Principles of Signals and Systems
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Lecture - 61
Discrete Fourier Transform: Properties – Conjugation, Frequency Shifts, Duality, Circular Convolution, Multiplication, Parseval's Relation, Example Problems for Fourier Analysis of Discrete Time Signals

Hello, welcome to another module in this massive open online course. So, we are looking at the DFT; that is the discrete Fourier transform all right, and its application for finite length in discrete time sequence.

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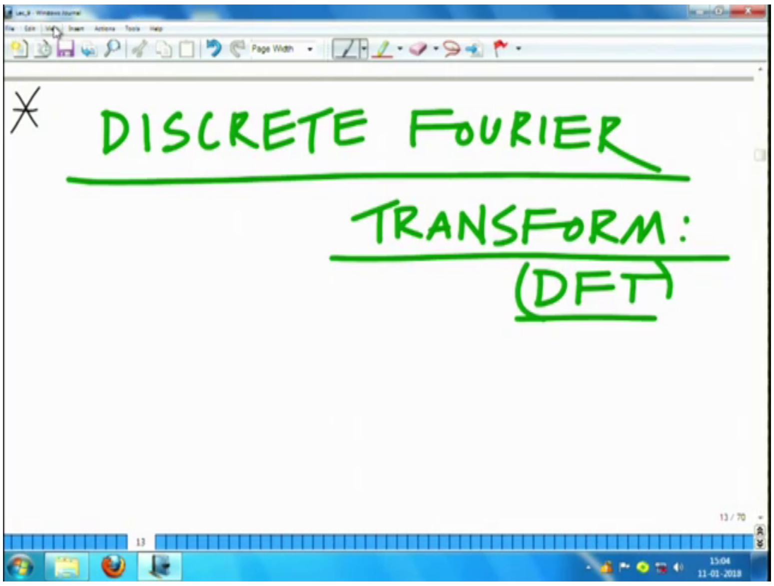

So, we are looking at the discrete Fourier transform which as I have already alluded to, is one of the most important tools which has revolutionized signal process.

So, this is the discrete Fourier transform. So, we have already seen and this is the DFT and in the DFT what we have is, we have the DFT of the finite length sequence and we are looking at the properties of the DFT.

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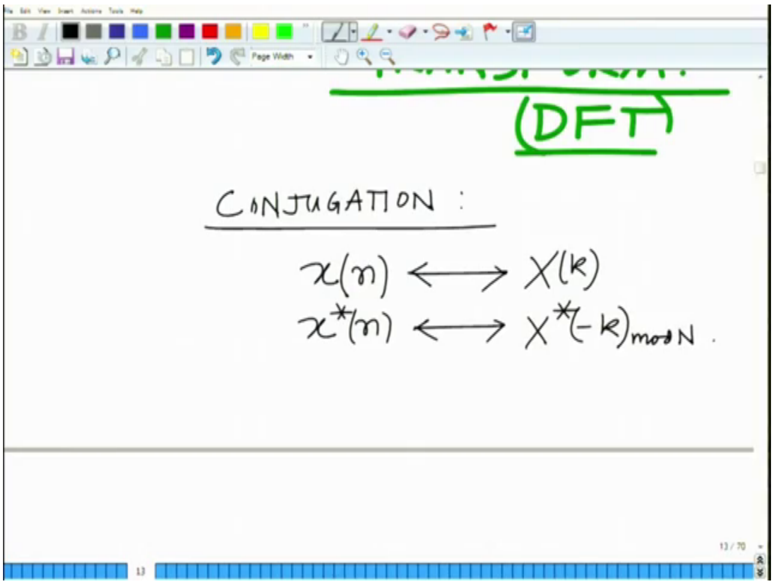

Let us continue our discussion with the properties. So, this is the next property that is conjugation. So, x; so we have a sequence xn which has the DFT

So, let us say X of k. Then you can show that x conjugate of n, it is not very difficult to see x conjugate of n has the DFT that is X conjugate of minus k, but remember the minus k has to be, has to be interpreted as modulo N all right. So, that is X conjugate of minus k modulo N in the next property. So, again this is a very simple property which you can verify yourself.

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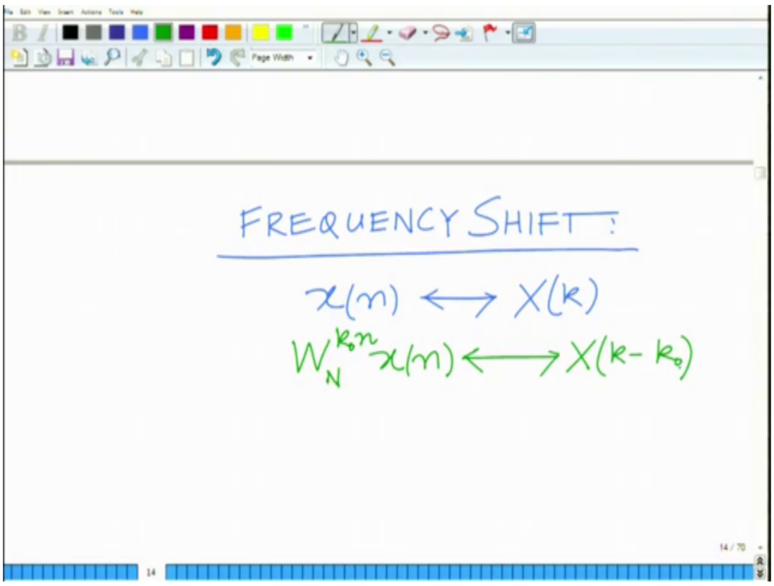

