Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 18
Introduction to Affine functions and examples: Norm cones l 2, l p, l 1, norm balls

Hello. Welcome to another module in this massive open online course. So, we are looking at the properties of convex set or basically operations on convex sets that preserve convexity, alright. Let us continue our discussion. Let us look at another important operation that preserves convexity which is known as an affine function, ok.

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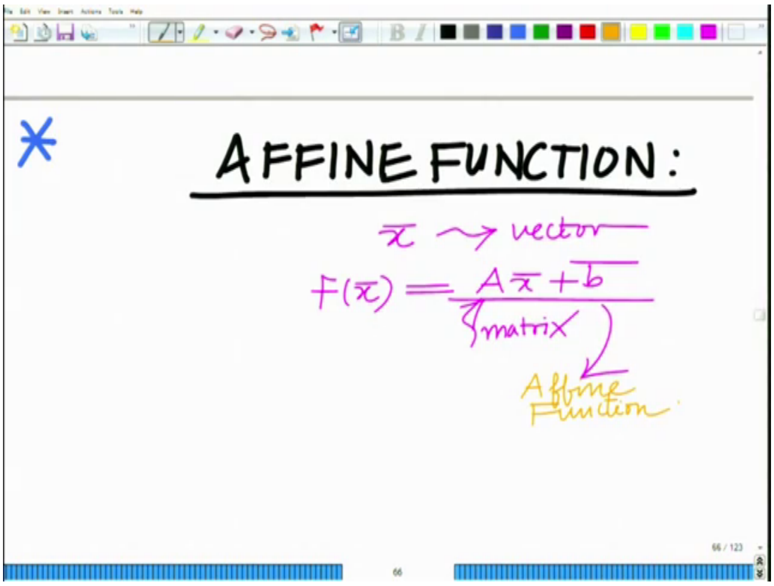

So, the next important transformation that preserves convexity and this arises fairly frequently is what is known as an affine. This is known as an Affine Function. For instance, what is an affine function? Now, if you have a vector that is for instance let us say x bar is a vector, let say this is your vector.

Now, an affine function is a function that is of the form A x bar plus b bar that is A is a matrix, that is multiplied by trans matrix and translated by the vector b bar. So, this, basically this function of this from this is termed as a function of this form is termed as an Affine Function.

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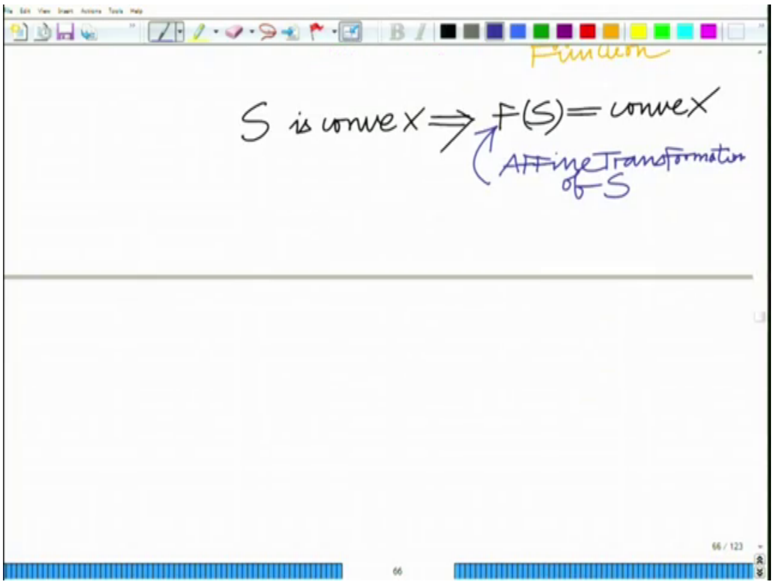

Now, under affine function now, the interesting property or it is relevant with respect to convex sets is that if S is convex, this implies that F of S that is affine transformation applied on as in the resulting set F of S is convex, ok. So, F of S implies affine transformation of S or affine transformation of all elements in it S that also results in a convex set ok. Typically, for instance we take a convex set, if you rotate it and translate it, that is which corresponds to basically an affine transformation, the resulting set is also convex.

