Applied Optimization for Wireless, Machine Learning, Big Data
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture – 11
Introduction to Convex Sets and Properties

Hello. Welcome to another module in this massive open online course. So, let us start our discussion on optimization by looking at some of the fundamental building blocks of optimization by first looking at convex sets the notion of a convex set and the various properties of convex sets ok.

So, we want to start our discussion of optimization.

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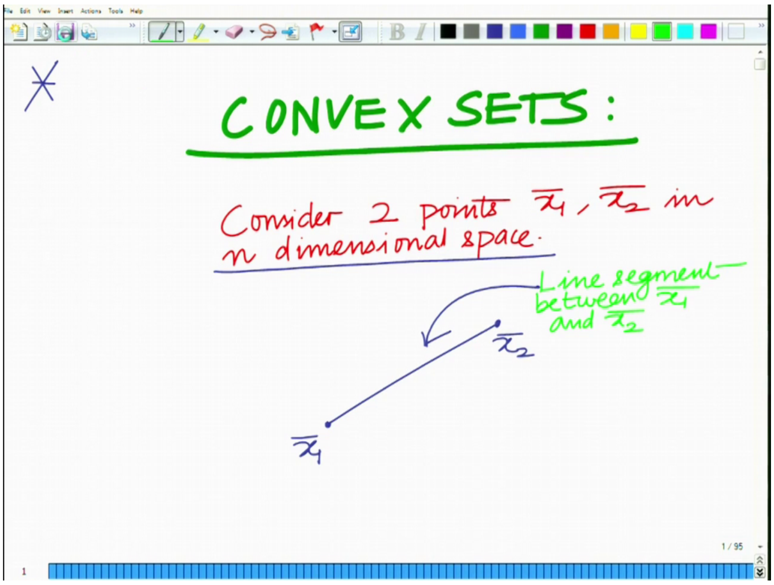

And one of the important concepts to understand in convex optimization is a convex set, the definition and properties of convex sets. So, a convex set is as follows now consider. So, define a convex set let us start with the following setup consider 2 points x 1 bar comma x 2 bar in n dimensional space ok, which means these are vectors in n dimensional space. These are points in these are general points in n dimensional space.

So, we are considering 2 points in n dimensional space let me describe these points. So, I have point 1, let us say this is your x bar and this is your x bar. And this is the line segment that is joining x line segment between x 1 bar between x 1 bar and x 2 bar ok. This is the line segment between x 1 bar and x 2 bar.

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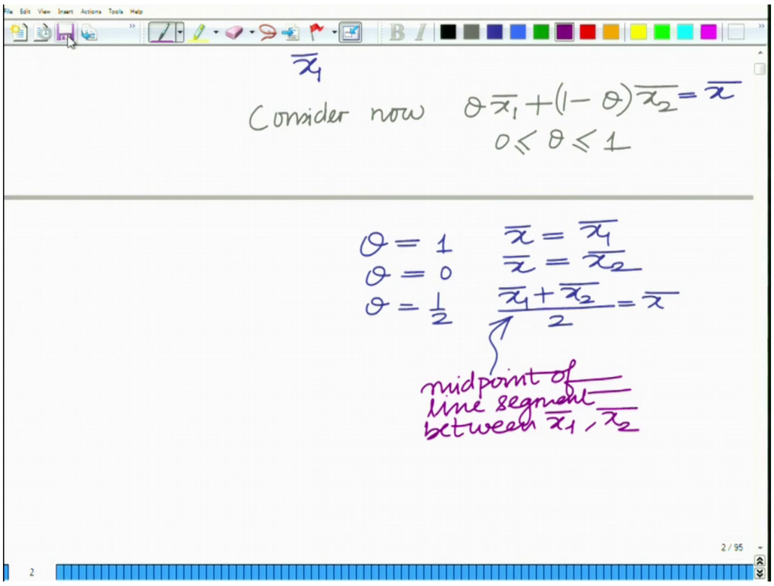

Now, consider now a linear combination of the form theta x 1 bar plus 1 minus theta x 2 bar 1 minus theta x 2 bar, such that well 0 is less than or equal to theta is less than or equal to 1 ok. So, we are considering a combination of x 1 bar and x 2 bar such with the weights theta n 1 weighs theta that is theta times x 1 bar plus 1 minus theta times x 2 bar. And another important aspect to note here is that we are not allowing any value of theta, but only values of theta lying between 0 and 1 ok.