There is a frequency shift property, the frequency shift property, shift in time results in modulation and frequency naturally one can expect, as expected modulation in time results in a shift in frequency. So, xn as the DFT X of k then WN k naught of n xn as the DFT Xk minus k naught ok. Then you have the duality all right.

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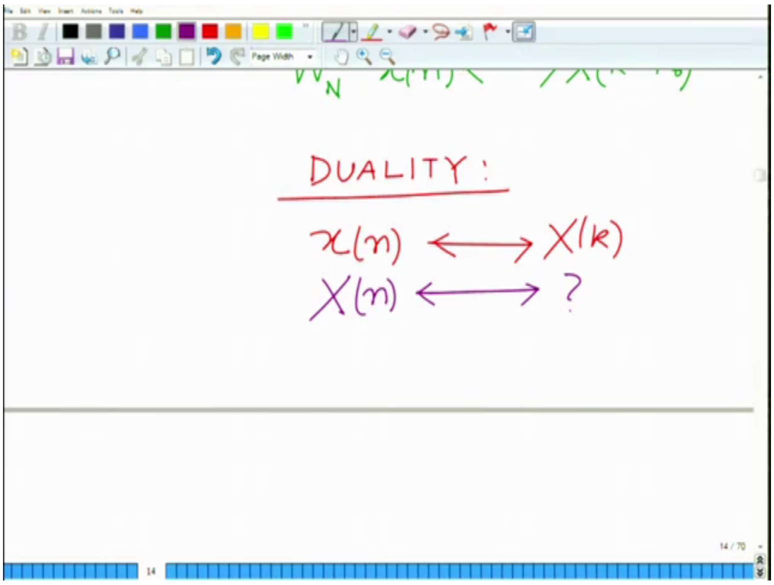

So, xn has the DFT X of k. Now what can we say about the sequence, remember duality is the time domain becomes frequency domain and what happens to the DFT of capital X of n then a small x of n has the DFT capital coefficients given by capital X of k. What can we say about the DFT coefficients, what can we say about the DFT of the time domain coefficients given by capital X of n. And that is also fairly easy to see we have you just write it as capital small x of n,.

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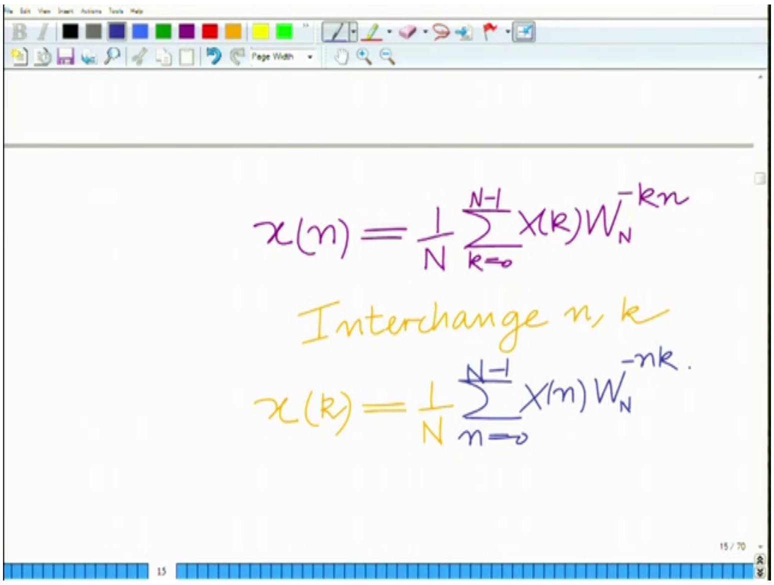

Remember you can write it is in the inverse DFT as 1 over N summation k equal to 0 to N minus 1 X k W N of minus k n, which is.

Now, we change. So, now, if you look at x of, now if change interchange n and k replace n by k, interchange the rules of n and k and then what you will have is, I can simply write x of k equals 1 over N summation k equal to 0 to N minus 1, or in fact, we are interchanging xn n and k. So, this is n equal to 0 capital N minus 1 X of n W N and this remains minus nk, which is the same as minus k n.

