Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture -12
Affine Set Examples - Line, Halfspace, Hyperplane and Application - Power Allocation for Users in Wireless Communication

Hello welcome to another module in this massive open online course. So, we are looking at the building blocks and the various fundamental definitions required to develop the optimization techniques and we have previously looked at the definition notion of the convex set the convex combination of the set of points and also the concept of a convex hull. So, let us continue this discussion by look at looking at something slightly different today that is the definition of an affine set.

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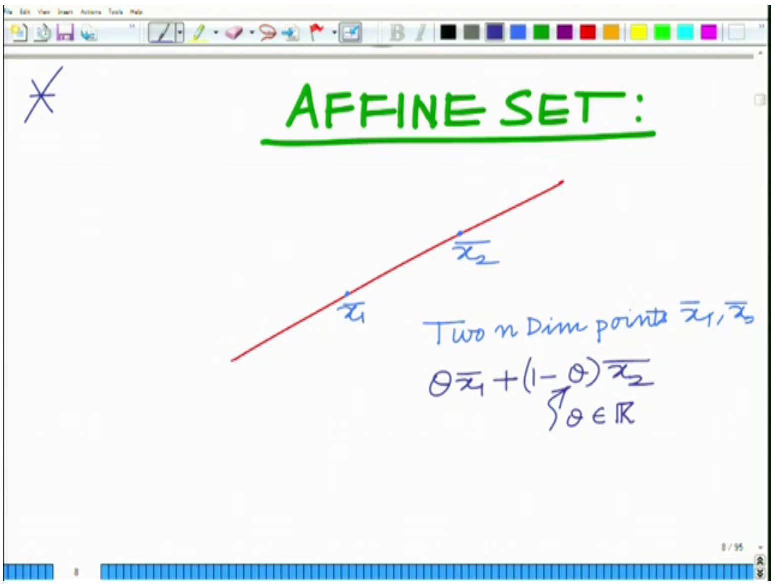

So, what you want to look at is the notion or the you want to learn the concept of an affine set. And this is very simple the affine set in the previous module we have seen the notion of a convex set. Now what an affine set is that is if you consider any two points x 1 bar and x 2 bar ok. So, similar to the definition of convex set consider two n dimensional points x1 bar comma x2 bar and now perform the combination theta x1 bar plus 1 minus theta x2 bar, but the theta can take and real value theta can take any remember in the convex for a convex combination we had restricted theta to lie between

0 and 1 however, in this case there is no such restriction and theta can take any real value ok.

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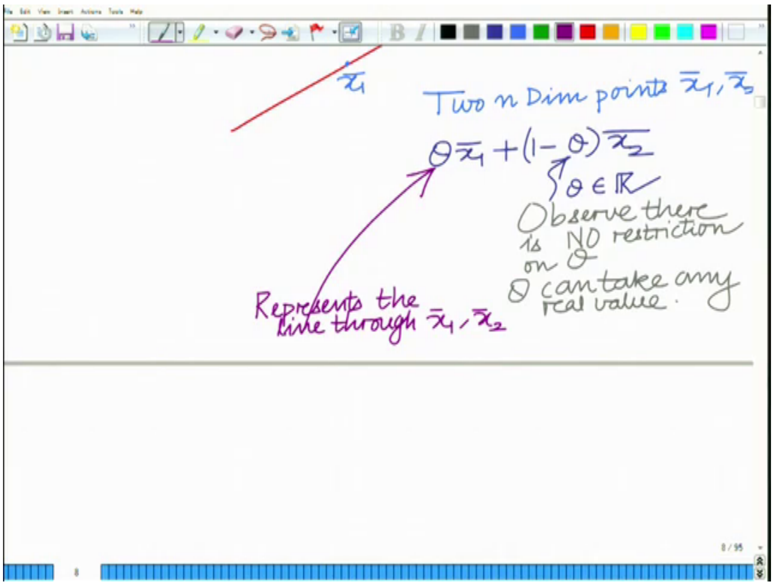

And this is an important observe there is no restriction on theta. In fact, theta can take. In fact, theta can take any real value and such a combination now this is an affine combination and basically you can say that for various values of theta this represent the entire line represents the line through x 1 bar comma x 2 bar.

So, previously when 0 is less than equal to theta is less than equal to 1 it simply represented the line segment between x 1 bar and x 2 bar. Now if you remove that restriction on theta, it represents the entire line that is any point on the line is captured by this combination theta times x 1 bar plus1 minus theta times x 2 bar. Now if this belongs to the set S whenever x 1 bar and x 2 bar belong to the set S that is the entire line all right, entire line joining the points x 1 bar and x 2 bar belongs to the set S for any two points x1 bar x2 bar belonging to the set S such a set is known as an affine set.