And now for instance let us take a look at the various such points generated by this combination. Now for instance, if theta equals 1, let us denote this point by x bar we have x bar equals well 1 times x bar plus 0 times x 1 times x 1 bar plus 0 times x 2 bar. So, this is x 1 bar if theta equal 0 on the other had x bar is simply you can check theta equals 0 times x 1 bar plus 1 times x 2 bar. So, this is x 2 bar if theta equals half this half times x 1 bar plus half times x 2 bar.

So, this is x 1 bar plus x 2 bar by 2 equals x bar which is basically you can see mid-point of the midpoint, of the line segment between x 1 bar and x 2 bar. So, this you can see is the midpoint, midpoint of the line segment between x 1 bar and x 2 bar. So, what you observe is that x theta varies from 0 to 1. This combination theta times x 1 bar plus 1 minus x 2 bar traces the line segment between x 1 bar and x 2 bar correct.

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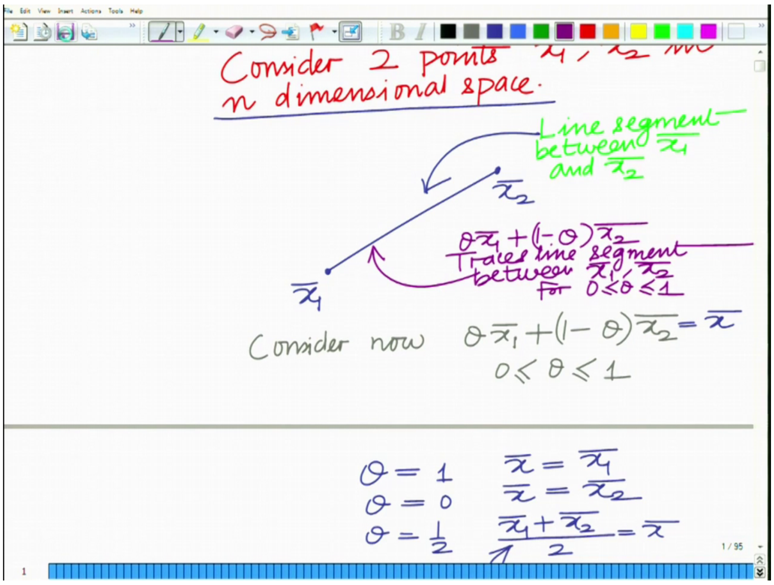

So, theta times x 1 bar plus 1 minus theta times x 2 bar traces line segment between x 1 bar comma x 2 bar, for various values of 0 less than equal to theta less than equal to 1, that is theta theta times.

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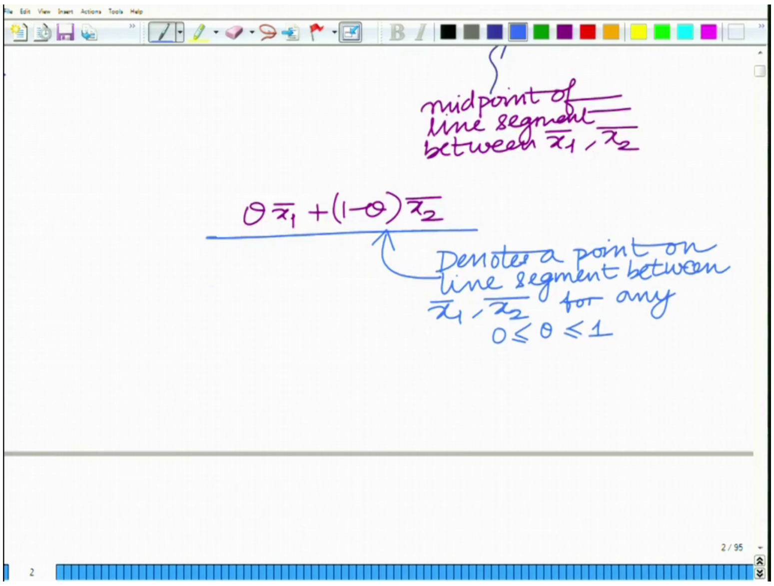

X 1 bar plus 1 minus theta times x 2 bar if you look at this combination which is also termed as a convex combination. This denotes a point on line segment between x 1 bar comma x 2 bar, for any particular value of theta lying between 0 and 1 ok.