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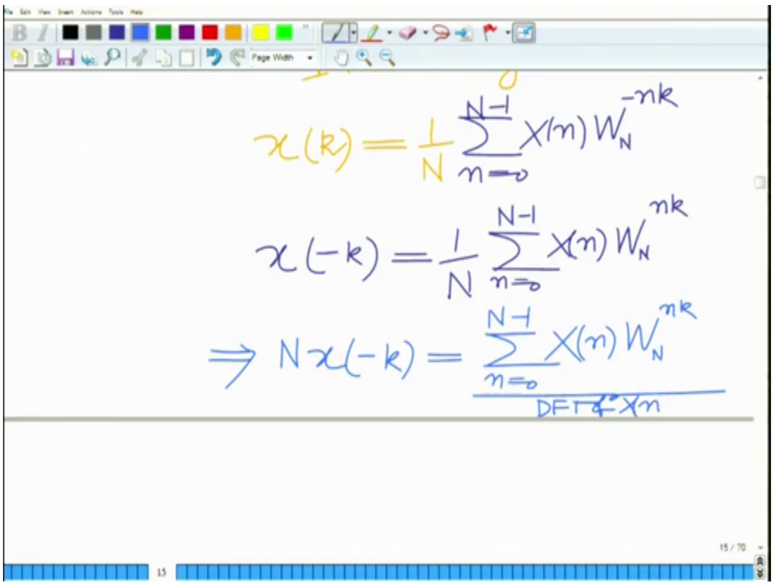

And now what I can do is if I consider x of minus k that becomes 1 over, that becomes 1 over N summation n equal to 0 to N minus 1 Xn W N raise to n k, whether replacing k by, replacing it by minus k ok, W N by star n.

Of course the minus k is interpolated modulo N and the final operation is to get the N to the other side and that implies N x of minus k is basically summation n equal to 0 to N minus 1 capital Xn W N to the power of n k, which is basically DFT of you can see Xn.

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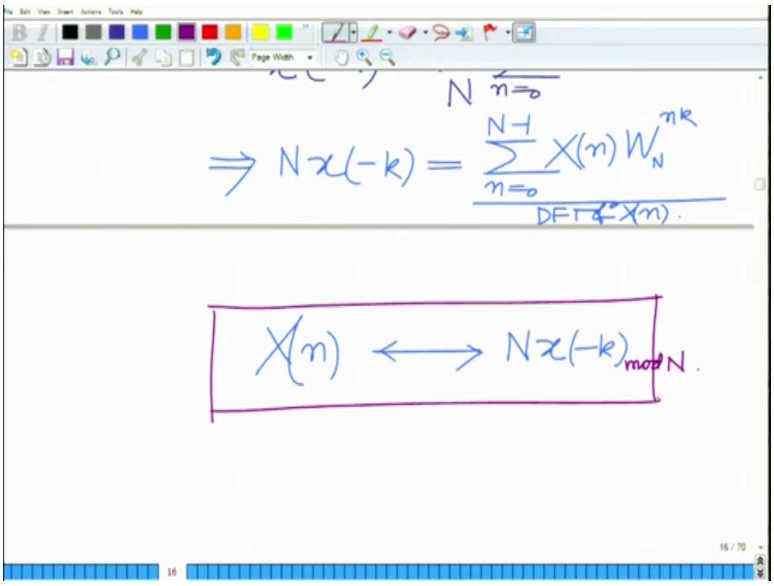

So, Xn, the capital Xn sequence you can see has the DFT coefficients which are N x of minus k. So,. And of course, the minus k is remember everything is always interpreted modulo N. So, a capital X of n has the DFT discrete time Fourier transform coefficients given by capital N x of minus k all right. And then the next property is also fairly important, which is a circular convolution

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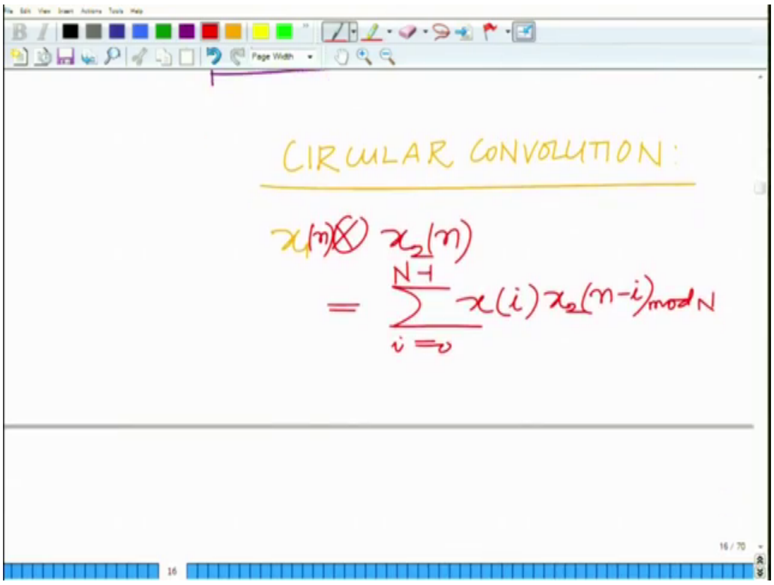

Which is basically x 1 circularly convolved with x 2 or x 1 circularly convolved with x 2 of n. This is basically equal to summation i equal to 0 to N minus 1 x i x 2 n minus i.

But the n minus i has to be interpreted modulo N uh, which is basically x for summation n equal to, so basically which is summation x i. So, summation x i x 2 n minus i modulo N. And now if you look at the DFT of this you want to ask the question.