Now, interestingly, what one can also show that an affine pre composition also results in a convex set, alright. So, what is the meaning of that, that is F inverse S; if S is convex then F inverse of S is also convex.

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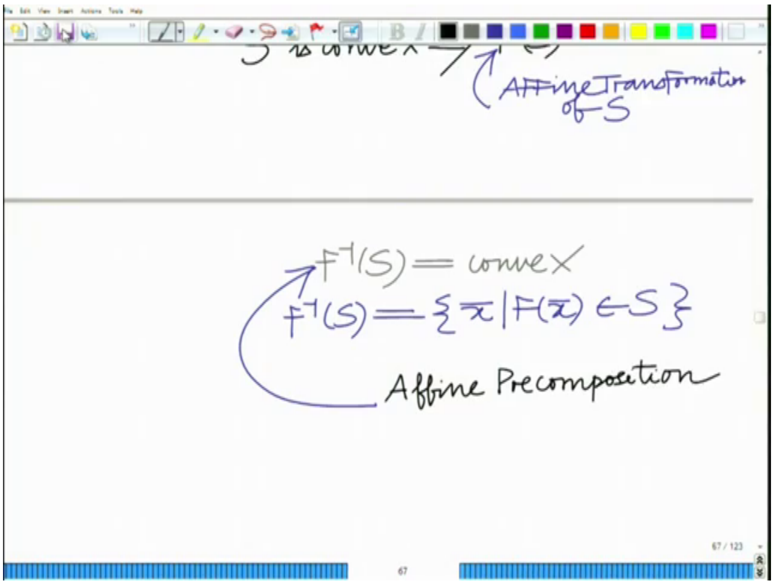

This implies, now what is F inverse of S; F inverse of S, that is the inverse of the set under this affined pre composition is the set of all vectors x bar such that F of x bar belongs to S. This is known as an affined pre composition. We have F of S which is affine composition, F inverse of S is the affine, this is the Affine Pre composition. For instance, an application can be demonstrated as follows.

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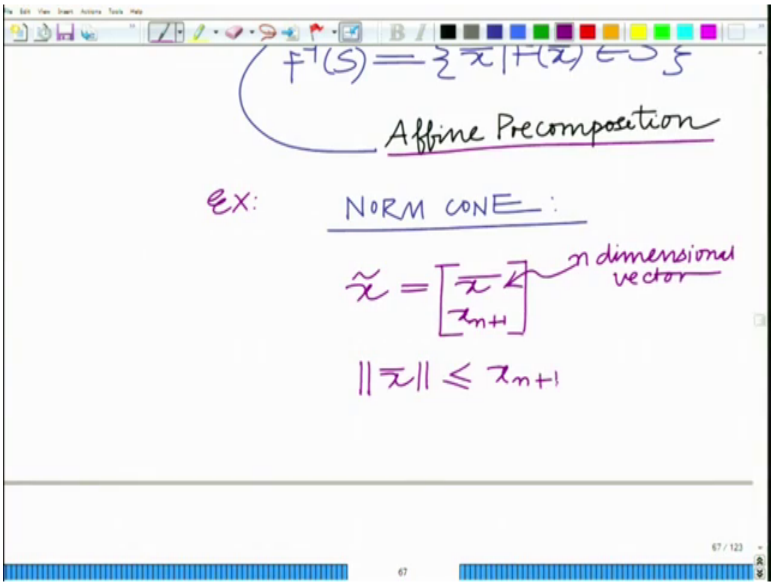

Consider the following simple example. We have already seen a Norm Cone. Let us go back to our illustration of the Norm Cone and what we have seen in the Norm Cone is that we have this vector x tilde which is of the form x bar x n plus 1, this is an n dimensional vector. We have an n dimensional vector x bar and another element n plus 1th element x n plus 1 and the norm cone is basically described by the set, norm of x bar is less than equal to x n plus 1 which basically implies that norm x bar square is less than or equal to x square of n plus 1; which basically implies that x bar transpose x bar is less than or equal to x square of n plus 1, correct because remember norm x bar square is simply x bar transpose x bar.

