Applied Optimization for Wireless, Machine Learning, Big data
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Lecture-20
Inverse of a Positive Define Matrix, Eigenvalue Properties and Relation between different norms

Hello, welcome to another module in this massive open online course. So, we are looking at examples; on matrices and also convex sets. So, let us continue a discussion, let us look at another example related to Positive Definite Matrices alright.

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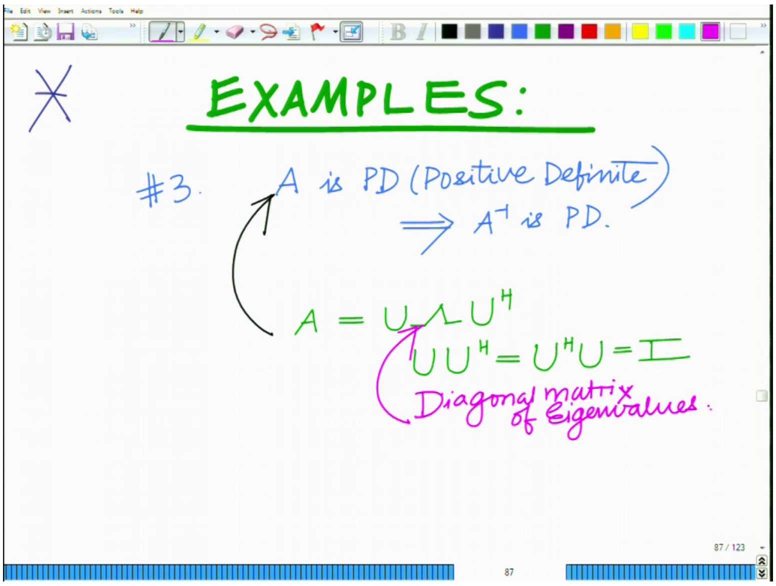

So, let us consider continue our discussion or continue looking at examples related to matrices and convex sets. So, this example number 3 what you have to show, that is A is a PD matrix that is positive definite, if A is positive definite this implies A inverse is also positive definite. We want to show that if A is positive definite A inverse is also positive definite, this can be shown as follows. If A, A can be written expressed as we already seen this U lambda U Hermitian where U is a unitary matrix satisfies U U Hermitian equals U Hermitian U equals identity.

And lambda is a diagonal matrix of eigenvalues, and also further we have seen that the eigenvalues of any positive definite matrix have to be greater than 0. The eigenvalues of a positive semi definite matrix are greater than equal to 0; the eigenvalues of a positive definite matrix they have to be strictly greater than 0, it cannot have any eigenvalues equal to 0. And therefore, now if you look at A inverse its rather easy to see A inverse equals U lambda U H U lambda U Hermitian inverse, which is basically U Hermitian inverse times lambda inverse times U inverse.

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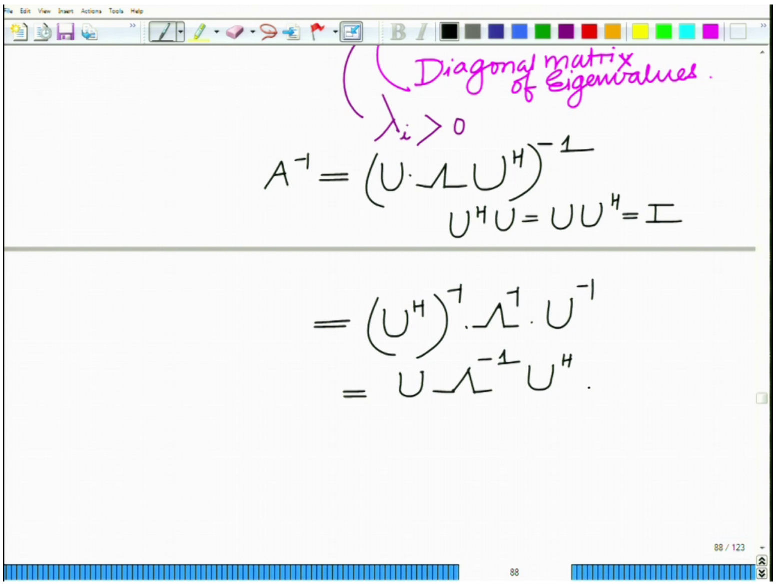

But we have seen that U is a unitary matrix which implies U Hermitian U equals U U Hermitian equals identity, well this implies U equals U Hermitian. So, what this implies is that if you look at you Hermitian inverse that is U itself because, U Hermitian to use identity times lambda inverse into U inverse is U Hermitian because again U U Hermitian equals identity. And therefore, this is again has the same structure except you can see with eigenvalues 1 over lambda 1 1 over lambda 2 1 over lambda n times U Hermitian.

