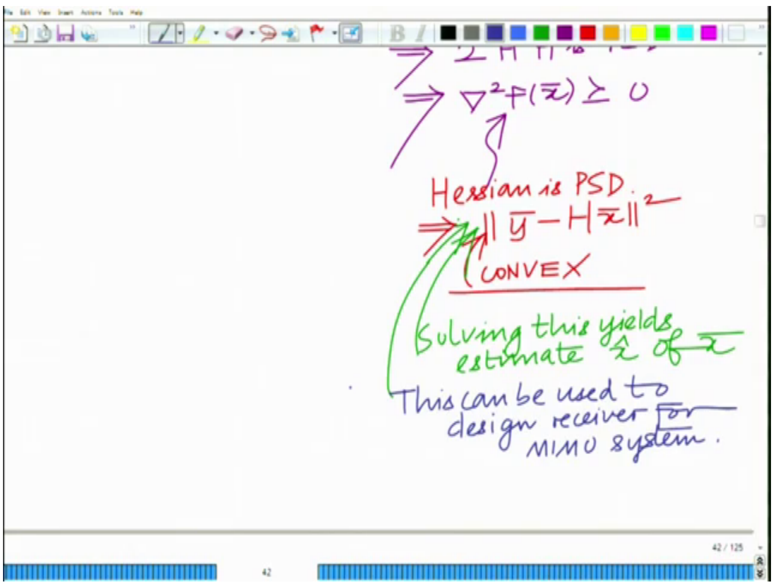

So, solving this can be used to design receiver this can be used to design the receiver for the MIMO system all right. So, this is important. So, this least squares problem, in fact, least squares problem occurs in several different scenarios all right. So, one sort of application of this least squares problem is to define an efficient receiver for MIMO system.

We are trying to find the best transmit vector x bar corresponding to a received vector y bar that minimizes the approximation error that is the best vector best estimate x hat which closely predicts or which is a closed or which is the which basically best explains or we can say or which is the best approximation for the receive vector y bar in this MIMO system all right. So, we will stop here and continue with other aspects.

Thank you very much.