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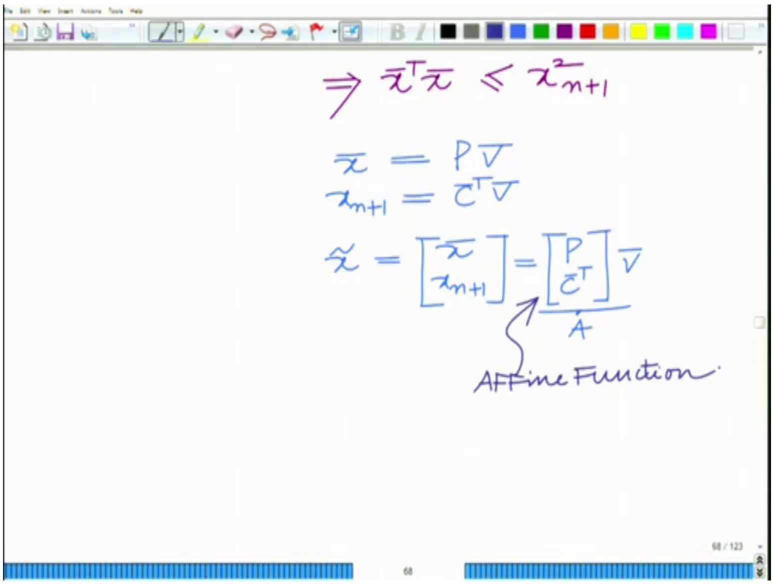

So, x bar transpose x bar less than equal to x square n plus 1. This is an alternative representation of the Norm Core. Now, let us see what is affine pre composition corresponds to. So now, let us consider x bar equals P times another vector V bar and x n plus 1 equal C bar transpose V bar. So, I can write x tilde equals this vector which is already we have seen x bar x n plus 1, this is equal to the matrix P stack matrix P C bar transpose and C bar. So, this is your matrix A, b bar is 0. So, this is an affine transformation, correct or rather this is an Affine Function.

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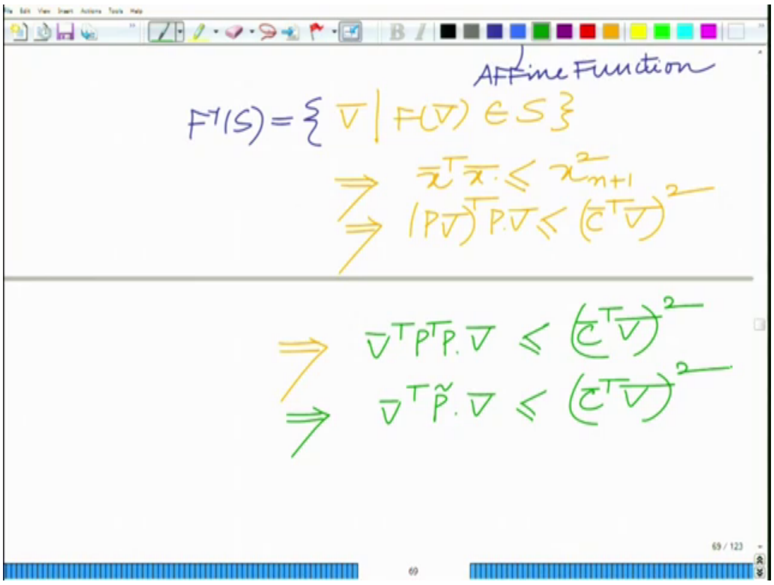

Now, we want to find the set of all F inverse of sets. So, this is our convex set S. Now, F inverse of S will be all S or will be all let say V bar such F of V bar belongs to S; implies, now if you look at F of V bar belongs to S implies well, we already seen x bar transpose x bar less than or equal to x square n plus 1. Now, substituting for x bar and x n plus 1, we have x bar is well P times V transpose into x bar which is P times V bar is less than or equal to x square of n plus 1 that is C bar transpose V bar.

Remember, x n plus 1 is C bar transpose V bar, square of that which basically implies that V bar transpose P transpose P into V bar is less than or equal to C bar transpose V bar whole square which basically implies V bar transpose V tilde V bar is less than or equal to C bar transpose V bar whole square, ok.