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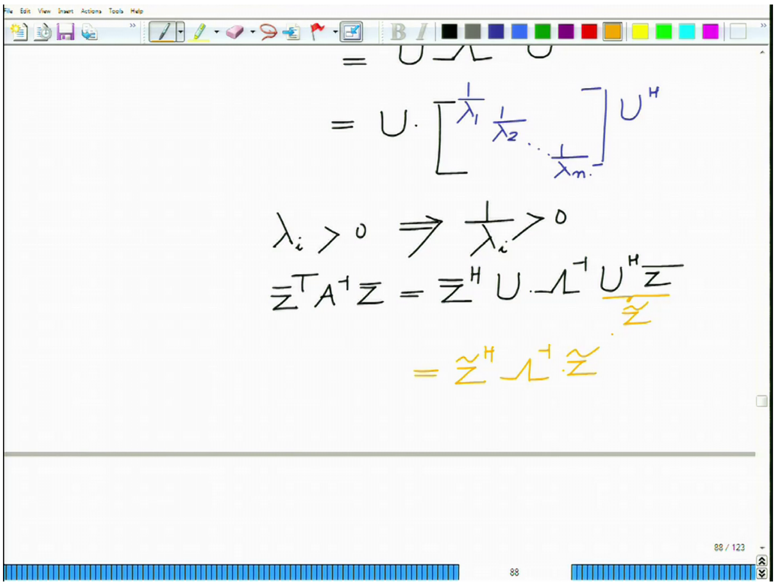

And therefore, now you can see if lambda i is greater than 0, this also implies that 1 over lambda i is greater than 0. So, eigenvalues of A inverse or also greater than 0 in effect A inverse can be expressed as U lambda inverse U Hermitian. And therefore, it has and it has positive eigenvalues alright and therefore, it is also a positive definite matrix.

And you can also check this as follows, for instance if you consider Z bar transpose for any real vector if you consider Z bar transpose A inverse Z bar, I can now write this as now since this is a real vector I can write this as Z bar Hermitian, A inverse we have seen is U lambda inverse U Hermitian Z bar. Now if u set U Hermitian Z bar if you set this is equal to Z tilde then this will become Z tilde Hermitian lambda inverse Z tilde.

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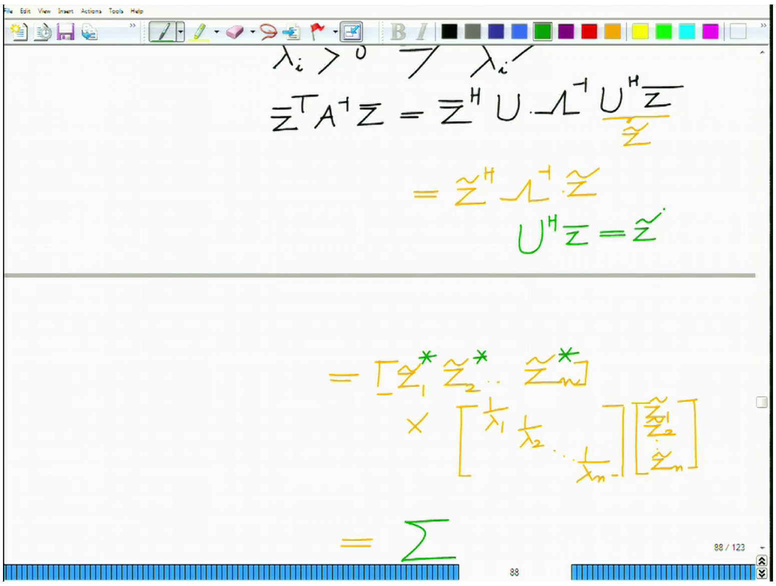

Which is equal to Z tilde 1 Z tilde 2 so on, up to Z tilde n into the product 1 over lambda 1 1 over lambda 2 so on, 1 over lambda n times Z tilde 1, Z tilde 2 up to Z tilde n. And if you look at this, this is nothing but well this will be Z tilde conjugate since this is the Hermitian of the vector ok. So, we are setting U Hermitian Z bar Z tilde. So, this will be summation over i equals 1 to n over lambda i Z tilde i conjugate into Z tilde i that is magnitude Z tilde i square.

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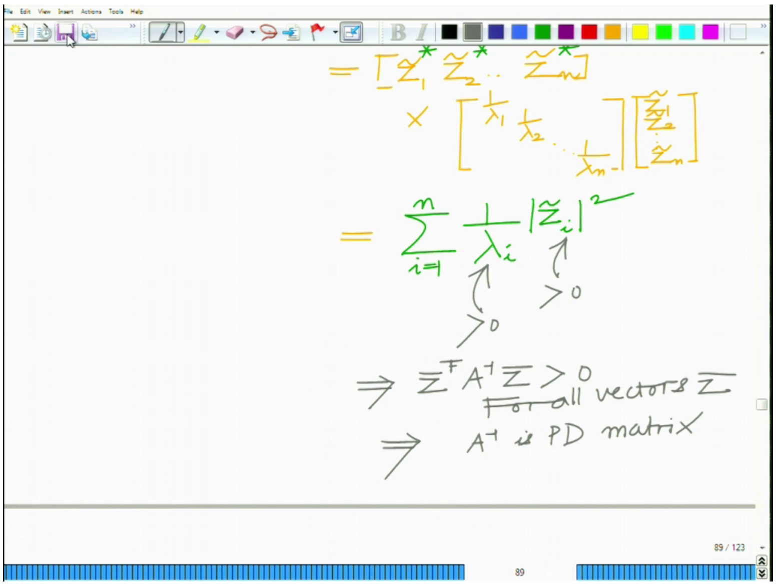

