Principles of Signals and Systems
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Lecture - 60
Discrete Fourier Transform-Definition, Inverse DFT, Relation between DFT and DFS, Relation between DFT and DTFT, Properties-Linearity, Time Shifting

Hello, welcome to another module in this massive open online course. So, we are looking at the Fourier analysis of discrete time signals. We have completed our discussion of the discrete time Fourier transform and in this module we will start a different concept that is the discrete Fourier transform ok.

So, we want to start looking at that discrete Fourier transform.

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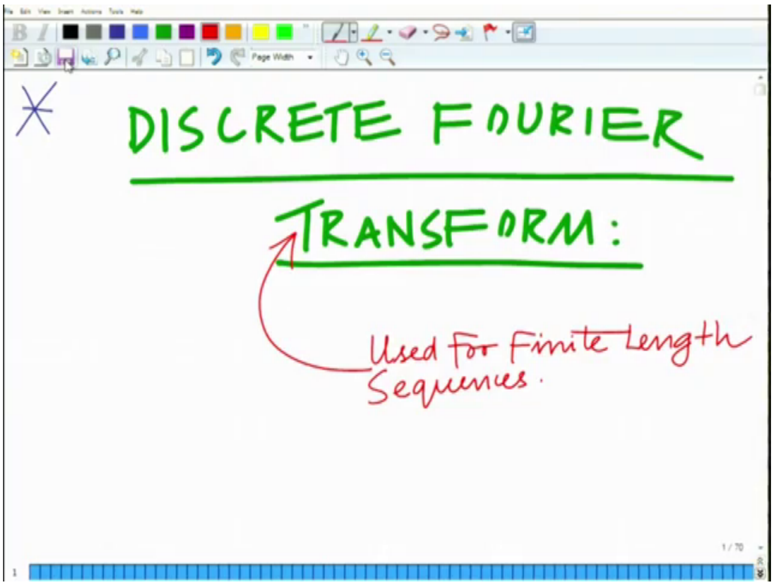

So, this is the Fourier analysis of discrete time signals and in particular of a very important transform for discrete signals is the discrete Fourier transform, which has a revolutionized digital signal processing because there is a very fast way to implement the discrete Fourier transform that is the DFT through the FFT that is the fast Fourier transform routine ok. And that has resulted in a revolution and signal processing cases. One of the most important kinds of transport transforms especially since much of the processing happens in the digital domain. So, it has discrete time signals which are being processed by the aid of the DFT ok.

So, this is used for finite length sequences. As I said it is a very key transform is used heavily in images audio and video processing and for that matter also for communication applications especially wireless communications applications and so on. So, it is used for finite length sequences as I already said correct and the definition is x n consider a sequence x n and this is a finite length sequence 0 less than equal to defined only for 0 less than equal to n less than or equal to N minus 1.

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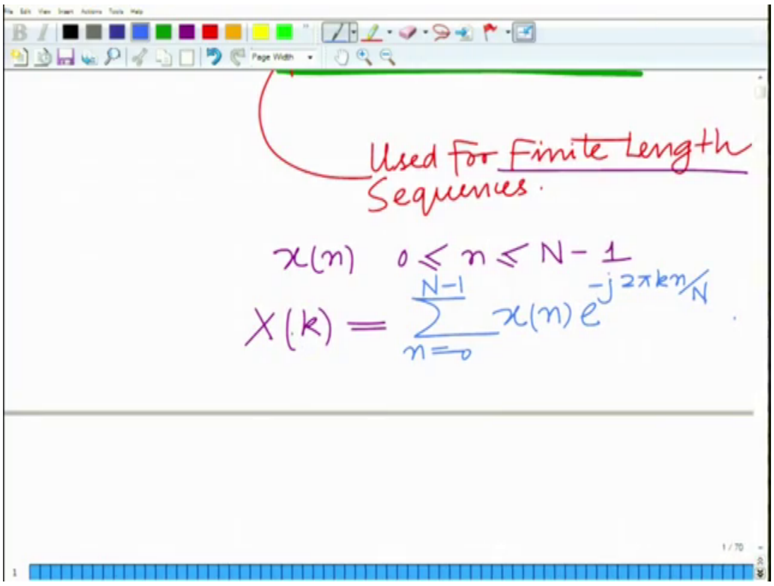

And X k the DFT the k-th DFT coefficient is defined as summation n equal to 0 equal to 0 to N minus 1 x n e raised to minus j 2 pi k n over N that is the expression for the DFT coefficient X k and this is defined for this is also a finite length sequence this is defined as for 0 less than equal to k less than or equal to N minus 1. So, the limits of k are 0 less than equal to k less than equal to N minus 1 ok.