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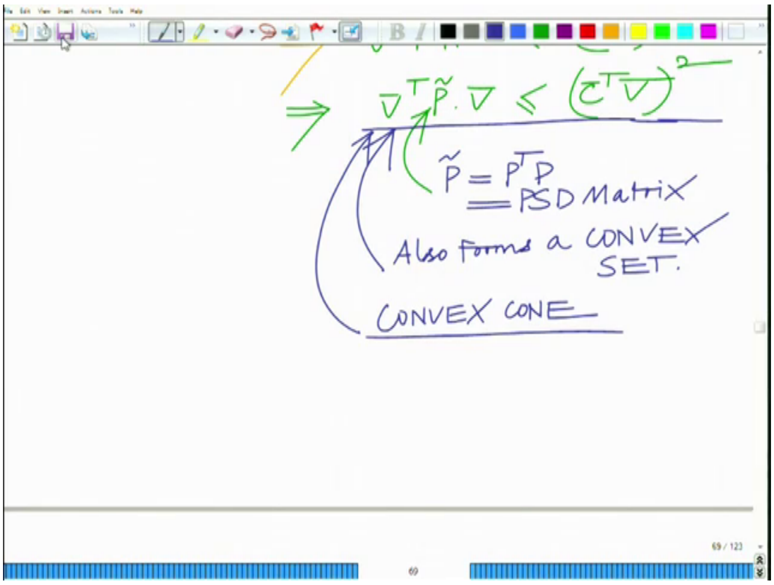

And this matrix V tilde is defined as P transpose P and you can see, this is a positive semi definite matrix. So, now, what you can see is this set alright V bar which satisfies this by the property of the affined composition, right since x bar correct since we said F of V bar that is x bar belongs to S, that is the norm cone. So, the V bar which is the affine pre composition which basically, which is the set corresponding to the affine pre composition alright that also forms a, that also forms a convex set. So, this set V bar such that F V bar belongs to S which is characterized by this relation also forms set of all V bars satisfying this also forms, this also forms a convex set, ok.

And in fact, this is a convex cone is, we can think of this as a general expression for a convex cone given by the affine pre composition ok, alright. So, these are very interesting properties; the first one is a rather simple and which is basically says that intersection of two sets, if two sets are convex or a finite number sets if or if any number of sets is convex, their intersection is also convex. And further, if you can consider an affine function F and a convex set S, then both F S and F inverse S are also convex, alright.

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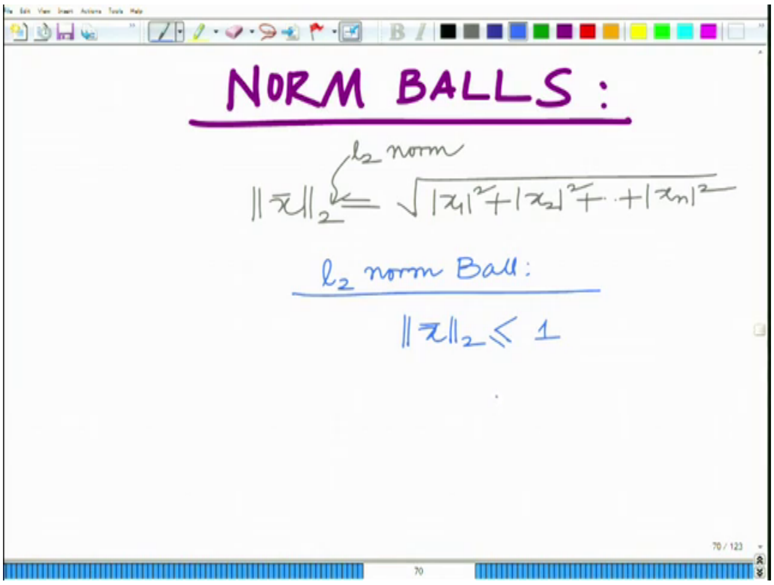

Let us now move on to another interesting aspect and let us re visit the concept of Norm Balls that we have seen previously. We had seen this concept of a Norm Ball, ok. What is a norm ball? Now, remember the norm ball was defined as follows. I have the 2 norm, this also known as the l 2 norm which you can write as magnitude x 1 square plus magnitude x 2 square plus 1, magnitude x n square, this is the l 2 norm and the corresponding l 2 norm ball that is given as norm of x bar that is l 2 norm less than or equal to for instance r let say equal to 1, ok. So, this is your l 2 norm ball.