Now, 1 over lambda i this is greater than 0, magnitude Z tilde i square this is greater than 0. And therefore, this implies that Z bar transpose A inverse Z bar is also greater than 0 for each for all Z bar for all vectors Z bar, and this implies that A inverse is a this implies that A inverse is a positive definite matrix. So, A is a positive definite matrix A inverse is also positive definite matrix.

In fact, the eigenvalues of a inverse are the inverse that is if lambda i is an eigenvalue of A then eigenvalues of corresponding eigenvalue of A inverse is 1 over lambda i. Eigenvalues of A are strictly greater than 0 if A positive definite matrix and similarly the eigenvalues of A inverse are also strictly greater than 0, if strictly greater than 0 correct. Since lambda is greater than 0 1 over lambda is also greater than 0 alright, let us continue our discussion let us look at another problem number example number 4.

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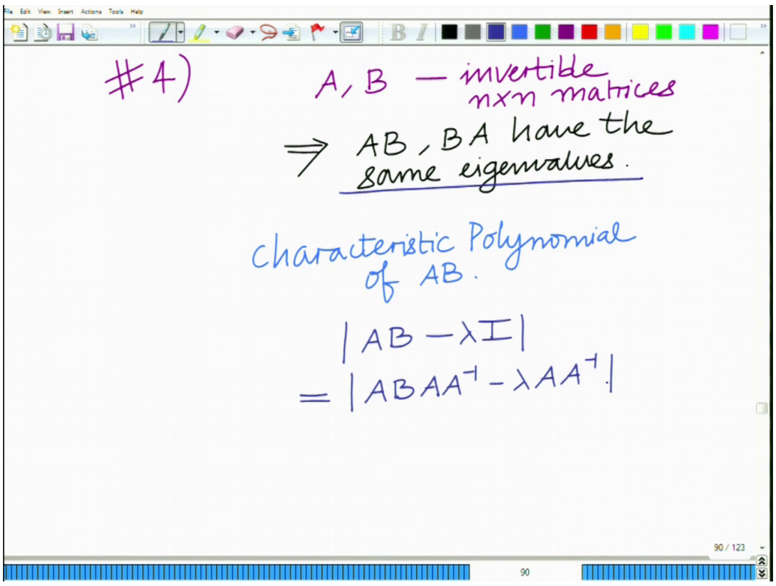

What we want to show is that if A, B these are 2 invertible n cross n matrices then, A B, B A have the same A B and B A have the same eigenvalues we want to show this property that A B eigenvalues of A B are equal to eigenvalues of B A well. We start with the characteristic polynomial remember to compute the eigenvalues of any matrix in this case the eigenvalues of the matrix A B. So, we start with the characteristic polynomial of A B, that is obtained by nothing but, that is basically obtained by looking at the determinant of A B minus lambda i remember the eigenvalues are computed as the roots of the characteristic polynomial.

The characteristic polynomial of A matrix A is A minus lambda i you want to look at the characteristic polynomial of the matrix A B therefore, that will be the determinant of A B minus lambda i. So, this is equal to I can write this as determinant of A B A A inverse minus lambda A A inverse because remember A is an invertible matrix both A and B are invertible matrices so, A A inverse equals identity.

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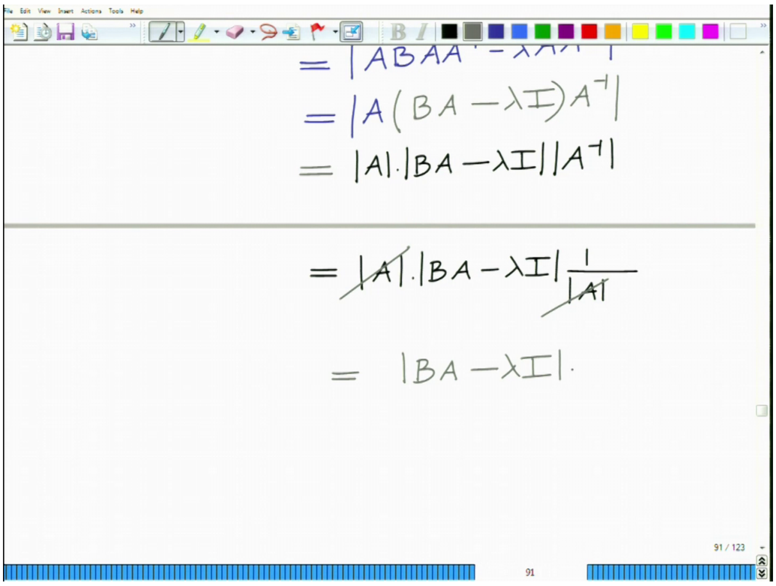