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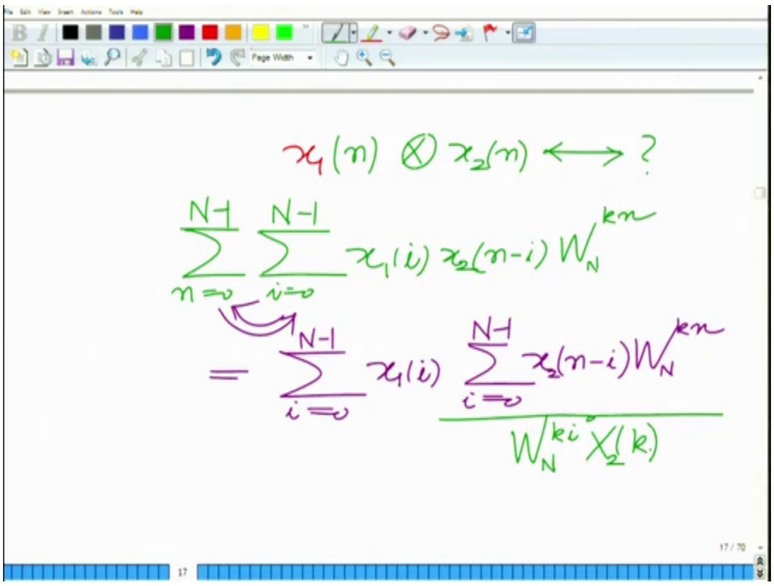

The question that we want to ask is x 1 of n circularly convolved with x 2 of n. What is the DFT of this quantity, and the DFT of this quantity can be found as follows, you have summation i equal to 0 to n minus 1, remember this is x 1 i, x 1 i x 2 n minus i W N k n summation n equal to 0 to N minus 1.

That is basically your or the DFT create, the DFT coefficient of the convolution circular convolution of x 1 n, spherically convolved with x 2 of n all right. And now again as usual we interchange the order of summation. So, that gives us summation i equal to 0 to N minus 1 summation n equal to 0 to capital N minus 1.

We have x, now the term with respect to i that comes outside. So, that is x 1 i summation i equal to 0 to N minus 1 x 2 n minus i W N of kn, and remember this is the shifted sequence, DFT of the shifted sequence of the x of x of n minus x 2 of n minus i. So, this is simply W N of k i times x 2 of k.

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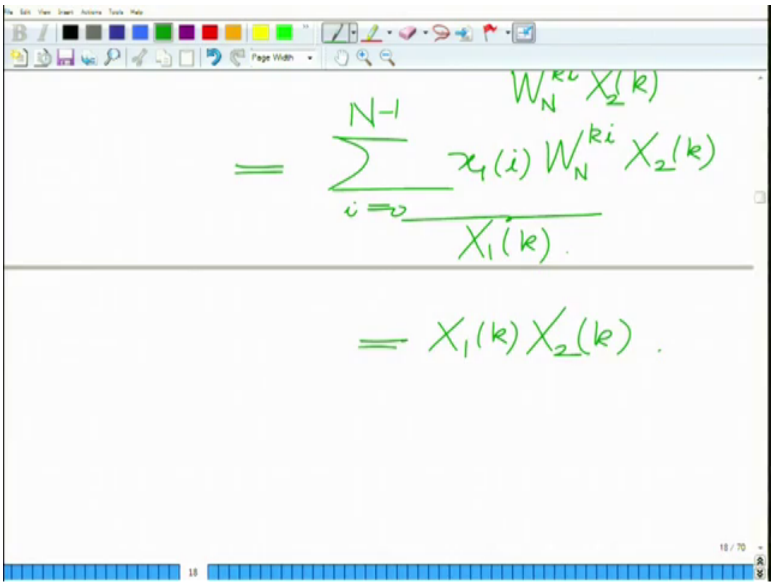

So, now we have this basically reduces to summation i equal to 0 to N minus 1 x 1 of i into W N of k i into X 2 of k and X 2 of k is constant depending only on k and you can see this is nothing, but X 1 of k. So, that reduces to X 1 of k times X 2 of k.

And this is something that is very important. Circular convolution of two finite length sequences right of equal length capital length, when you circularly convolved them the DFT domain they respect, the respective DFT coefficients get multiplied and this is a very important property, it is used by practical wireless communication system.

In fact, most of the four, all of the four g phones that you use are based on the LTI standard which uses OFDM orthogonal frequency division multiplication multiplexing, which is based on this phenomena correct.