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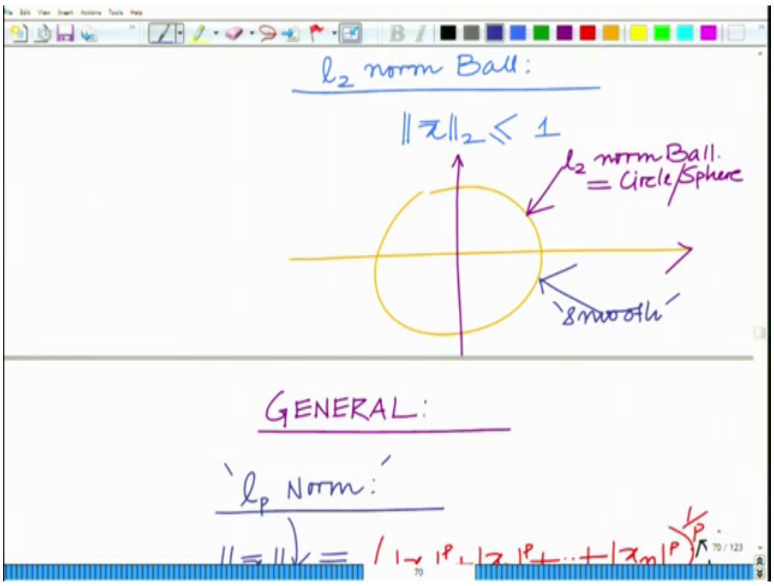

And we have also seen that this l 2 norm ball for instance in two dimensions, this corresponds to a this is l 2 norm ball which is basically equal to circle slash sphere in n dimensions, it is a sphere ok. So, this is your l 2 norm ball, ok.

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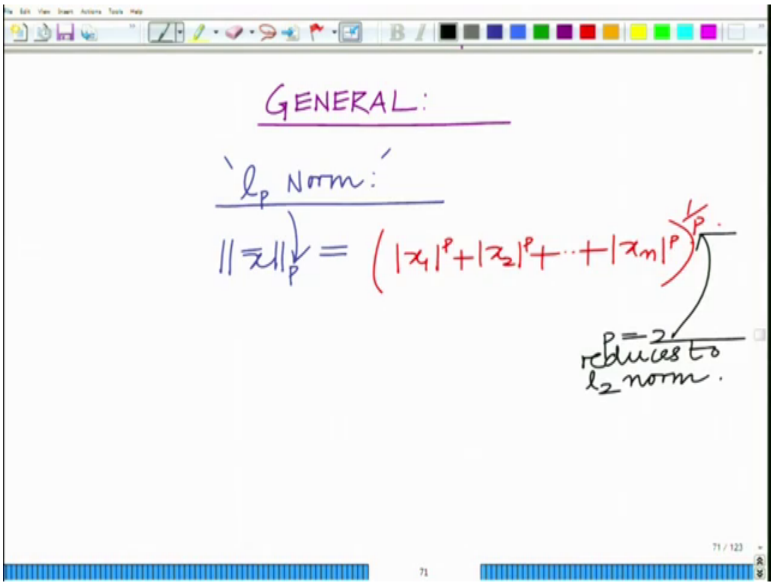

Now in general, we would also we one can define; now in general, one can define what is known as an l P norm. What is this l P norm? If you take a vector x bar, the l P norm indicated by this P here is basically given as magnitude x 1 to the power of P plus magnitude x 2 to the power of P plus magnitude x n to the power of P whole raise to the power of 1 over P.

Now, you can see if you set P equal to 1, P equal to 2, it reduces to, reduces to the l l, l 2 norm; therefore it is general. So, for P equal to 2, it reduces to magnitude x 1 square plus magnitude x 2 square so on up to magnitude x n square 1 over 2, that is square root of the whole thing which is nothing but the l 2 norm. Now, this can be now used to construct other very interesting norm.