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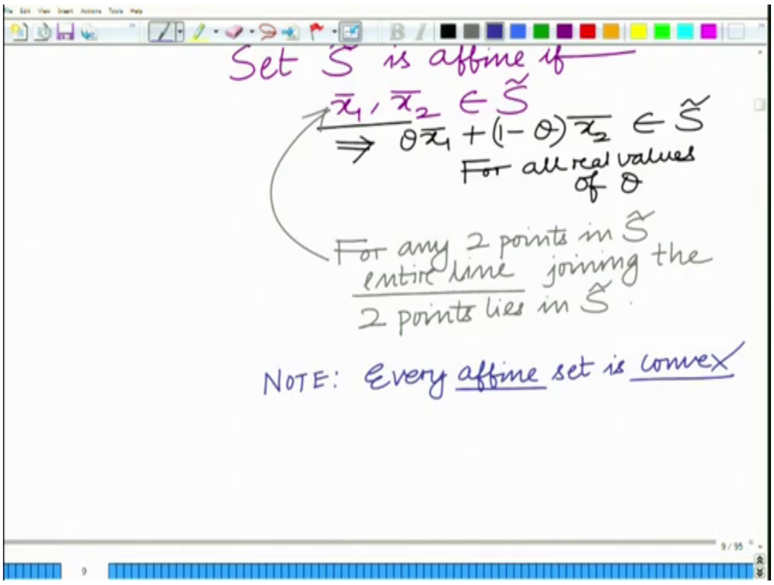

That is set S tilde is affine if x 1 bar comma x 2 bar belongs to S tilde ok. Remember we saw we seen this symbol belongs to S tilde implies theta times x1 bar plus 1 minus theta times x2 bar also belongs to x tilde for all real values of theta, that is for any real value of theta. That is for any two points, for any to given any two points in S tilde entire line joining the two points entire line joining the two points the entire line joining the two points lies in S lies in S tilde.

And note that the affine set is convex and that you can note this is an interesting property every affine set every affine set is convex correct. The reason is very simple because if it contains the entire line joining the two points, naturally it contains the line segment that is for any x1 bar x2 bar belonging to S tilde since it contains the entire line or if it is affine, it naturally also contains the line segment all right.

So, the convex set is a special case of an affine set all right. So, every affine set is a is also a convex. I am sorry affine set is a special case of a convex set all right. So, every affine set is a convex set, but note that every convex set need not be an affine set all right note that also, but all convex sets are not affine.

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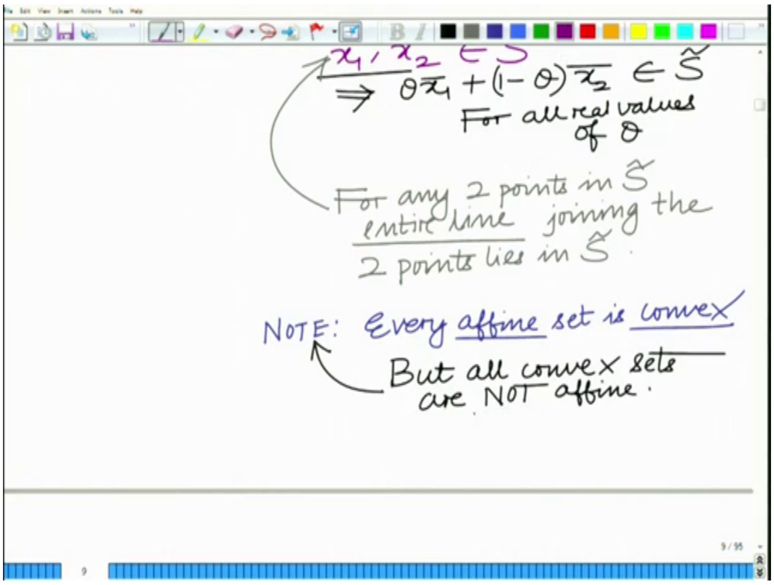

All convex sets are not affine and that is very easy to see take a simple example for instance if you consider a circle correct.