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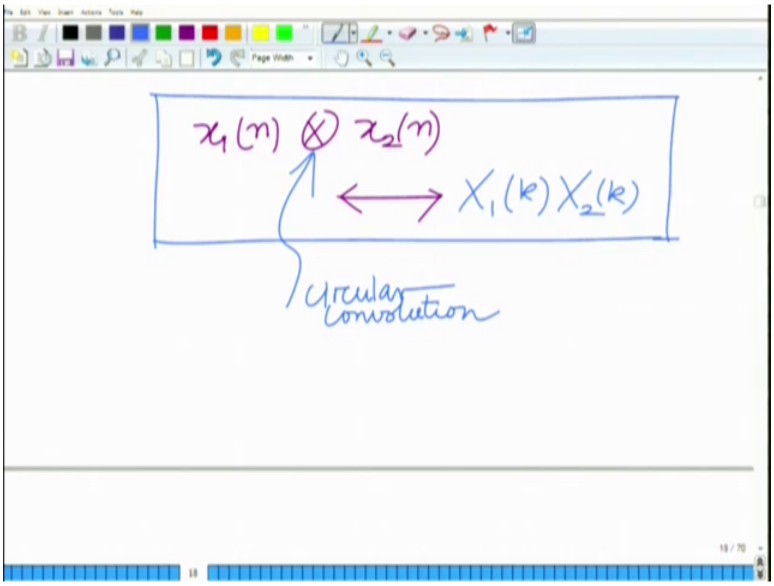

So, we have to finite time domain sequences x 1 n circularly convolved with x 2 of n and that basically has the DFT coefficients X 1 k into multiplication in the DFT domain ok. So, these are the coefficients X 1 k in to X 2 all right; so circular converter.

So, this is, remember this is your circular convolution, just so that you are not confused; one has to use the appropriate sense of convolution correct, for periodic sequence for periodic signals or finite length signals its always circular convolution, for infinite signals its always the regular m the linear convolution all right, and on the other hand when you have multiplication in the time domain again as a dual property multiplication in time, when you have multiplication in time.

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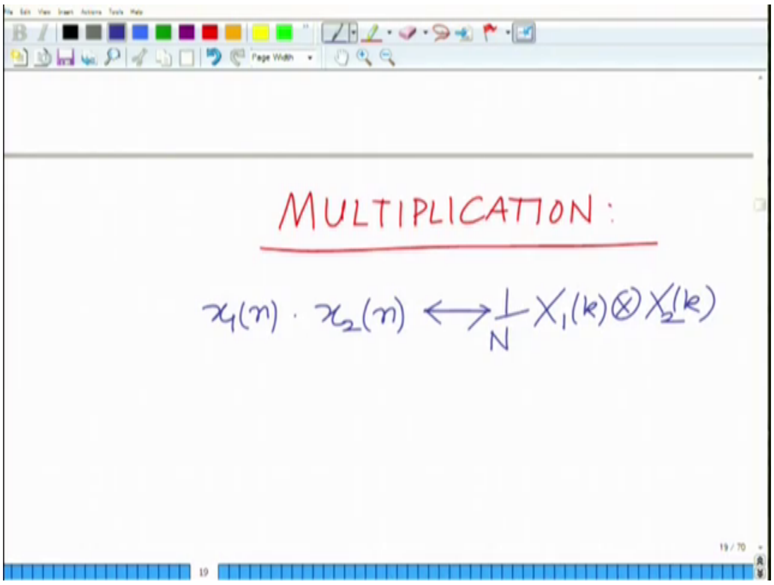

So, x 1 n into x 2 n that has the DFT that is 1 over N and you can show this X 1 k circularly convolved with X 2 k, because remember capital X 1 capital X 2 are also finite length, DFT sequences so its circular convolution between X 1 k and X 2 k and divided by N; that is the rule for multiplication of 2 time domain signals. And now we can derive the Parseval's relation from this.

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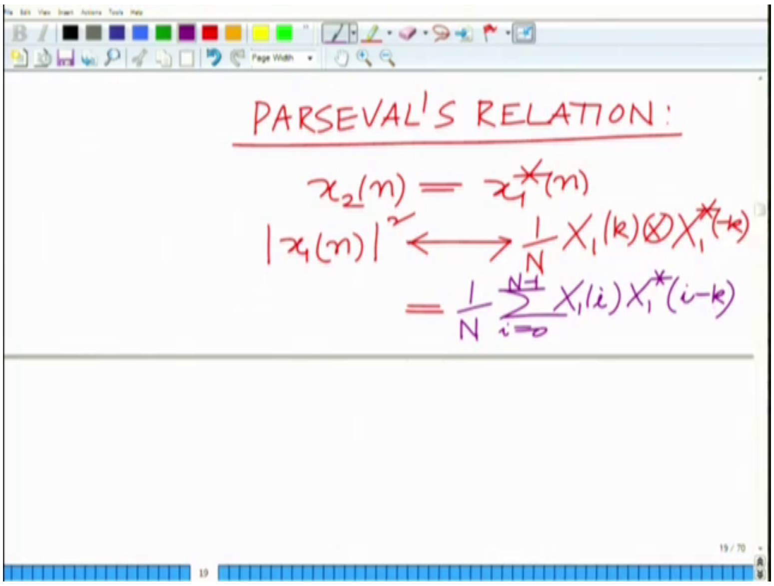

The Parseval's relation, we can derive the Parseval's relation, Parseval's relation we can set x 2 n equals x 1 conjugate n. So, if you multiply x 1 with x 1 conjugate n, then we have magnitude x 1 square that has the Fourier transform 1 over N X 1 X 1 k.