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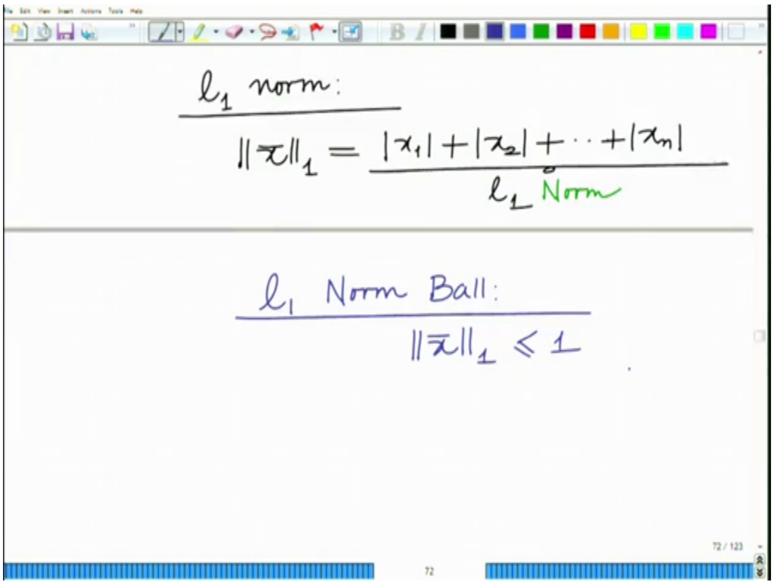

So, for instance, the l 1 norm which is one of the most fundamental and widely applied the l 1 norm is norm of x bar 1. You can see that simply reduces to magnitude x 1 plus magnitude x 2 that is each to the power of P which is 1 plus magnitude x n whole to the power of 1 over P which is again 1. So, this is simply magnitude x 1 plus magnitude plus magnitude x n, this is the l 1 norm. And the l 1 norm sphere or the l 1 norm ball, this is given by norm x bar of 1 less than or equal to 1, this is your l 1 norm ball. And for instance to look at this, let us consider a 2 D example, consider 2 dimensional case.

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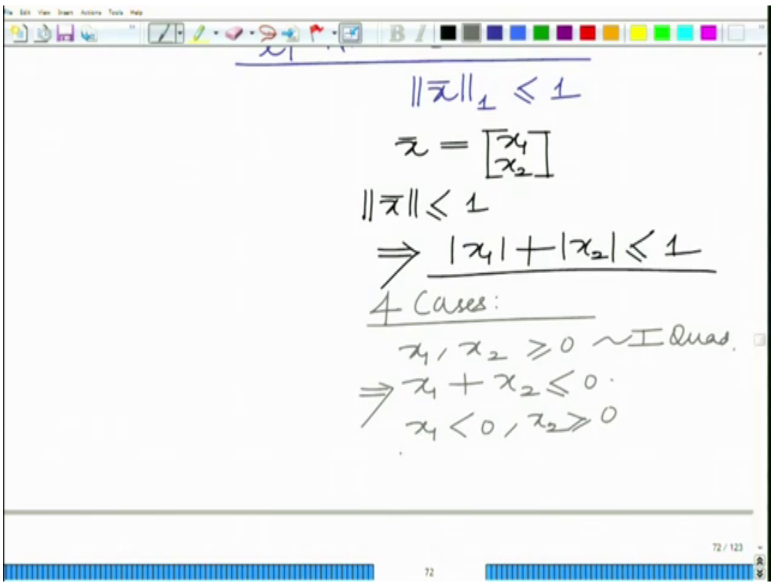

If x bar equals x 1 plus x 2 norm x bar less than equal to 1, this implies magnitude x 1 plus magnitude x 2 less than equal to 1, ok. Now, how to find this norm ball? You can consider four cases; one is x 1 comma x 2, both greater than equal to 0 in which case magnitude x 1 is nothing but x 1, magnitude x 2 is x 2 less than equal to 0. So, this corresponds to the first quadrant.