So, as theta varies from 0 to 1 it traces the line segment between x 1 bar and x 2 bar ok. That is the first basic concept. Now what is the definition of a convex set a set is known as a convex set?

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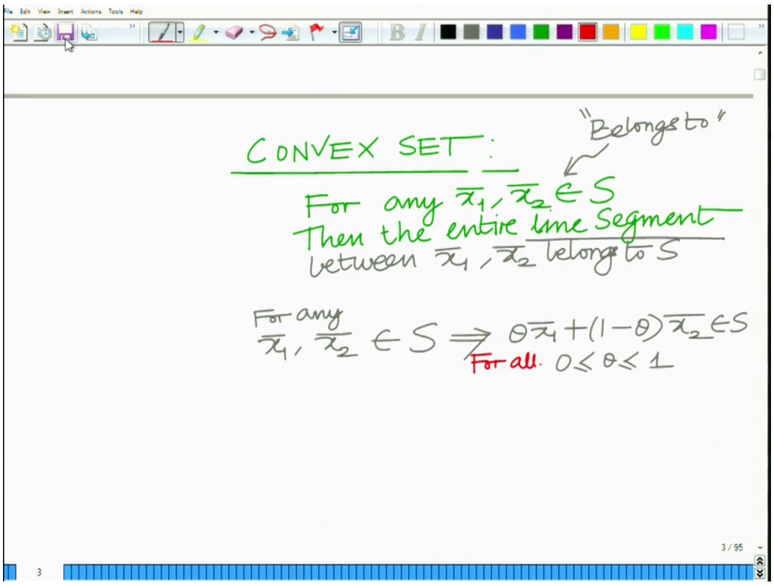

A set is termed as a convex set if for that is if any 2 points x 1 bar comma x 2 bar belong to s, then the entire line segment not the line the entire line segment between x 1 bar and x 2 bar, comma x 2 bar belongs to S ok. This symbol is basically belongs to belongs to alright.

So, what this means is the following if mathematically writing, if x 1 bar comma x 2 bar belongs to S implies that the line segment. Remember, we just demonstrated that the line segment is denoted by x 1 bar theta times x 1 bar plus 1 minus theta times x 2 bar 0 less or equal to theta less than equal to 1 also belongs to S; for all that is for any x 1 bar for all that is for all 0 less or equal to theta less or equal to 1. What this says is that that is if you pick any 2 points x 1 bar x 2 bar belonging to the set S, then if the entire line segment between x 1 bar and x 2 bar belongs to S. And this is true for any such set of points x 1 bar x 2 bar such a set is known as a convex set.

And the mathematical way of stating this is that if you pick any 2 points x 1 bar x 2 bar construct theta times x 1 bar, plus 1 minus theta times x 2 bar which represents a point on the line segment for various values of theta between 0 and 1 this point represented by theta times x 1 bar plus 1 minus theta times x 2 bar must belong to x must belong to the set S ok.

It is a very simple example; it is a very simple definition.

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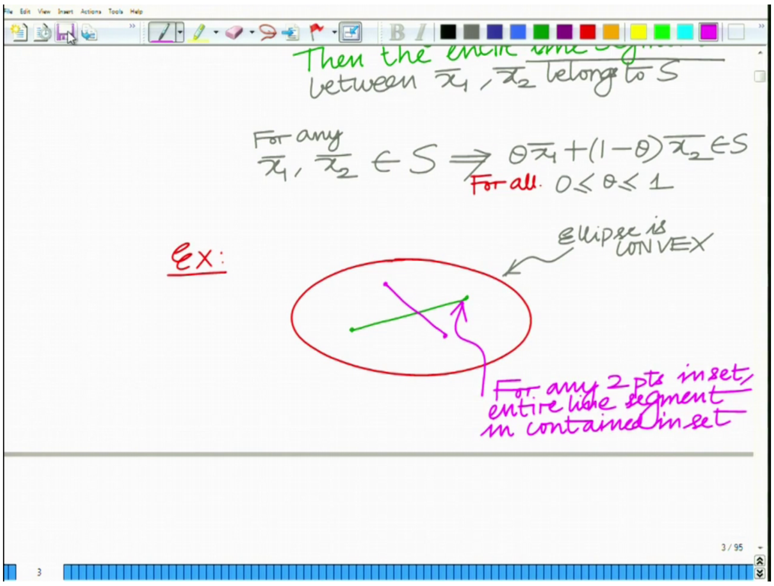

For instance, you can readily see that an ellipse is a convex set. This can be formally also shown which we will show later. That is, you can choose any 2 points and the entire line segment joining the 2 points you can check lies in the set. So, for any 2 points, entire line segment is contained in set. So, for any 2 points in set entire line segment is contained. For any 2 points in the set the entire line segment between them is contained in the set.