And the Fourier transform of x 1 conjugate of n; that is X 1 conjugate of minus k which is if you look at this; that is going to be summation 1 over i equal to 0 to N minus 1 X 1 i X 1 conjugate of minus k minus i which means X 1 conjugate of i minus k; of course, the i minus k is modulo of N.

So, this is basically the Fourier transform of magnitude X 1 square, which means if you look at summation of that k-th DFT coefficient, summation of n equal to 0 to N minus 1; that is if you look at the k-th DFT coefficient of this magnitude x 1s n square W N to the k n.

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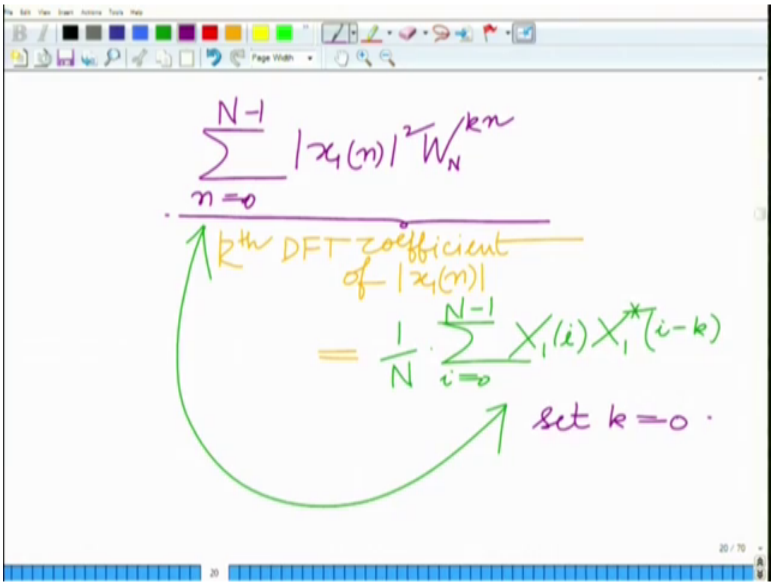

This is the k-th DFT coefficient of magnitude x 1 n square. Remember this is the k-th DFT coefficient. And this is equal to remember 1 over N summation i equal to 0 to N minus 1 X 1 X 1 i X 1 conjugate of i minus k. Now in this both sides what we do now is you set k equal to 0. When you set k equal to 0 what I have is this reduces to W N raise to 0 which is 1.

So, on the left hand side you have summation n equal to 0 to capital N minus 1 magnitude xn x 1 n square. On the right hand side you have 1 over N summation i equal to 0 to N minus 1 X 1 i into X 1 conjugate i minus k, but k is 0. So, X 1 i into X 1 conjugate i, which is magnitude X i square summation i equal to 0 to N minus 1 divided by N. So, that is your Parseval's relation.

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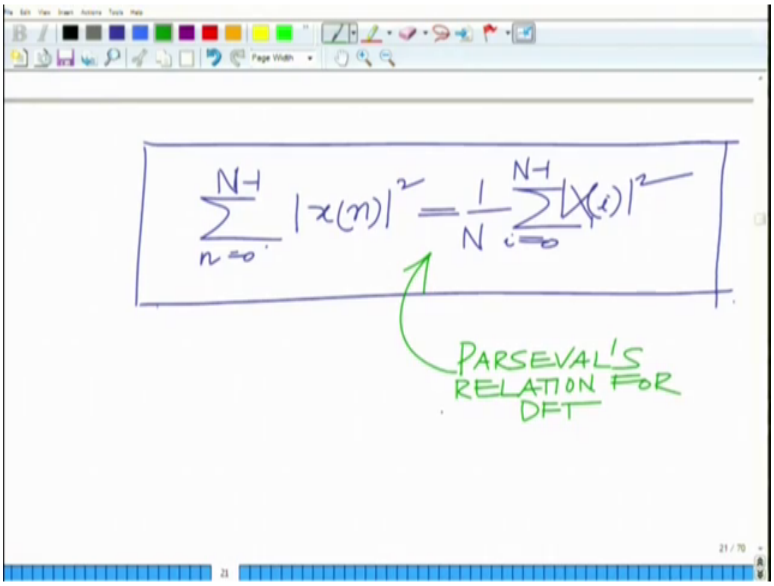

So, set k equal to 0 that gives you summation, that gives us something very interesting, just a very handy relation that gives you summation n equal to 0 to capital N minus 1 magnitude xn square its 1 over N summation i equal to 0 to N minus 1 magnitude X 1 i square and this is the Parseval's relation for, this is the Parseval's relation for DFT.