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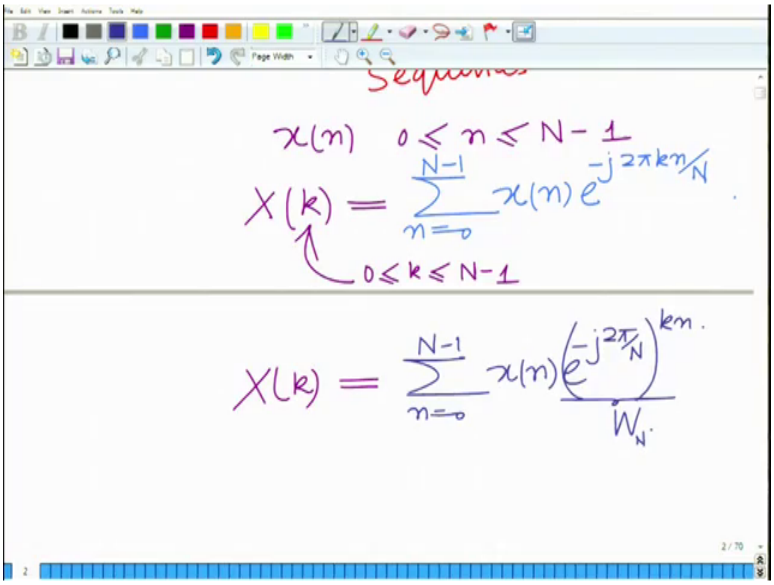

And this can also be written as follows this can also be written as X k equals summation n equal to 0 to N minus 1 x n e raised to minus j 2 pi over N raised to the power of k n.

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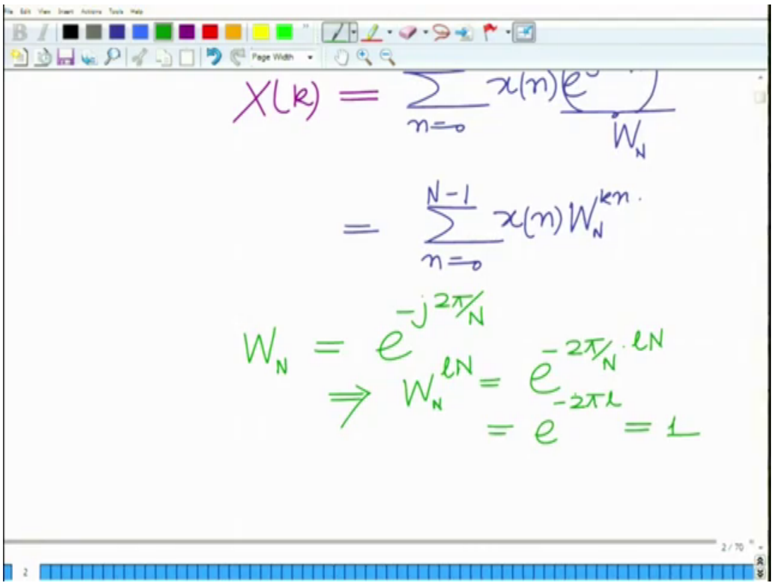

Now, if I call this denote this as W N then I can write this in a very succinct fashion as n equal to 0 to N minus 1 and this now, I can replace by W N to the power k n ok, where W N note that W N equals this is a fundamental quantity e raised to minus j 2 pi n. And also note that W N to the power of any integer k or any integer that is any let us say l times n equals e to the power of minus j 2 pi over N into l N e raised to minus j 2 pi l which is equal to 1 ok.