So, I can always write this as A B determinant of A B A inverse minus lambda A A inverse. Now, I can extract the A on the right so, this will become B A minus lambda I times A inverse this is determinant of A times B A minus lambda I times A inverse. The determinant of A matrix product is the product of the determinants that is determinant of A times determinant of B A minus lambda I times determinant of A inverse. Determinant of A inverse is basically 1 over the determinant of A because A times A inverse is identity.

So, this is B A minus lambda I into 1 over the determinant of A determinant of A times 1 over determinant of A these cancels. So, this becomes the determinant of A minus lambda I. Therefore, we have this interesting property that is A B determinant of A B minus lambda I equals determinant of B A minus lambda I implies characteristic polynomials of B A this implies the characteristic polynomial, the characteristic polynomial of A B which is determinant of A B minus lambda I. This equals the characteristic polynomial of B A.

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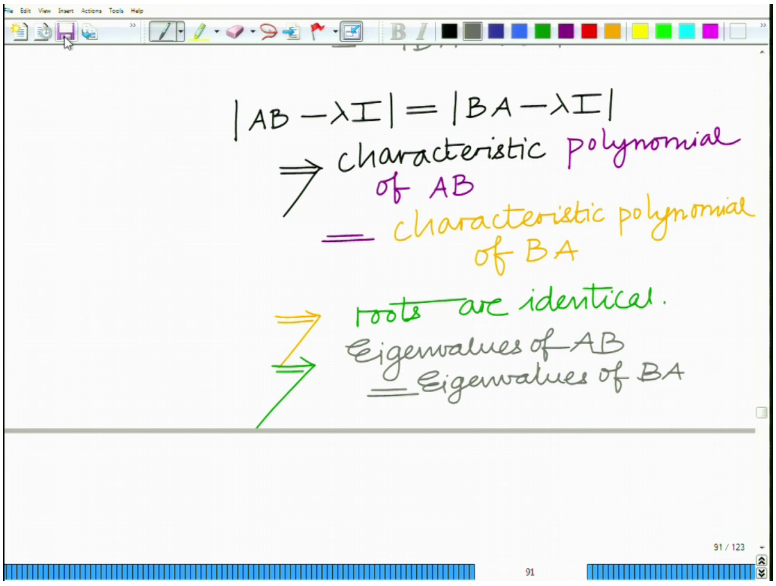

The characteristic polynomial of A B equals the characteristic polynomial of B A this implies the roots are equal roots are equal or identical and this implies. So, characteristic polynomials are equal implies the roots are identical. And this implies therefore, eigenvalues of A B equal eigenvalues of this implies eigenvalues of A B equals this implies that eigenvalues of A B equal eigenvalues of B A ok.

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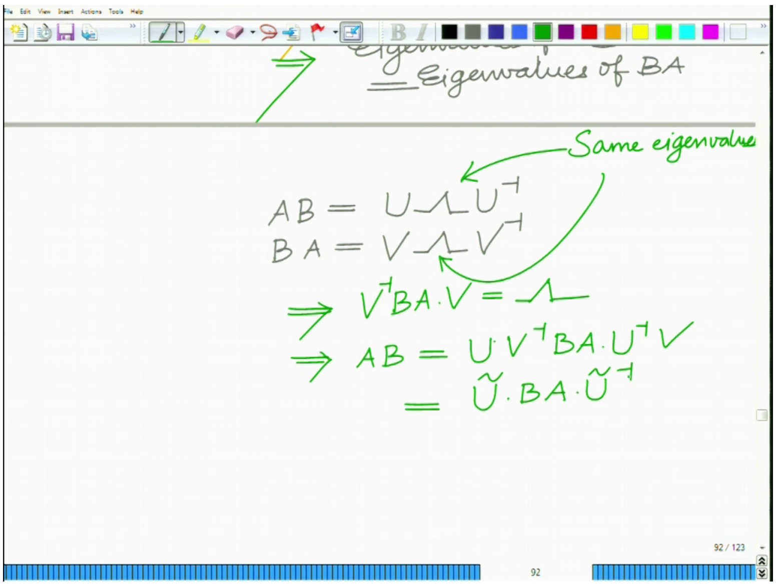

And in fact, you can see, if you write A B equals U lambda U inverse remember this eigenvalue decomposition I can write these are eigenvalues are equal I can write B A equals some V times lambda times V inverse because remember their eigenvalues are equal. So, that diagonal matrix of eigenvalues will be the equal or identical similar eigenvalues they have the same eigenvalues.