This is a Parseval's relation for DFT discrete Fourier transform all right, and that basically completes our discussion for the DFT that is the discrete Fourier transform. All right. So, what we can do now is, we can start looking at some example problems for the Fourier analysis of discrete time

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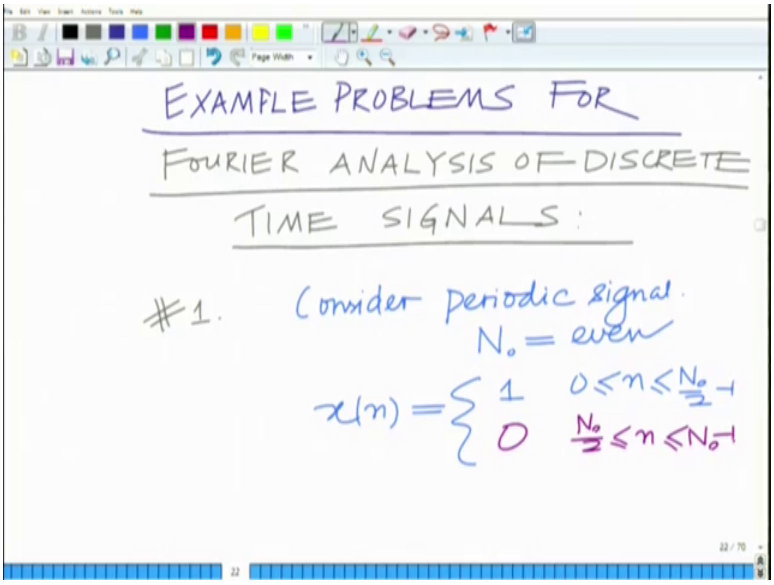

So, we have example problems. In fact, several example problems for Fourier analysis of discrete time signals. The Fourier analysis of discrete time signals, Let us start with the first problems, first similar to our discussion here, we are going to start with the discrete Fourier series. First we are going to the discrete time Fourier transform and then ultimately the discrete Fourier transform, just the way we have covered this in the various modules.

Let us look at our first problem that is considered a periodic signal period N naught equal to 0, which is even period N naught is even. So, xn equals 1, in a single period is given as xn equal to 1 for 0 less than equal to n less than or equal to n naught over 2 minus 1, and it is 0 for N naught by 2 less than equal to n less than or equal to N naught minus 1.

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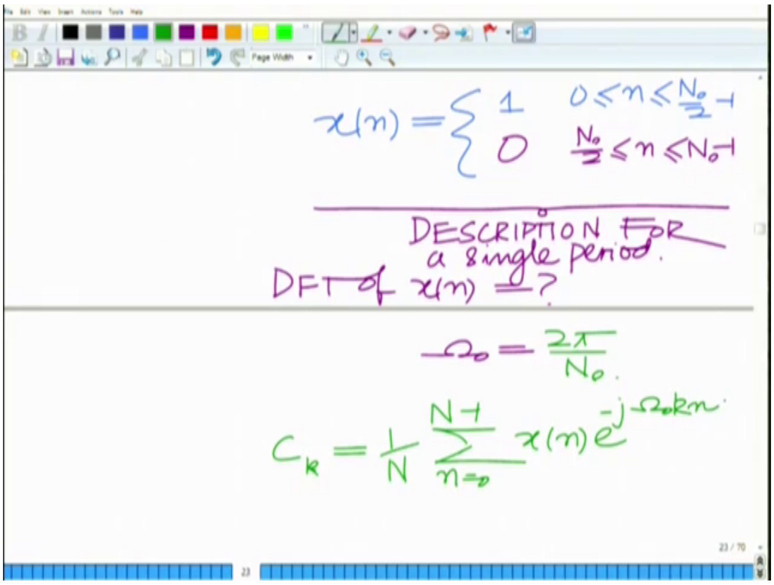

So, this is basically description of it in a single period and then it repeats ok. This is for 1 period a naught and then again it repeats ok, description for a single period, this is the description for a single period. Now what is that discrete DF, the DFS, what is that discrete Fourier series coefficients of xn. Now we know.

That n naught is the period. So, the fundamental frequency omega naught can be defined as 2 pi by N naught and the DFS coefficient is Ck equals summation 1 over. Remember the DFS coefficients are given as 1 over N summation n equal to 0 to capital N minus 1 to xn e raise to minus j omega naught kn equals 1 over N naught summation n equal to 0 to.

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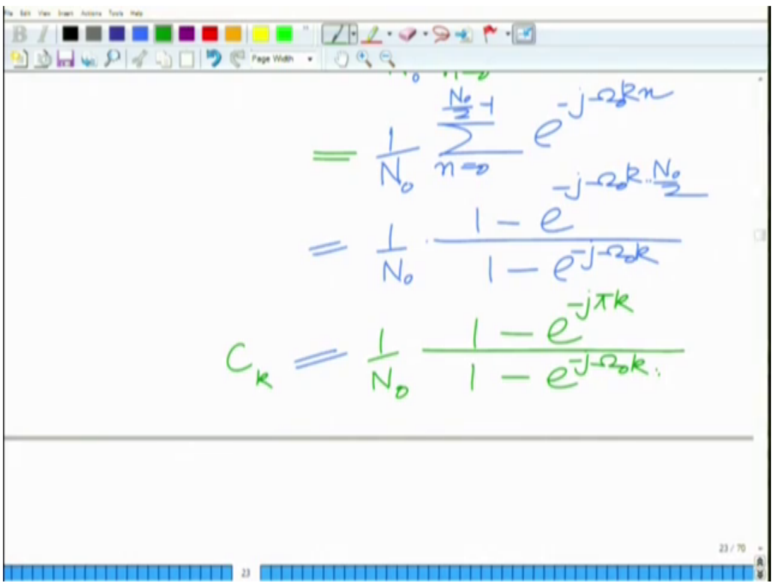

Well, now it is only 1 from 0 to N naught by 2 minus 1. So, N naught by 2 minus 1 xn is 1 e raise to minus j omega naught kn, which is basically 1 over N naught 1 minus e raise to minus j omega naught k e raise to minus j omega naught k into N naught by 2 into over 1 over 1 minus e raise to minus j omega naught k.