So, this is in fact, one of the roots of unity that is what you can see ok. So, W N is e raised to minus j 2 pi over N and the DFT coefficient k is summation n equal to 0 to N minus 1 x n W N raised to the power k m ok. So, that gives an expression for the DFT coefficient X k ok.

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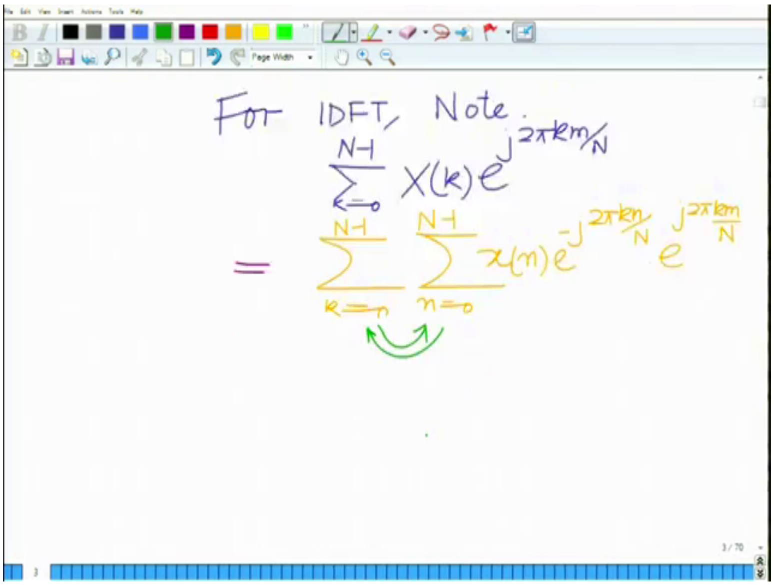

Now, for the IDFT note that, for IDFT note if you perform summation k equal to 0 to N minus 1 X k e raised to j 2 pi k m over N that is equal to now, substituting the expression for X k that is equal to summation k equal to 0 k equal to 0 to N minus 1 summation n equal to 0 to N minus 1 x n e raised to minus j 2 pi k n over N times e raised to j 2 pi k m over N and now, interchanging the order of summation ok.

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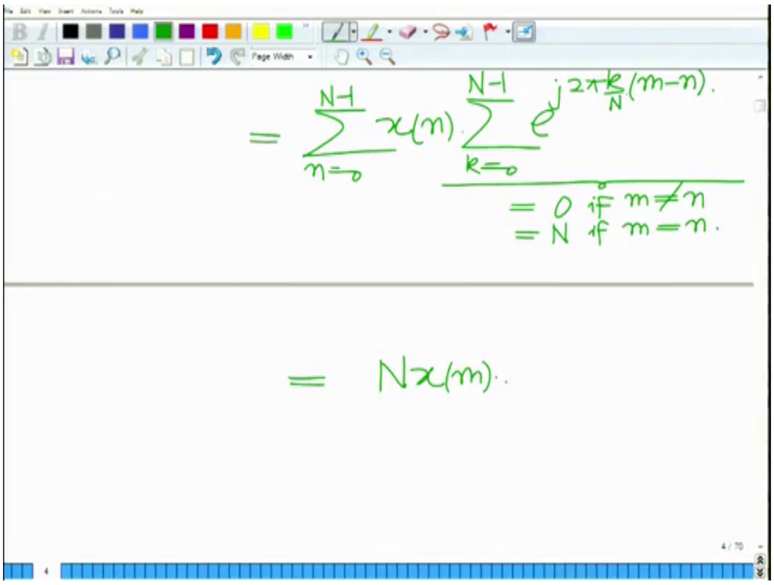

So, interchanging the order of summation we will have, this will be summation n equal to 0 to N minus 1 x n depends on only on n. So, that comes out summation k equal to 0 to N minus 1 e raised to j 2 pi k over N into m n. And now, you can see this is equal to 0 this is basically the sum of the roots of unity is equal to 0 if m is not equal to n and this is equal to this in fact, equal to n if m equal to n ok. And therefore, this is equal to only the term n corresponding to m survives. So, this is equal to Nx m.