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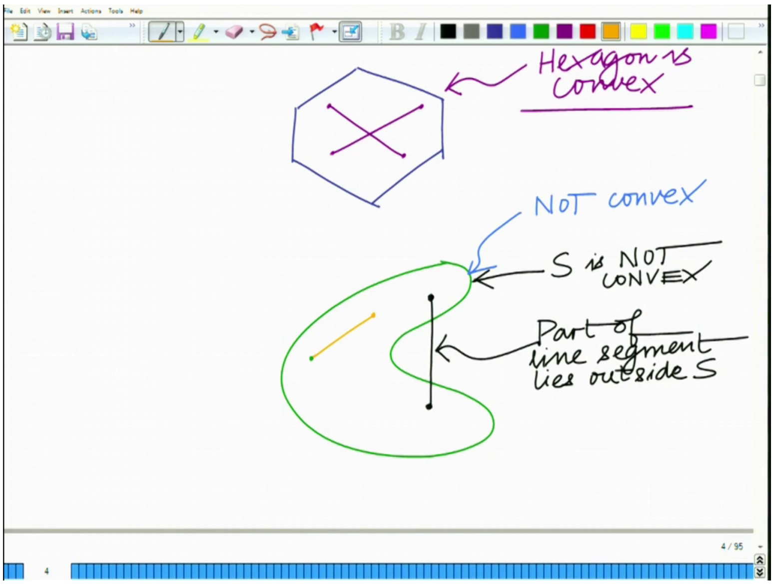

For instance, you can also quickly check many other sets such as hexagon for instance.

These need not be regular shapes for instance a hexagon you take any 2 points; you join them by the line segment. So, the hexagon you can clearly see is a hexagon is also a convex set ok. On the other hand, if you have a region like this, you can clearly see this kind of a set this kind of a set is not convex, because if you take any 2 points and join it by a line segment then the line segment does not is not entirely contained in S. The line segment you can quickly verify the line segment part of the line segment lies part of the line segment lies outside the set s. So, if this is your set S, S is not convex ok.

So, you take 2 points and join them part of the line segment lies outside the set S that is a line segment is not entirely. Remember that it is not just a few points of the line segment the entire line segment has to be contained in S. Only then and that has to be true for any set of points in S.

Now, for instance you can look at this even though if you choose these 2 points here then the line segment is contained in S right, but it is not only for a particular set of points this has to be true for any set of points. And the entire line segment between that chosen set of points all right that has to be completely contained in S.

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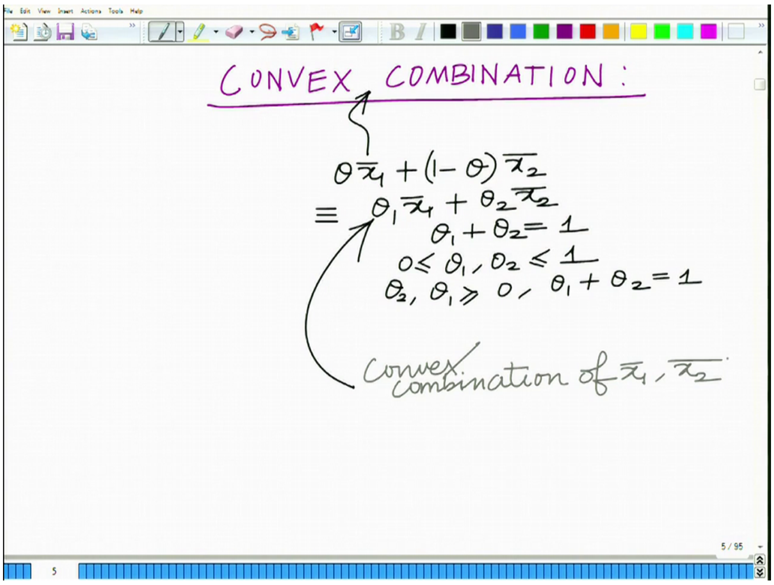