And now here in the numerator you can see omega naught equals 2 pi by N naught. So, omega naught N naught over 2 is basically simply pi. So, in the numerator I have basically, this is your Ck, left Ck right as 1 over the right hand side is 1 over N naught times 1 minus e raise to minus j omega naught N naught over 2 is pi 1 minus 1 1 minus e raise to minus j pi k 1 minus e raise to minus j omega naught k.

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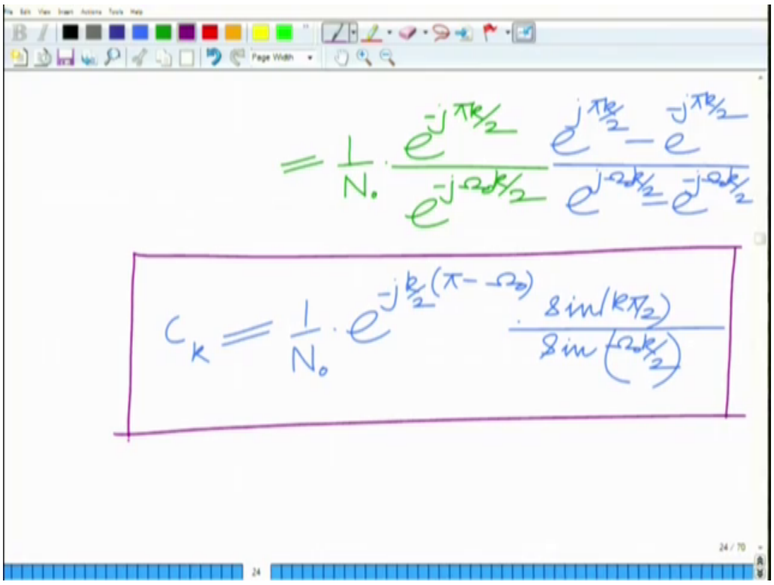

Which is basically 1 over N naught e raise to minus j pi k divided by 2 e raise to minus j omega naught k divided by 2 e raise to minus j pi k over 2 minus e raise to.

I am sorry e raise 2 j pi k over 2 minus e raise to minus j pi k over 2, denominator i raised to minus j omega naught k over 2 or 1 e raise to 0 omega naught 0 2 minus e raise to minus j omega naught k over 2.

Which is basically now you can simplified the this as 1 over N naught e raise to minus j k over 2 times pi minus omega naught times sin k pi over 2 divided by sin omega naught k over; that is our expression for.

So, this is your expression for Ck; that is the DFS discrete Fourier series coefficients original periodic signal xn oh all right.

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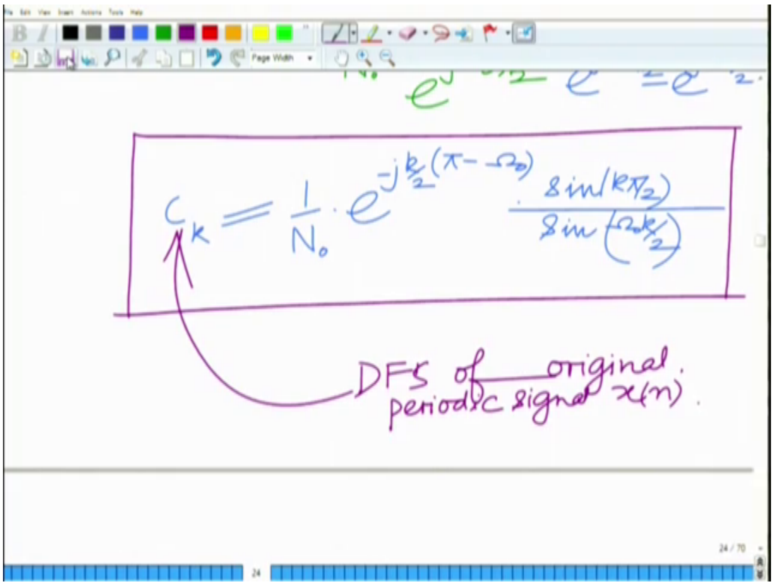

So, we have basically what you can see or basically what we have done in this module, is we have completed or discussion of the DFT, the various properties of the DFT as, such as the duality the convolution, the circular convolution and also the Parseval's relation followed by an example for the discrete Fourier series.

So, we will stop here and look at other examples in the subsequent modules.

Thank you very much.