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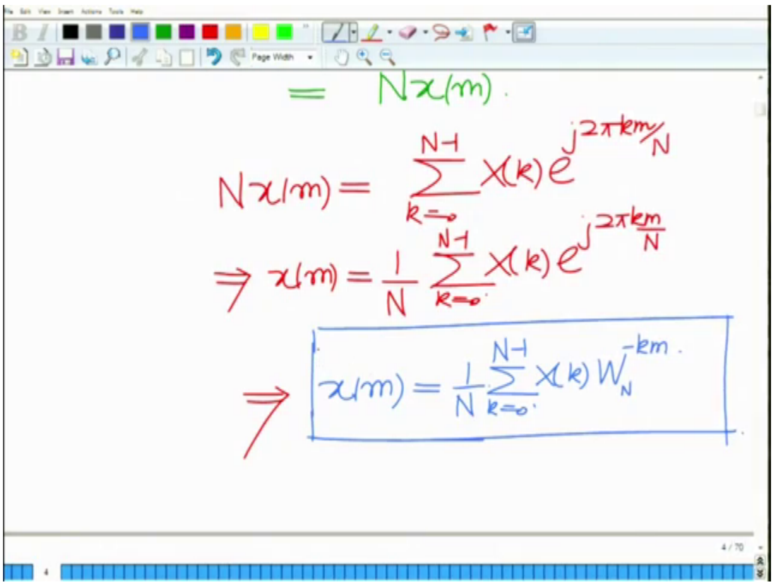

So, we have from this manipulation what we have seen is basically that Nx m equals summation k equal to 0 to N minus 1 X k e raised to j 2 pi k m, divided by m which implies that x m equals 1 over N summation k equal to 0 N minus 1 X k e raised to j 2 pi k m over N and in fact, e raised to j 2 pi over N is W N raised to minus.

So, this implies basically you have x m equals summation equals 1 over N summation k equal to 0 to N minus 1 X k W N to the raised to minus k m ok. So, this is the expression for the IDFT inverse discrete Fourier transform. This is your inverse; this is the expression for your inverse discrete Fourier transform ok.

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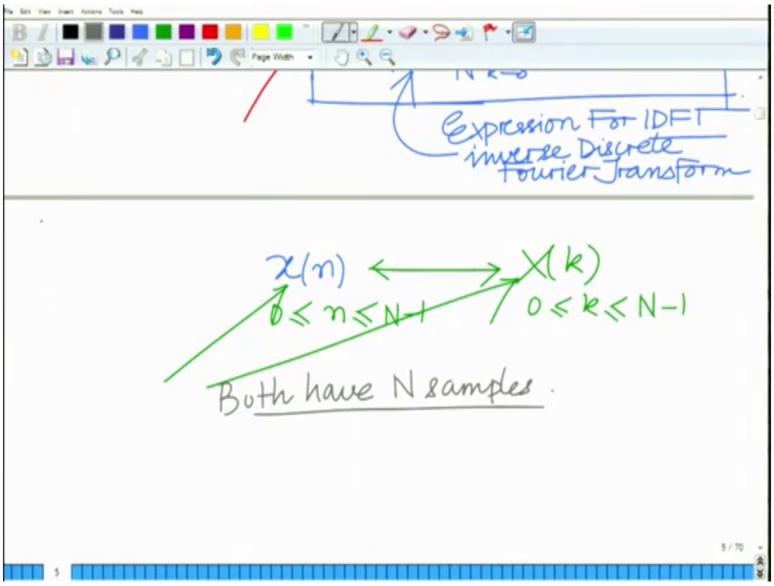

And this is of course, represented by similar to what we have seen many times before you have x n plus x x n and x a for my DFT pair x n in the range 0 to capital N minus 1 k also in the range 0 to capital N minus 1 ok, so these for my DFT pair. So, both have n samples ok. So, observe that both have both the both the original samples and their DFT coefficients are n in number ok, all right.