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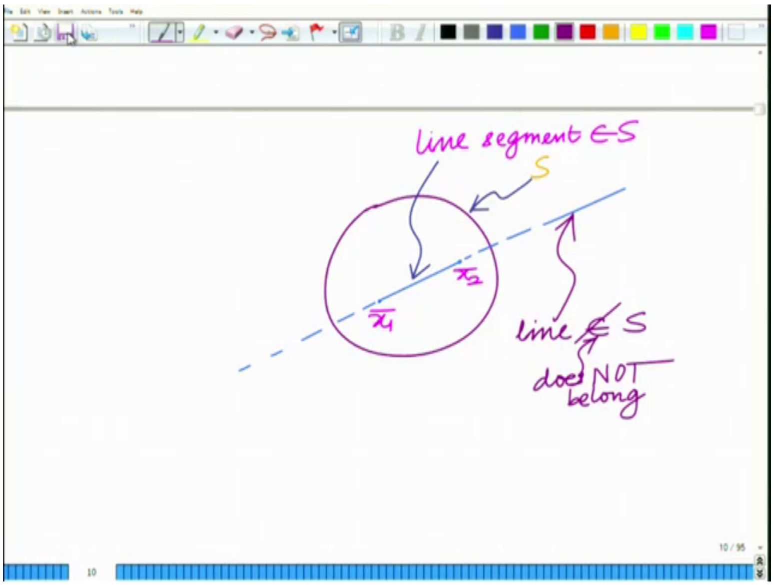

Which we saw yesterday is a convex set because if you took take any two points correct. The line segment joining the two points which is contained, but if you extend that to form the line the you can see that the entire line.

So, the line segment is contained. So, this is your S line segment is contained in S this is these are your points x1 bar x2 bar the line segment belongs to the S , but the line does not belong to S that is it does not the entire line does not belong to S it is a very simple thing ok. So, every affine set is convex, but every convex set is not affine ok. So, these are the, that these are the this is the interesting relation between affine sets and convex sets ok. All right now let us look at some examples to understand these better examples.

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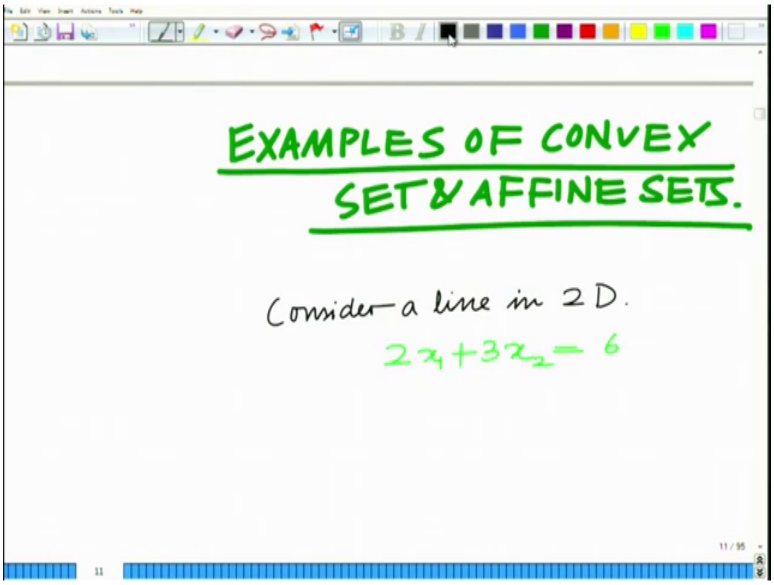

Examples of convex sets and affine, let us look at some examples of convex and affine sets and consider a simple line for if simple example for instance; consider a simple line in 2 dimensions the line let us say is given by the equation 2 x1 plus 3 x2 equal to 6 ok. Now if you plot that line it looks something like this.

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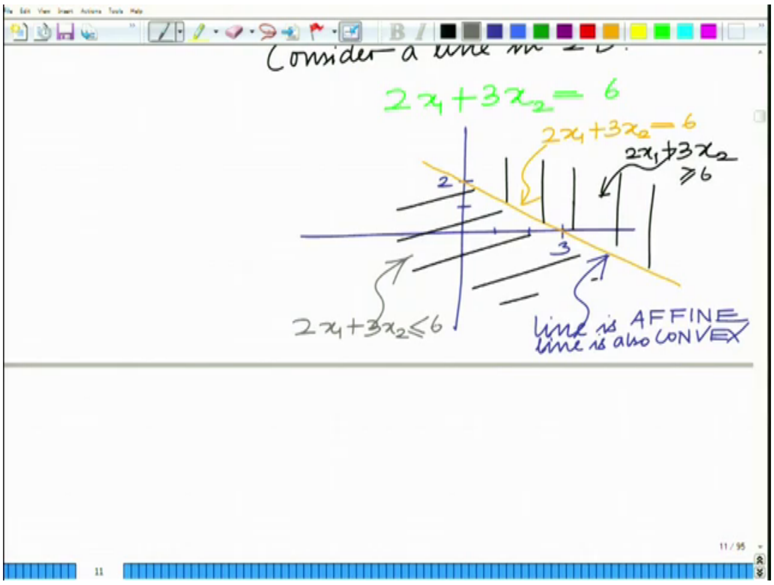

For x1equals for x2 equals 0 x1 equals you can see this is 3 and when x1 equal to 0 x2 equals 2. So, if you plot the line that will look something like it will look something like this ok.