This is the same eigenvalues the diagonal matrices lambda right, the diagonal matrix of eigenvalues will be the same for both A B and B A in their eigenvalue decomposition ok. So now, this implies that B A equals V lambda V inverse so, this implies V times or V inverse times B A into V equals lambda. Now, substitute lambda in the first one this implies A B equals U times lambda, but lambda is V inverse B A into U inverse V and this is nothing, but U V inverse let us call this as U tilde B A. If U inverse U V inverse is U tilde then U inverse B becomes U tilde inverse.

So, I can write the matrix A B as some matrix U tilde I can write this as you tilde times B A times U tilde inverse such matrices are said to be similar matrices. So, A B implies A B is similar to B A. In general C similar to D if there exist M such that C equals M inverse D M. So, if there exists a matrix M, such that you can write C equals M inverse D into M, then the matrices then the matrices C and D are said to be similar matrices. So, in this case you can see these 2 matrices A B and B A in fact, which of the same eigenvalues alright these are similar matrices alright.

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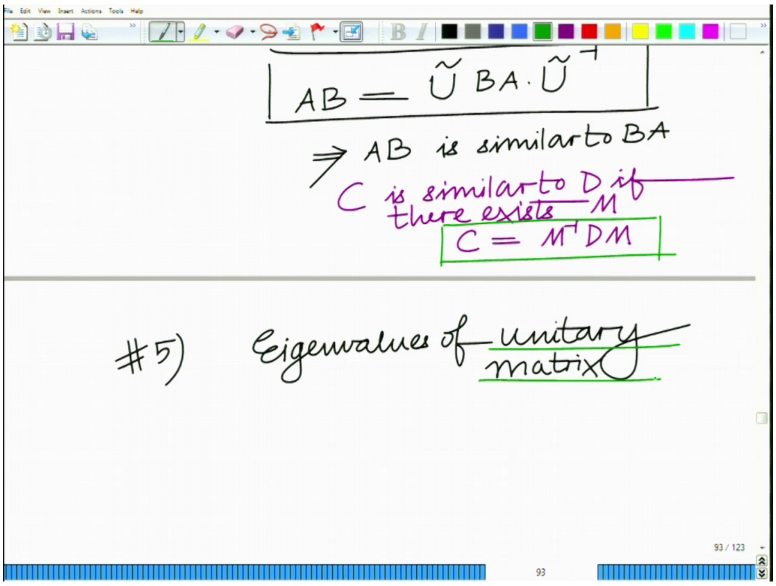

Let us now look at another interesting property that is the eigenvalues of unitary matrix. So, this is our example number 5 again another simple property, what can we say about the eigenvalues of a unitary matrix, now let U be a unitary matrix.

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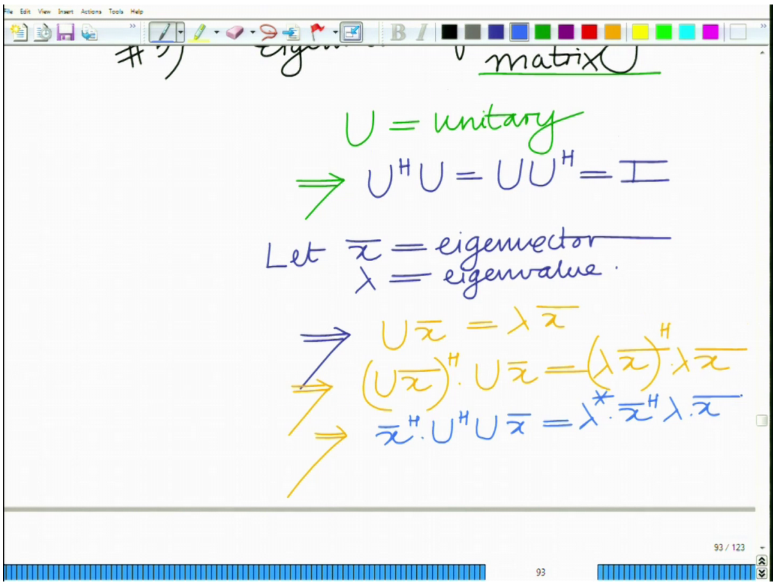

Remember the unitary matrix is defined by the property U Hermitian U equals U U Hermitian equals identity. This is the property of the unitary matrix now let, x bar be the eigen vector and lambda equals corresponding eigenvalue. Now this implies, what this implies is that, U x bar equals lambda times x bar correct, which implies now you can multiply U x bar Hermitian U x bar that will be equal to lambda x bar Hermitian. Because U x bar equals lambda x bar U x bar Hermitian equals lambda x bar Hermitian multiplied by lambda x bar.