Let us come to the notion of a convex combination another very useful notion is that of a convex combination. And a convex combination is as follows that is if we have remember we said, theta times x 1 bar plus 1 minus theta times x 2 bar consider this combination, I can equivalently represent this as theta 1 times x 1 bar plus theta 2 times x 2 bar, but they have to satisfy the property remember theta 1 plus because remember we have theta and 1 minus theta. So, the implies theta 1 plus theta 2 equals to 1 and 0 less than theta lies between 0 and 1, which means we have to have the property 0 less than or equal to theta 1 comma theta 2 less than or equal to 1. Or you can equivalently say that theta 1 comma theta 2 theta 2 comma theta 1 greater than or equal to 0 and theta 1 plus theta 2 equals one ok.

So, a convex such a combination is known as a convex combination. This is known as a termed as a convex combination of x 1 bar and x 2 bar. Now I can generalize this notion of convex combination.

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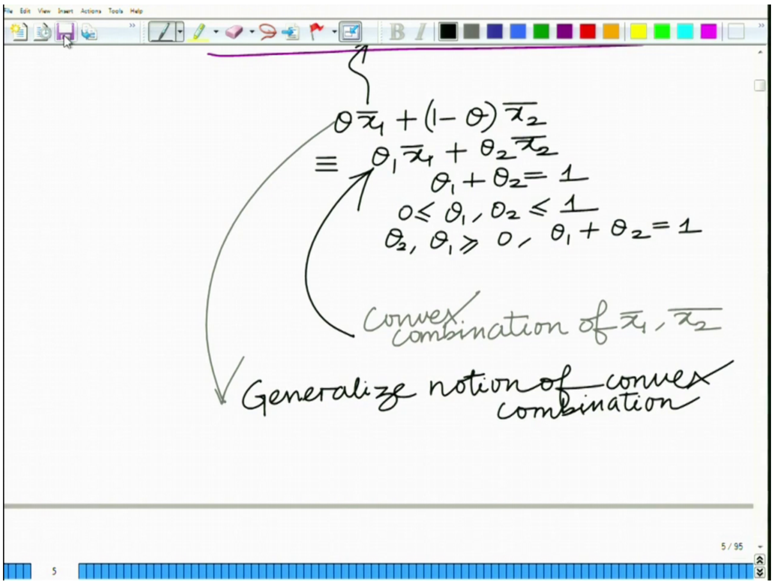

Now, I can what I can do is, I can generalize starting from here I can generalize. I can generalize this notion of convex combination to include n points. So, you consider k points.

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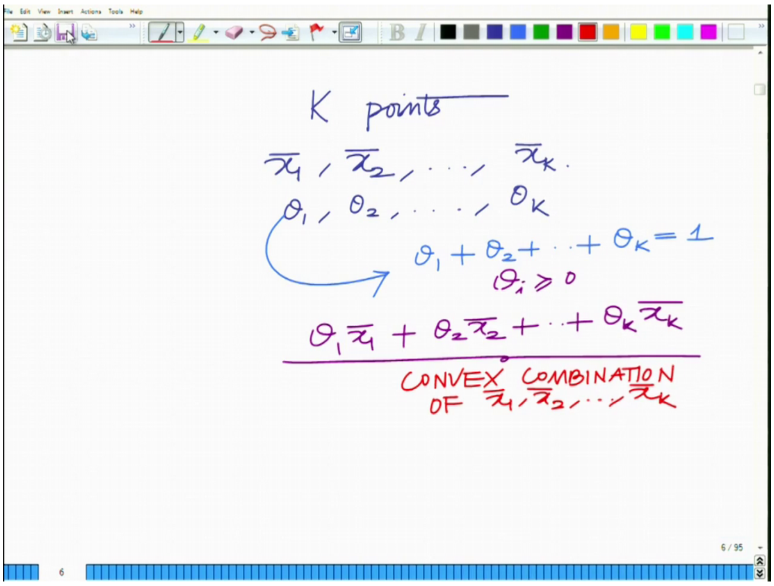

K points in n dimensional space that is x 1 bar x 2 bar, so on up to x k bar. And you consider theta 1 theta 2 theta k such that, theta 1 plus theta 2 plus theta k equals 1 and each theta i is greater than or equal to 0 and perform the combination theta 1 x 1 bar plus theta 2 x 2 bar plus so, on theta k x k bar. This is termed as a convex combination of x 1 bar x 2 bar and up to x k bar.