So, this is the line 2 x1 plus 3 x2 equal to 6 and remember line is a trivial example of an affine set correct because if you take any two points on the line all right if you take the line as a set if you take any two points on the line and join the line correct. Naturally the entire line, which is the same line belongs to that set all right. So, the line is a trivial example. So, any line is a trivial example of an affine set ok. So, let us note that. So, this line is affine line is affine and it is also convex because every affine set is convex the line is also line is also convex.

Now, the interesting thing occurs when you look at these regions now the line is partitioning this plane into two regions if you look at this ok. Now this region is the region 2 x1 plus 3 x2 greater than or equal to 6 and this region is the region 2 x1 plus 3 x2 is less than or equal to 6 and these regions are known as half spaces.

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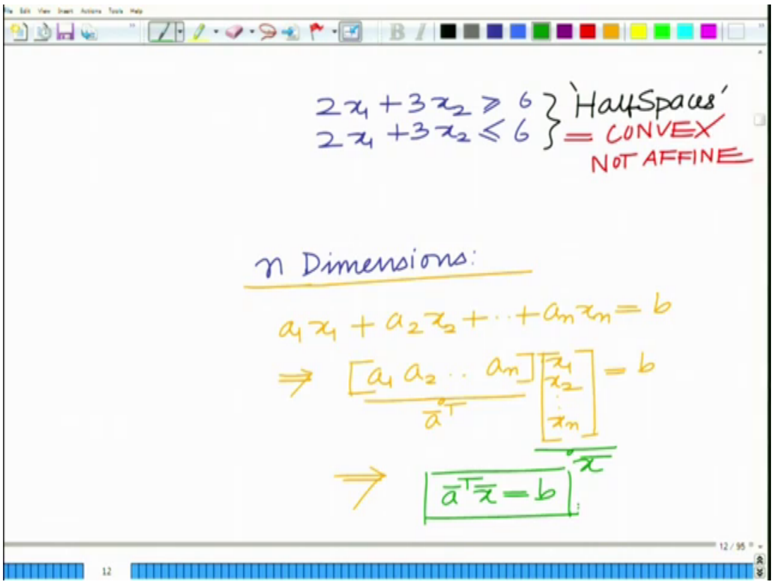

So, the line divides the plane into two regions 2 x1 plus 3 x2 greater than or equal to six or 2 x1 plus 3 x2 less than or equal to 6 and these are known as half spaces these are known as half spaces ok. So, we have a line and the line divides the plane into two regions. So, the line is convex and also affine. In fact, and it divides the plane into two regions are half spaces and note that half spaces are only convex they are not affine ok.

So, these half spaces these are convex and these are not affine. So, a line is affine which implies it is also a convex, but half space is only convex and not affine now if we generalize this. So, n dimensions in n dimensions one can consider an n dimensional equation which is of the form a1 x 1 plus a 2 x 2 plus an xn equals b which implies if I write it in vector notation that we are familiar with a one.

I can write it as the row vector a one times the column vector x1 x2 xn equals b which implies now I can denote this by a bar transpose and this by x bar this I can denote by x bar. So, I can write this as a bar transpose x bar equals b and this equation in n dimensions this represents what is known as a hyper plane in n dimensions this is a hyper plane which is in fact, you can see it is affine ok.

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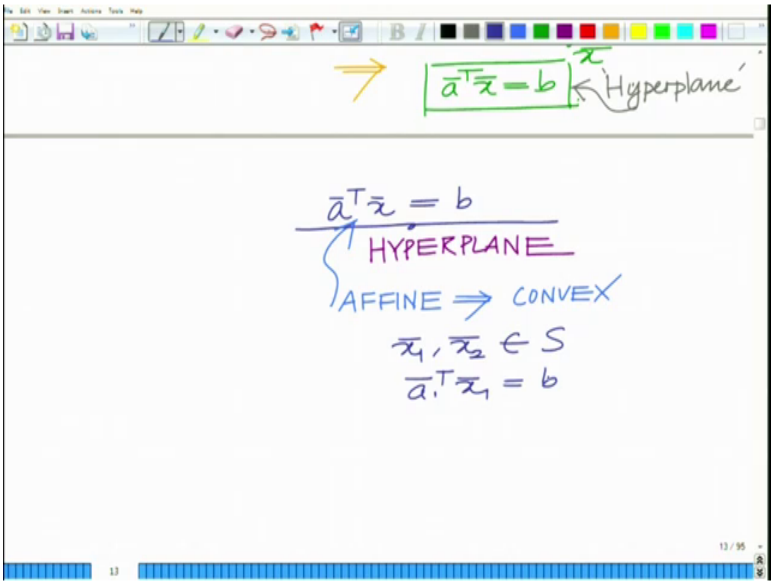

So, a bar transpose x bar equals b this is a hyper plane a general equation for a hyper plane this is an equation for a hyper plane and you can see that this is affine a hyper plane is affine which also implies that this is convex as well right for instance if x1 bar comma x2 bar belong to the set S that is you can quickly verify this that is a bar transpose x1 bar equals b a bar transpose x2 bar equals b this.