And this implies x bar Hermitian U Hermitian U x bar equals lambda Hermitian, but lambda is a number so lambda Hermitian is simply lambda conjugate x bar Hermitian lambda into x bar. Now U Hermitian U is identity because, U is unitary matrix that, leaves x bar Hermitian x bar which is remember norm of x bar square this is equal to lambda conjugate lambda, that is magnitude lambda square times x bar Hermitian x bar, which is again once again norm x bar square which implies cancelling the norm x bar square on both side. This implies magnitude lambda square equals 1 which means magnitude lambda equals 1.

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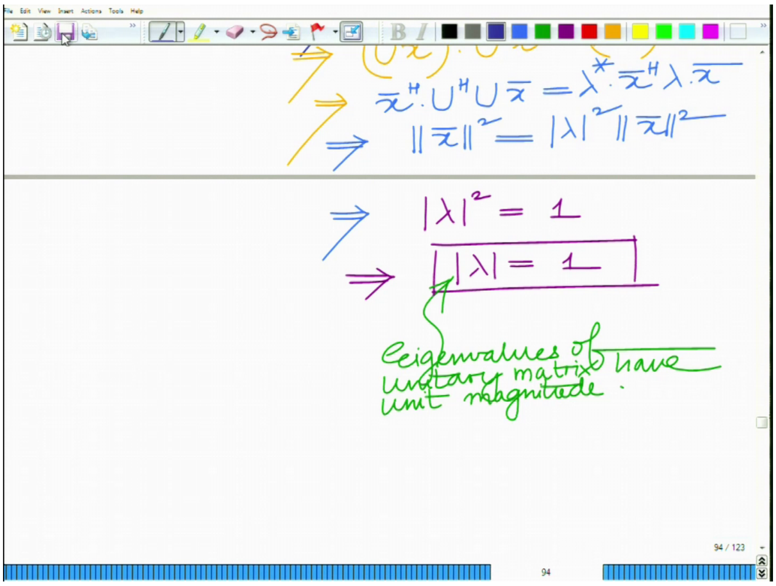

So, this implies basically that eigen so, this basically shows a very interesting property eigenvalues of unitary matrix, eigenvalues of a unitary matrix have unit magnitude that is the interesting property that this shows alright. And now similarly, if you consider the determinant of the unit the magnitude of the determinant let us consider the magnitude of the determinant. Remember we have seen that the determinant is nothing, but the product of the eigenvalues so, the magnitude of the product of the eigenvalues, which is nothing, but the product of the magnitudes of the eigenvalues.

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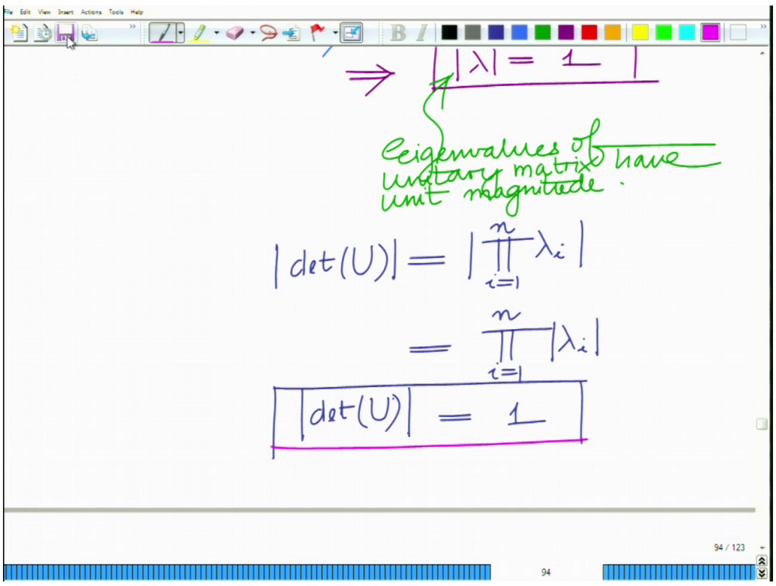

And each eigenvalue is unit magnitude so this is equal to the product of 1's which is one which shows this ancillary property or you can also think of this as an axiom that the determinant of a unitary matrix is identity. All the eigenvalues of a unitary matrix are magnitude 1 and the determinant of unitary matrix has the magnitude of the magnitude of the determinant of a unitary matrix is 1 as well alright. Let us continue a discussion let us start with another example let us consider the norm relation between the 1 norm of a vector we want to show that the 1 norm of a vector x bar is less than or equal to square root of n times the 2 norm.