Now, the next thing we want to look at is we want to look at the relations between these various transforms correct. So, we have seen several transforms we are now, seeing the DFT we have earlier we have seen the Fourier series as well as the discrete time Fourier transform. So, want to look at start by looking at the relation between this DFT and the discrete time Fourier series that is defined for a periodic signal periodic discrete time signal ok.

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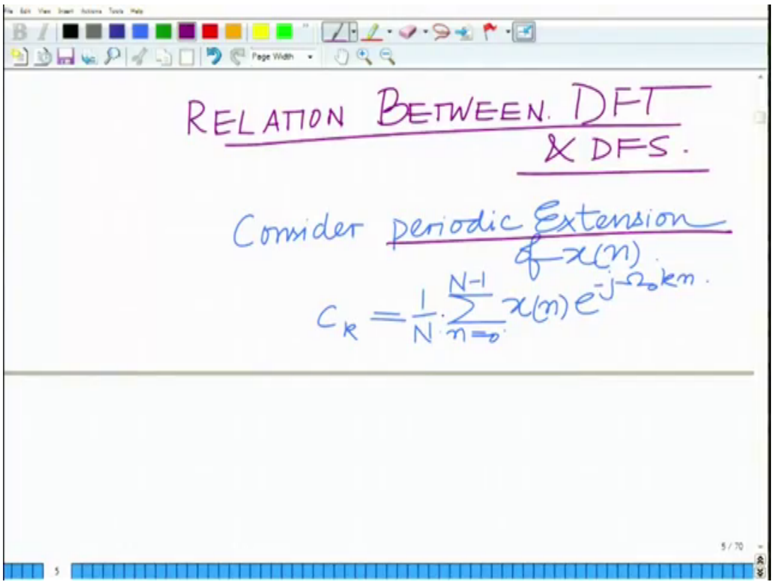

So, what we want to look at is basically relation, we want to look at the relation between DFT and DFS ok. So, consider the periodic extension of x n consider the periodic extension of x n and then we have C k equals 1 over N summation n equal to 0 to N minus 1 x n e raised to minus j omega naught k n ok. So, remember C k this is the, so we considering a periodic extension of the signal x n which is defined for 0 less than n less than equal to capital N minus 1.

So, we are extending it periodically which means we are repeating the same sequence x n periodically all right. So, this becomes a periodic sequence. And then we can one can consider the DFS of this periodic extension of x n and let us C k denote its DFS coefficient then you can see this implies that N C k equals you can readily see that N C k equals n equals 0 to N minus 1 x n e raised to minus j omega naught k n.

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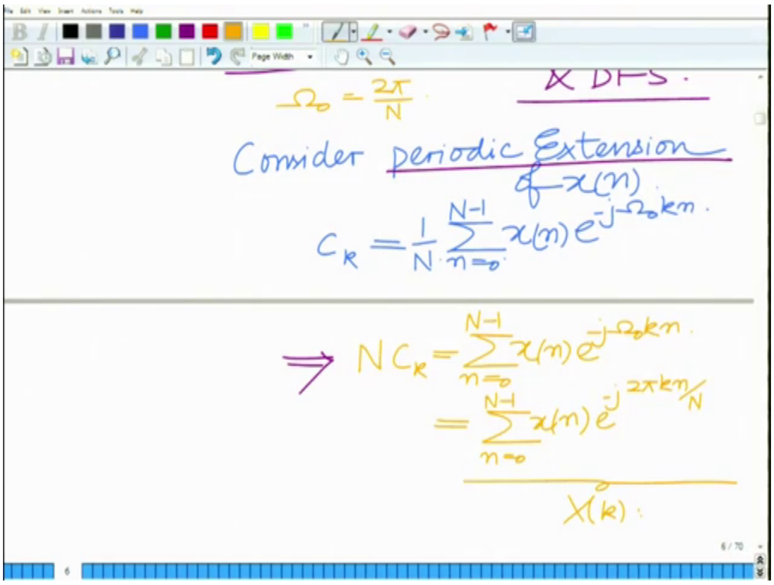

In fact, we have omega naught equals 2 pi by n. So, this becomes summation n equal to 0 to N minus 1 x n e raised to minus j 2 pi k n over N which you can see is nothing, but basically the DFT coefficient x of k.