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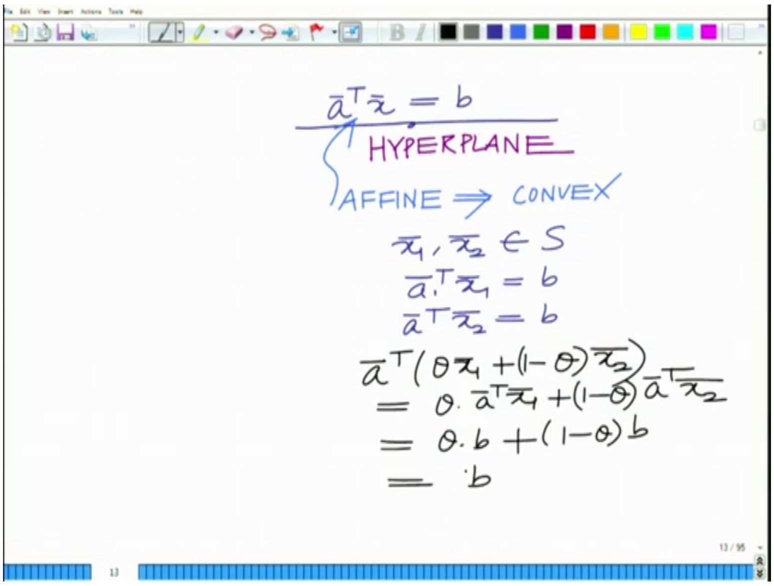

Now, consider theta times x1 bar plus 1 minus theta times x 2 bar theta times x1 bar plus 1 minus theta times x2 bar this equals theta times a bar transpose x 1 bar plus 1 minus theta times a bar transpose x2 bar which is theta times b plus 1 minus theta times b that is equal to b note that there is no restriction on theta valid for any theta.

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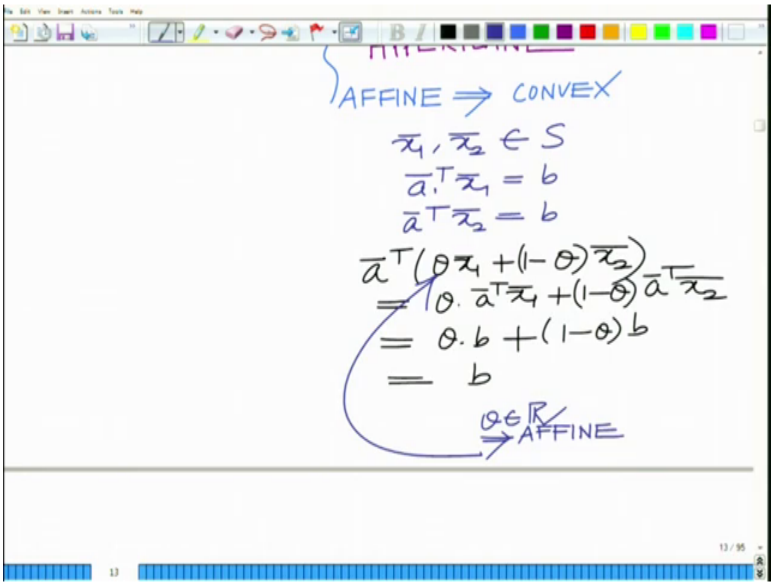

Element on R implies this as affine. If it is only valid for zero less than equal to theta less than equal to 1 it is convex in this case there is no restriction on theta. So, this is affine.

So, you can see that hyper plane is an affine set, now this hyper plane divides the space into 2 the n dimensional space into two regions a bar transpose x bar greater or equal to b a bar transpose x bar less than equal to b these two regions are known as half spaces ok.