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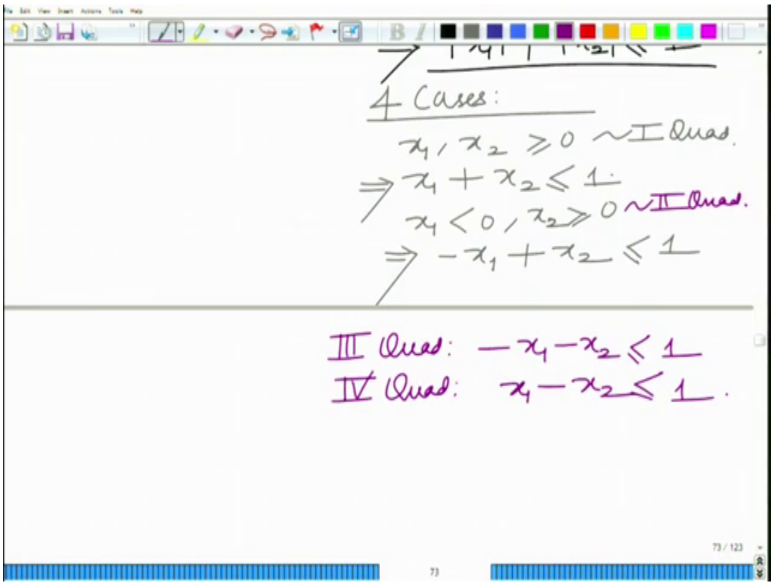

Second quadrant, we have x 1 less than 0, x 2 greater than equal to 0; this corresponds to the case magnitude x 1 is minus x 1. So, this will be minus x 1 plus x 2 less than equal, I am sorry this is not 0, this is 1 less than equal to 1. This is the second quadrant. Then in the third quadrant, you will have both are negative, you will have minus x 1 minus x 2 less than equal to 1. And in the fourth quadrant, you will have x 1 because x 1 is greater than equal 0, minus x 2 because x 2 is less than 0 less than equal to. So, these are the four cases and if you plot it, you will find something very interesting.

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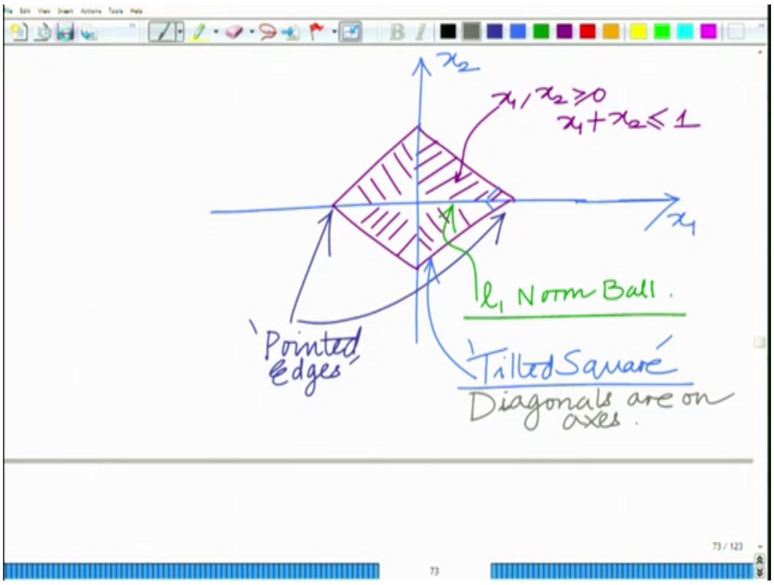

If you plot the l 1 norm ball and what you will observe is if you look at the first quadrant that corresponds to x 1 plus x 2 less than equal to 1 which is basically this region; x 1, x 2, x 1 comma x 2 greater than equal to 0 and x 1 plus x 2 less than equal to 1. And similarly, this will be the corresponding region in the second quadrant, third quadrant, fourth quadrant. And therefore, if you look at this, what you will observe is that this is the region corresponding to the l 1 norm ball, it is very interesting. It is very different from the l 2 norm ball in the sense that you can see that it has pointed edges, something very interesting. So, you can see and this simple observation which means it is non differentiable if you see, if we observe it, the simple observation leads to in fact profound implications.

So, if you look at the l 2 norm ball, you can see this is smooth, it has no (Refer Time: 17:47) or edges. So, the l 2 norm is something that is very amenable for analysis that is it can be easily differentiated and so on whereas, if you look at the l 1 norm ball, something very interesting that is a square with the diagonals along the axis. So, it is a tilted square and being a square, it has the sharp edges at which it is not differentiable. So, is something that is very interesting.

It is an very interesting shapes. So, this is not what you think of when you think of a so, this is basically your tilted square and it is 90 degrees, it is angles are 90 degrees is symmetric and the diagonals, the diagonals are aligned with the axis or diagonals are on the axis that is your x and y axis or your x 1 and x 2 axis, ok. So, this is the l 1 norm ball. Now, related to this is this notion, now we have seen the l 1 norm ball. Now, something very interesting is what is known as the l infinity norm that is what happens when P tends to infinity.