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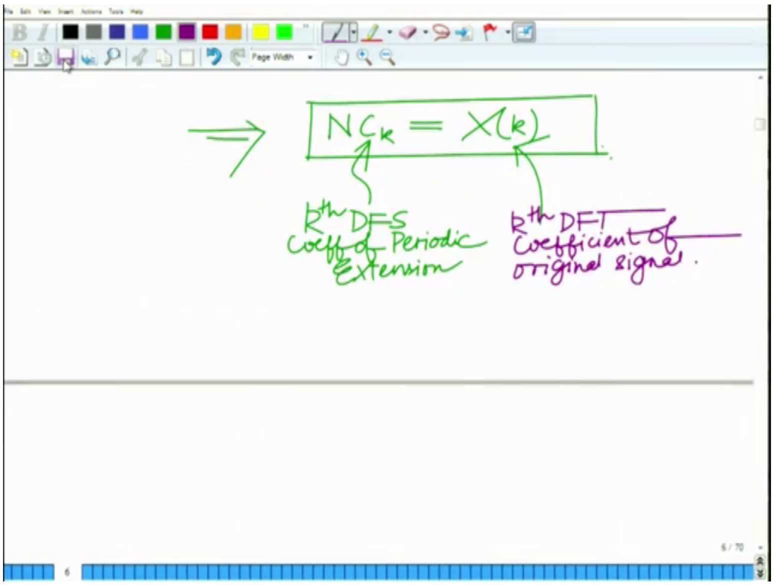

So, this implies basically that the DFS coefficient, so if C k denotes the coefficient the k th discrete Fourier series coefficient of the periodic extension then N times C k equals X k ok. And note once again that this is the kth this is the kth DFS coefficient of the, of the periodic extension and this is the kth DFT coefficient of the original signal, the original time limited signal with capital n samples ok.

And the other thing that you can see is the relation between the DFT and the DTFT that is a relation between the discrete Fourier transform and the discrete time Fourier transform and that is also fairly simple.

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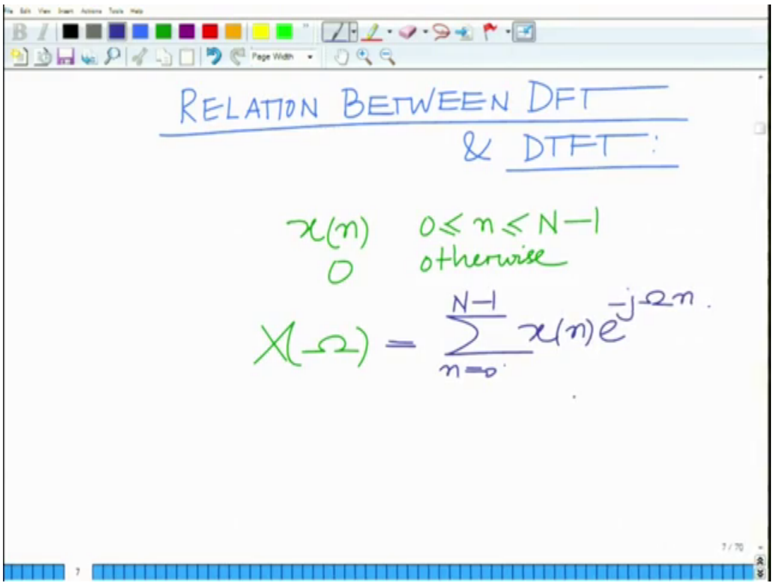

So, the next thing that we want to see is the relation between the DFT and relation between the DFT and DTFT ok. And now, consider x n equals now, you form a signal such that x n equal to that is a consider x n for 0 consider a signal x n which is 0 less than equal to for x n for 0 less than equal to n less than equal to N minus 1 and 0 otherwise ok, and for this signal.

If you look at the DTFT now, the DTFT will naturally be x of omega equals sum of because this is 0 outside n equal to 0 to N minus 1. So, this will be n equal to 0 to N minus 1 x n e raised to minus j omega n.