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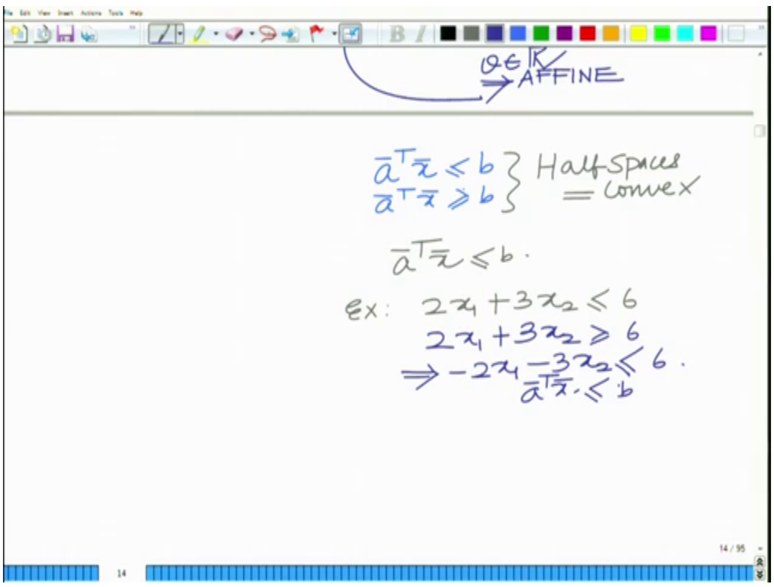

So, the hyper plane divides it into two regions correct a bar transpose x bar less than equal to b. These are half spaces that are. In fact, the general equation of half space you can always remember represented by a bar transpose x bar less than equal to b for instance. Example you have 2 x1 plus 3 x2 less than equal to 6, you also have the other half space that is 2 x1 plus 3 x2 greater than equal to 6, which basically implies minus 2 x1 minus you take the negative minus 2 x1 minus 3 x 2 less than or equal to 6.

So, the general equation of a half space which is of the form again a bar transpose x bar less than or equal to b ok.

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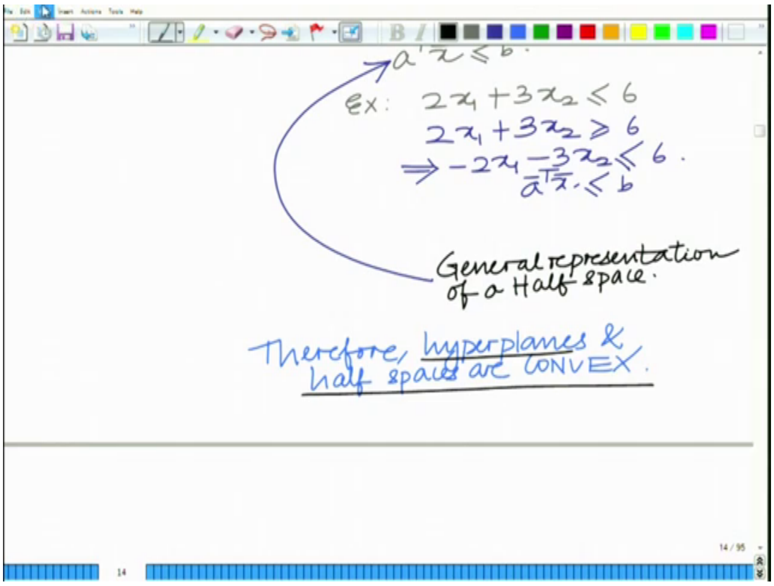

So, the general equation of a half space. So, a bar transpose x bar less than equal to b is the general expression for general representation of a half space ok. Thus these half planes this hyper planes and half spaces are complex therefore, the important thing to realize here therefore, hyper planes and half spaces. Hyper planes are affine as well, but for our purposes it is enough to note that hyper planes as well as half spaces are convex ok. Hyper planes as well as half spaces are convex ok.

Now, what we want to do is we want to explore a practical application because remember we want to also explore practical right. Applications of the concepts that we learn for optimization let us look at the practical application let us look at one of the practical applications of the concepts that we have just learned regarding convexity and how these influence practical optimization problems that are arise in wireless communications scenarios ok.

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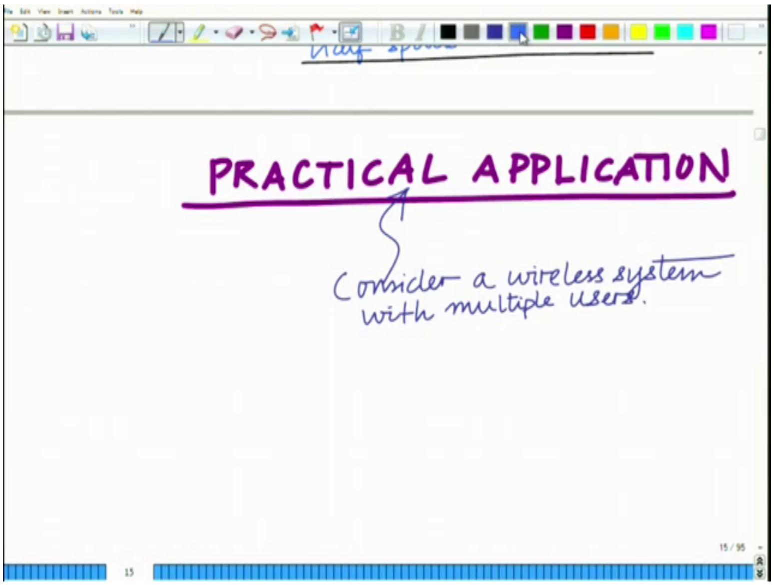

So, what we want to look at is we want to look at practical aspect practical application. So, for instance consider a wireless system with multiple users. So, what you want to do is we want to start by considering a wireless system with multiple users consider a wireless system with multiple users and for instance let us say you have a base station correct.