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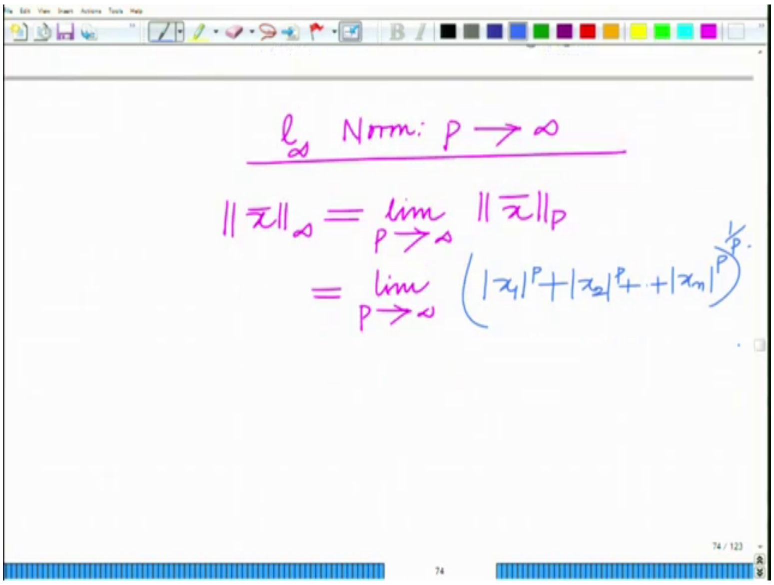

So, the l infinity norm that is norm of x bar infinity that is defined as limit P tending to infinity norm of x bar which is limit P tending to infinity under root of not under root, this is magnitude x 1 raise to the power of P plus magnitude x 2 raise to the power of P plus magnitude x n raise to the power of P whole to the power of P which can be basically shown to be magnitude of more x i, 1 less than equal to i, less than equal to n which is basically simply the maximum of magnitude x 1, magnitude x 2, so on up to magnitude of x n.

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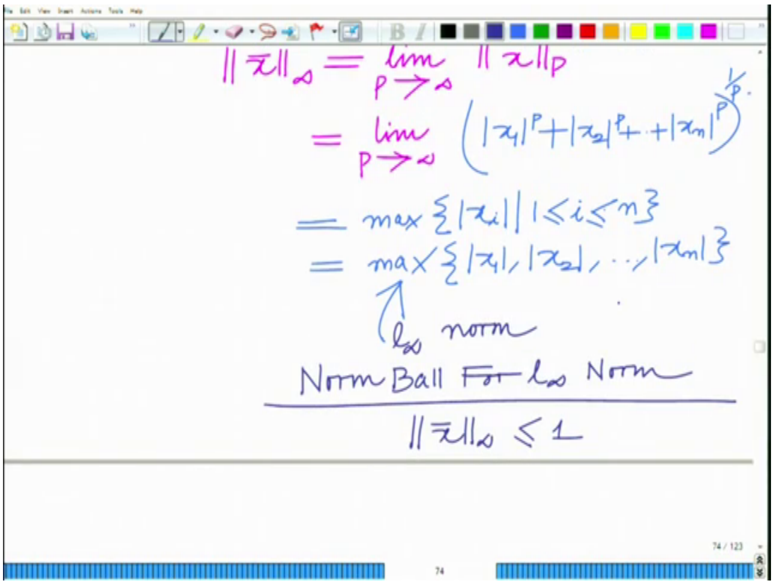

So, this is the l infinity norm, something that is very interesting. So, this is the l infinity norm. And now, one can corresponding derive the norm ball corresponding to l infinity norm, the norm ball and that is naturally given as norm of x bar infinity less than equal to 1. So, this is an interesting norm.

So, norm of vector, the infinity norm is simply in the maximum of the absolute values of the components of that vector and the l infinity norm ball is basically simply norm, the infinity norm of vector, the region corresponding to the infinity norm of a vector x bar being less than or equal to for instance, any radius. In this particular case, you can say the radius is equal to, alright.

So, we will stop here and continue with this discussion in the subsequent module.

Thank you very much.