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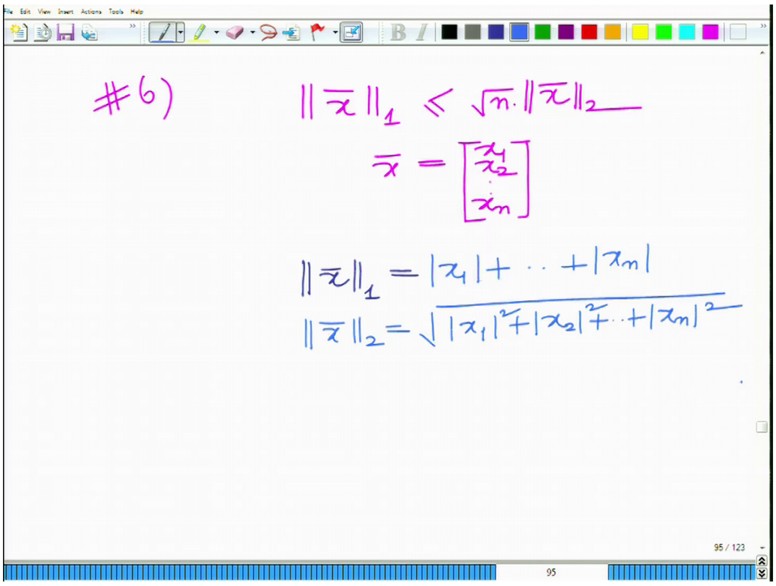

Now, if x bar is a vector with elements x 1, x 2 up to x n. Now, remember the 1 norm, this is simply the sum of the magnitudes, the magnitude of magnitude x 1 plus magnitude x 2, magnitude x n and the 2 norm is the square root of magnitude x 1 square plus magnitude x 2 square plus so on, plus magnitude x n square. Now, to show the property about what we will do is, we will consider two different vectors will construct 2 vectors u bar and components the elements of u bar are magnitude x 1 magnitude x 2.

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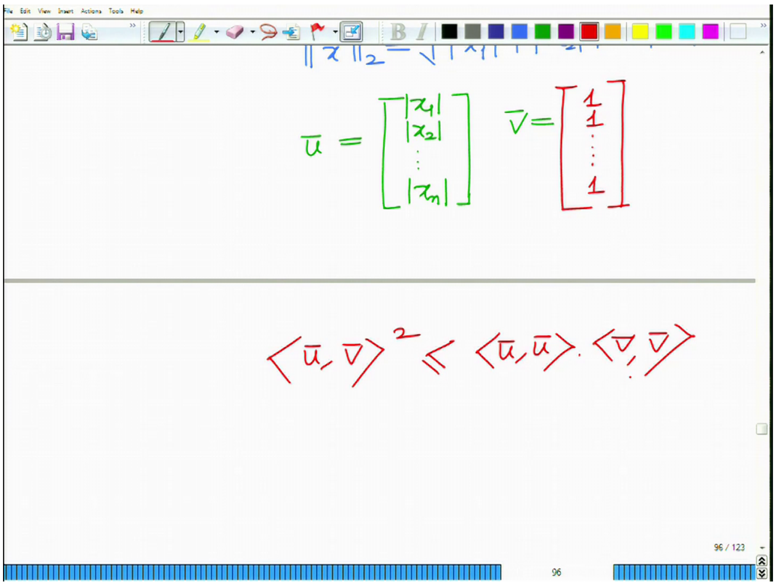

So, I am constructing two different vectors magnitude x 1 magnitude x 2 so on, magnitude x n and v bar is a vector n dimensional vector of all 1's. Now what I going to do is, I am going to apply the Cauchy–Schwarz inequality, remember we have seen the Cauchy–Schwarz inequality which states that the inner product square u bar v bar that is less than u bar, that is norm u bar square into norm b bar square.

This implies that if you look at u bar transpose v bar square, that is less than or equal to norm u bar square, norm v bar square. And this also implies that u bar transpose v bar is less than or equal to norm u bar into norm v bar we know this property.

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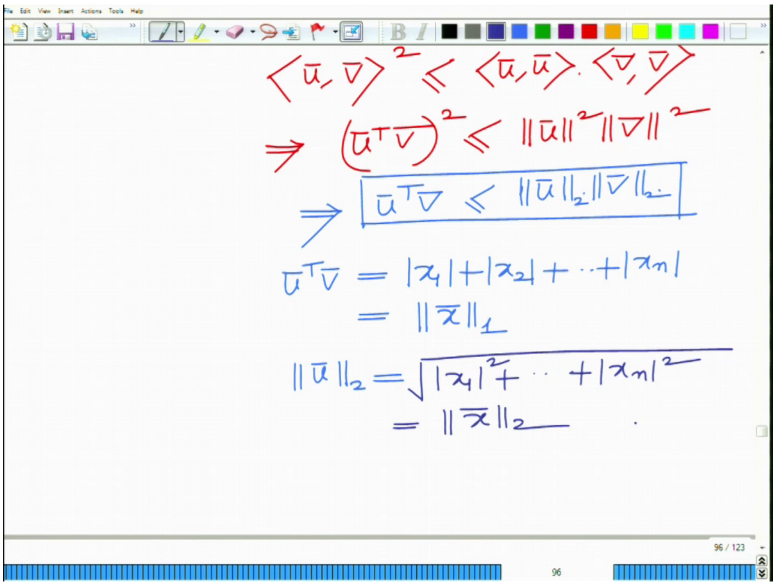

Now, all we have to do is substitute the definition from the definition above substitute u bar and v bar, you can see u bar transpose v bar is nothing but, magnitude x 1 plus magnitude x 2 so on, up to magnitude x n. Which is basically norm x bar of 1 and norm u bar that is the 2 norm remember the 2 norm u bar is square root of magnitude x 1 square plus so on magnitude x n square which is nothing, but the 2 norm of x bar.