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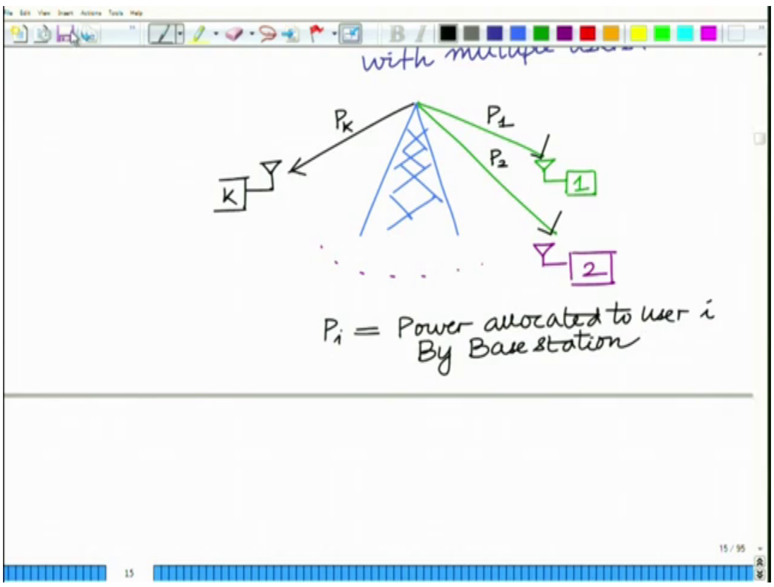

And you are transmitting signals to multiple users this is let us say user 1, you have another let us say user 2 somewhere and so on and so forth at some other point you have user k, we are considering a downlink scenario with a base station is transmitting to different users ok.

Now, let P 1 P1 denote the power to user 1, P 2 denote the power to user 2 so on and so forth P k denote the power to user k ok. So, we have P 1 P i equals power of signal power, allocated you can say power allocated to P i is the power allocated to user i by the base station, P i is the power allocated by user i to the base station then, now we need that.

So, P 1 P 2 P k are the powers that are allocated to the different users 1 to k, but this total power allocated to different users has to be less than or equal to correct the sum total of the powers of the different users has to be less than equal to the total power the maximum power of the base station available at the base station all right. So, that is the constraint that we have in a practical wireless scenario.

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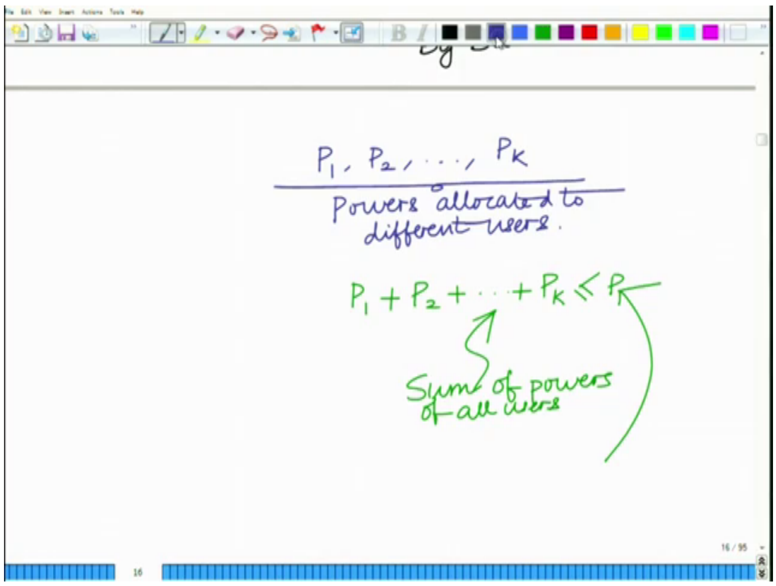

So, the power that is allocated to the different users that is P 1 P 2 P k these are the powers allocated to different users. These are the powers that are allocated to different users. Now when we consider P 1 plus P 2 plus P k this has to be less than or equal to P all right. So, the sum power of all the users sum of powers of all users has to be less than or equal to P which is the total power of the base station so, that has the total base station powers.