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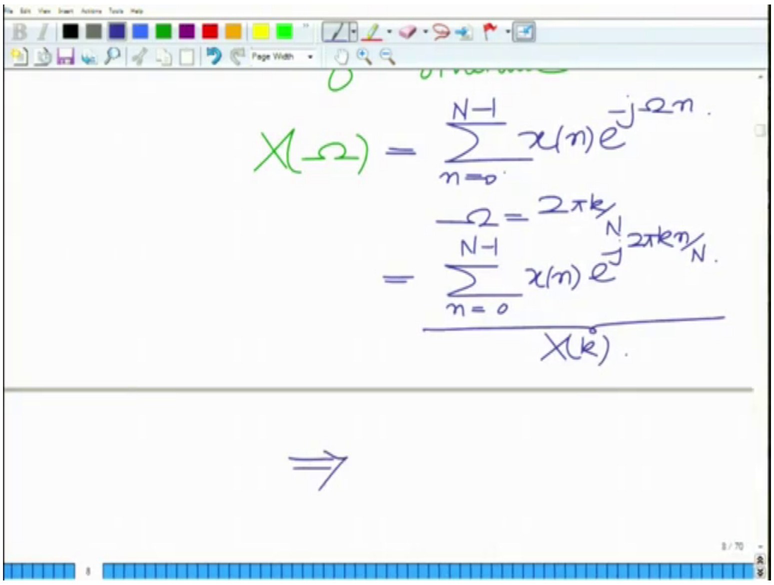

Now, if you substitute omega equal to 2 pi k over N over N this will become summation n equal to 0 to N minus 1 x n e raised to minus j 2 pi k n over N and which is nothing, but X k. Now, you can see this is nothing, but the DFT coefficient X k.

So, this implies that your X k this implies something very interesting that your X k is nothing, but the DTFT of x omega evaluated at omega equals 2 pi k over alright.

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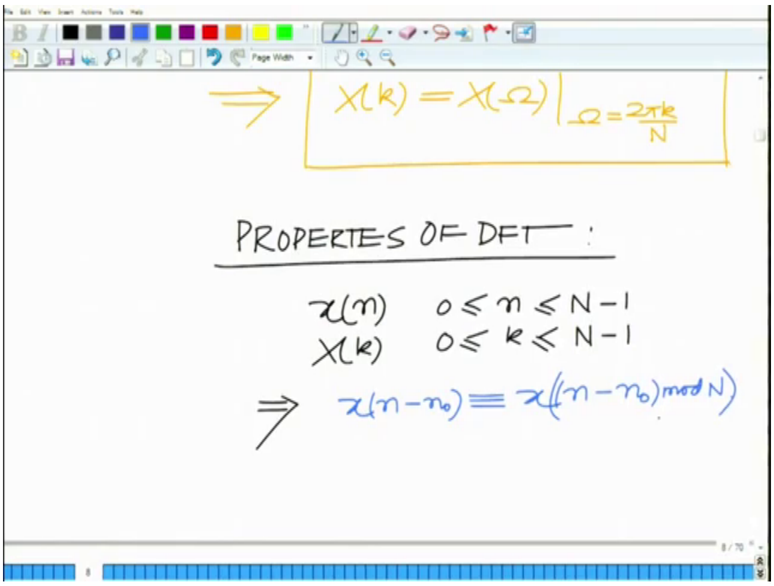

So, that is basically that basically summarizes the DFT the DFT definition the DFT its relation to the discrete Fourier series as well as the DTFT the discrete time Fourier transform. Let us now, look at some properties of the DFT. So, want to look at the properties of the DFT.

Now, we have x n 0 less than equal to n less minus 1 and X k 0 less than equal to k less than equal to N minus 1 n comma k are restricted. Now, of course, when we consider x n minus n naught, now it might go outside the range of 0 or N minus 1. So, what one can do is all such things can be restricted to module of n in this scenario ok. So, what we mean when we me when we say x n minus n naught in this case is that the n minus n naught has to be interpreted modulo n all right, where n is the total number of samples for instance.