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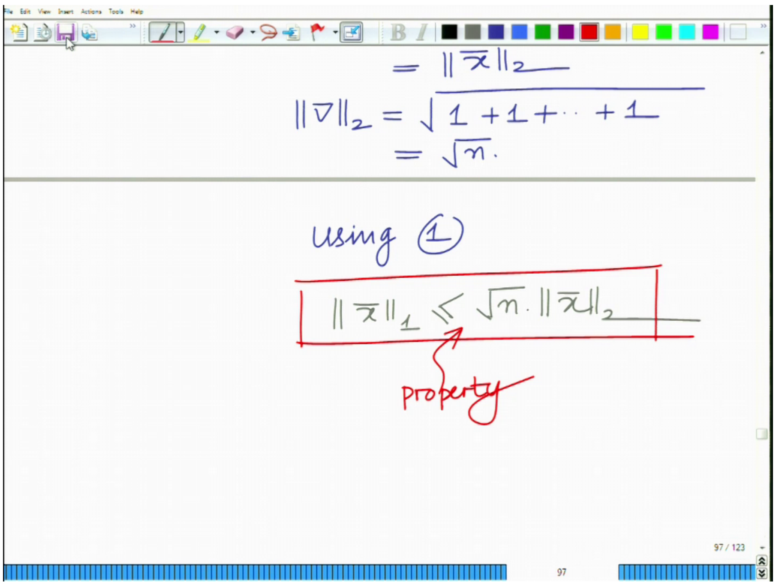

And finally, the 2 norm l 2 norm of v bar is square root of this is 1 plus 1 plus 1 n times this is nothing, but square root of n. And now using this property using 1 substituting all these in 1, this basically yields norm x bar 1 that is u transpose v bar less than or equal to norm v bar that is square root of n into norm u bar that is a 2 norm of x bar.

So, this is an interesting property that we have ok. So, this is the property or the relation you can say characterizes the relation between the 1 norm and the 2 norm. In fact you can also show something between the relation between the 2 norm the infinity norm.

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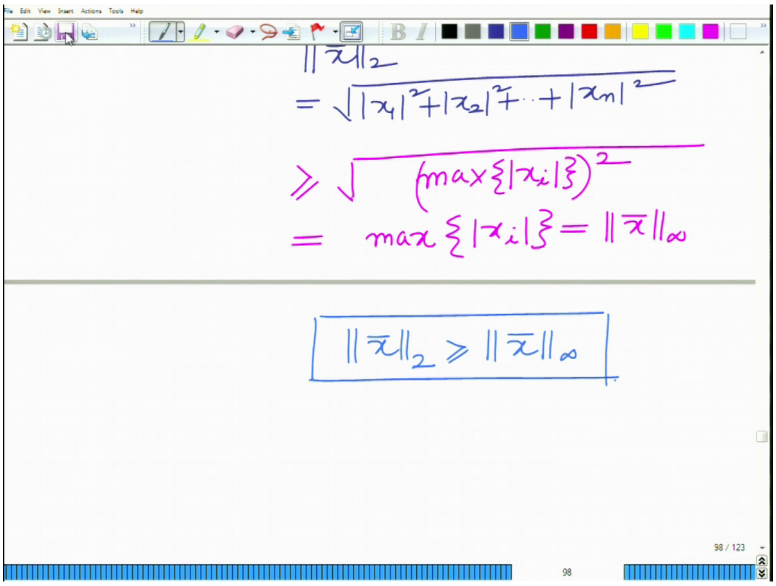

You can also show; that the infinity norm that is if you look at the 2 norm, now this is equal to well we have seen magnitude x 1 square plus magnitude x 2 square plus magnitude x n square, this is a sum of the squares of the magnitude is all the elements. Now, this is greater than or equal to you simply take the maximum of the maximum of the magnitude correct. This is the sum of the squares of the magnitude of all the elements, which is greater than equal to the square of simply the magnitude of the maximum of these elements, which is equal to now you take the square root all you are left with is the maximum of magnitude x i which is nothing, but the l infinity norm ok.

So, therefore, this shows that so, this shows that basically your this thing is greater than equal to the a l 2 norm is great. So, this basically shows that your l 2 norm is greater than or equal to the l infinity norm alright. So, let us stop here and we will continue with other aspects in the subsequent modules.

Thank you very much.