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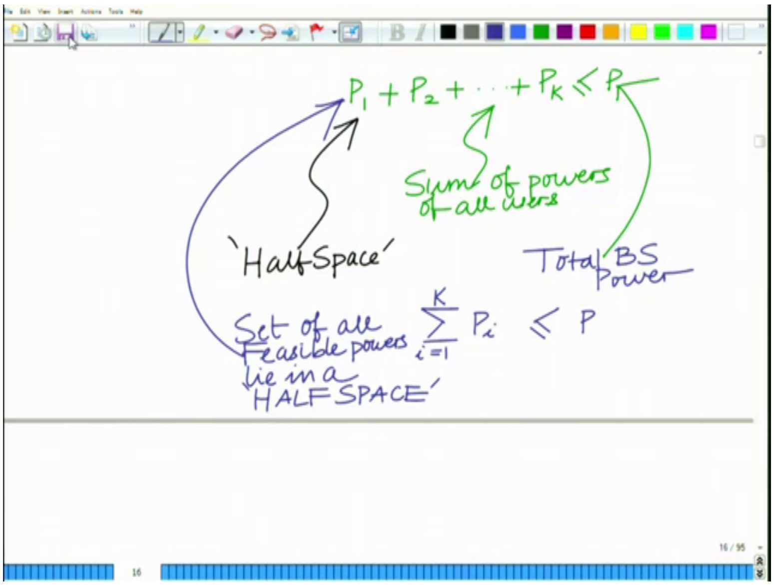

So, the sum of the powers of all the users or basically if you look at sigma P i summation P i i equal to 1 to k that has to be less than or equal to P. And you can see this constraint is basically P 1 plus P 2 plus P k less than equal to P this is nothing, but half space constraint because it is a linear combination of P 1 plus P 2 plus P k all right all right and you can consider the weighting coefficients a 1 a 2 a k to be unity that is one and therefore, we have P 1 plus P 2 plus up to P k less than equal to p this. In fact, represents a half space.

So, this is a very important constraint in wireless communication this is nothing, but a half space. So, basically the set of all feasible powers possible powers that satisfy this constraint, lie in a half space that is the interesting interpretation that is that one can make here all right. So, the set of all feasible powers in the wireless scenario this is an important notions set of all feasible, feasible in the sense that satisfy the constraint.

Set of all feasible powers lie in a half space the set of all feasible powers lie in a half space. Now you can also have an equality power constraint, that is you do not want to waste any power and you want to set the power of all the users equal to p that is P 1 plus P 2 plus P k equal to P and this is an equality power constraint.

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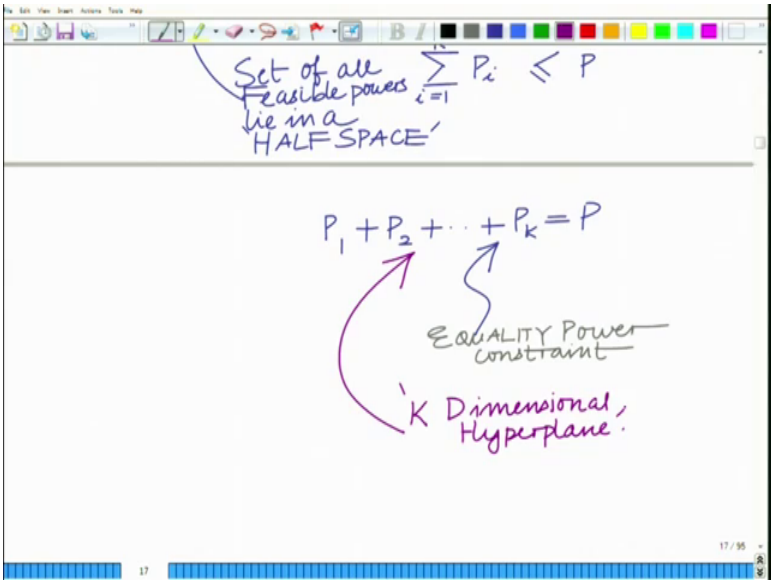

That is you are less than or equal to by equal to. So, this is a equality this is an equality power constraint and note that this represents the hyper plane so all the feasible powers lie on a hyper plane represents a k dimensional. This represents a k dimensional hyper plane all right.

So, we have this power constraint in a wireless communication system that can either you have any equality that is a sum total of the powers of the different users is less than or equal to P that is the total power of the base station. That is basically half space constraint and when your equality power constraint that is sum total of powers of all the users has to be equal to the power of the base station that basically, represents a hyper plane which means the set of all feasible powers corresponds to a hyper plane in k dimensional space all right.

So, this is an interesting practical perspective to the theoretical concepts of convex sets and affine sets that we have just seen. And we will explore several more links between the various theoretical concepts or the theoretical building blocks of optimization and it is relation to practical applications in several fields as wireless communications be it signal processing or so on. So, we will stop here and continue in the subsequent modules

Thank you very much.