So, this is a convex combination this is a convex combination of x 1 bar x 2 bar up to x k bar. So, let us now look at the convex hull of a set.

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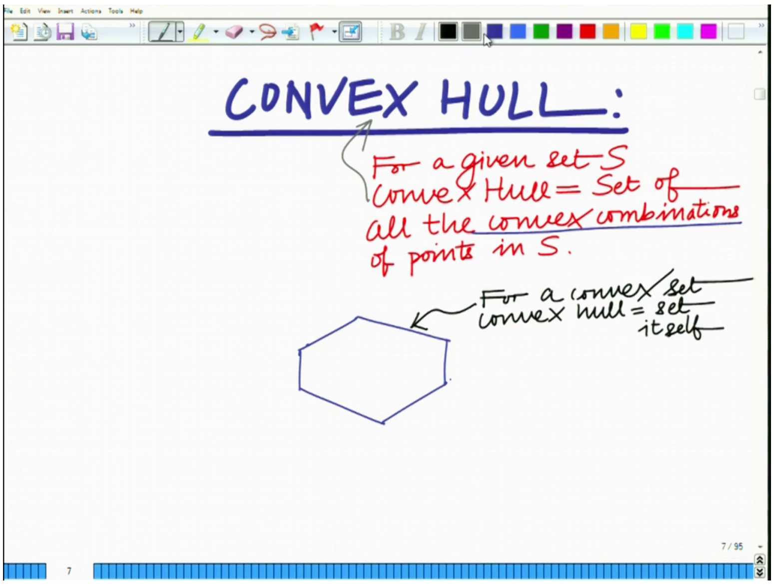

So, the other concept we want to look at is the notion of a convex hull. And the convex hull of a set is simply it is nothing but basically this is the set of. So, given set S for convex the convex hull is the set of all of convex combination set of all convex combinations set of all the convex combinations all the convex combinations of points in S.

So, you take a set for any given set consider the set of all the convex combinations of the points in S, S and that gives the convex hull of the set. Now naturally observe that for any convex set S the convex hull is the set itself because S if S is a convex set; then it already contains all the convex combinations of the points in S ok. So, for a convex set, for instance we saw yesterday that the hexagon is a convex set correct for a convex set convex hull equals the set itself, because it already contains all the combinations convex combinations for a non-convex set.

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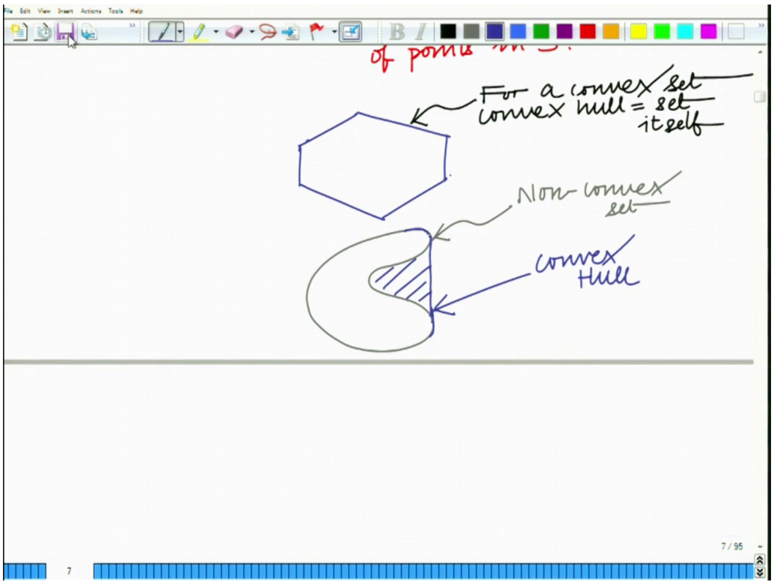

Such as this kind of region that we looked this is a non-convex set for a non-convex set the convex hull simply fills this region to make it a convex set.

So, the earlier one is a non-convex set. So now, once you fill this what you get that is this entire set that you get is now a convex hull. So, the convex hull that makes it basically that converts that you will see, you can say including includes all the convex combinations of all the original points in the set S to convert it into a convex set. That is the convex hull of a given set S alright.

So, let us stop this module here. And we will look at other aspects in the subsequent modules.

Thank you very much.