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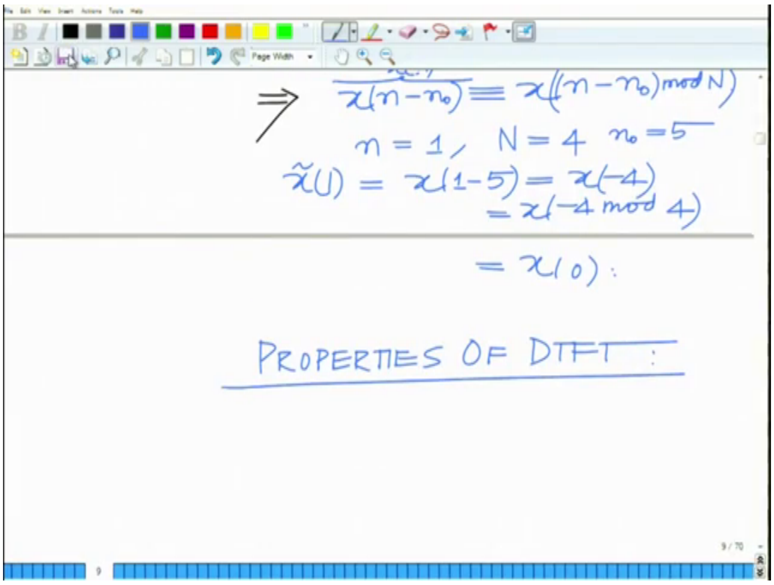

Let us say we have n equal to 1 and capital N equal to 4. So, we have x tilde of one because let us say we call this as x of n minus n naught as x tilde ok. So, we have x tilde of 1, is x of 1 minus 4 I am sorry n equals 5. Let us say n equals or n naught equals 5 n equals 4 n naught equals 5 which is x of N minus 1 minus 5 equals x of minus 4 this is x of minus 4 modulo 4 which is equal to 0 ok.

So, this is basically your x of 0. So, x x tilde of 1 where x is shifted by n naught equals 5. So, x tilde 1 that is the first sample of the shifted signal is actually 0. So, basically you are sort of circularly shifting this alright the shifting is basically modulo, modulo 4 where n of 4 is the n is equal to 4 is the total length of the sequence all right.

And now, we look at the properties of the DTFT. And the properties of the DTFT are as follows again the standard property we start with linearity which is simple yet fairly powerful and useful properties.

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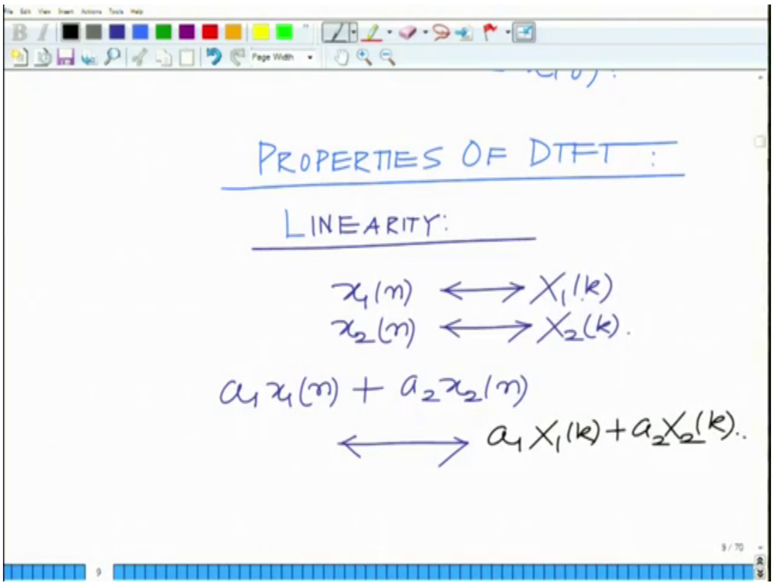

So, linearity tells us that if x 1 n has the DFT coefficients x 1 k and x 2 n have the DFT DFT coefficients x 2 k and both have the same length obviously, 0 less than equal to n less than equal to capital N minus 1 then a 1 x 1 n plus a 2 x 2 n has the DTFT a 1 x 1 k plus a 2 x 2 k all right. So, this is the linearity which we have seen many times before.

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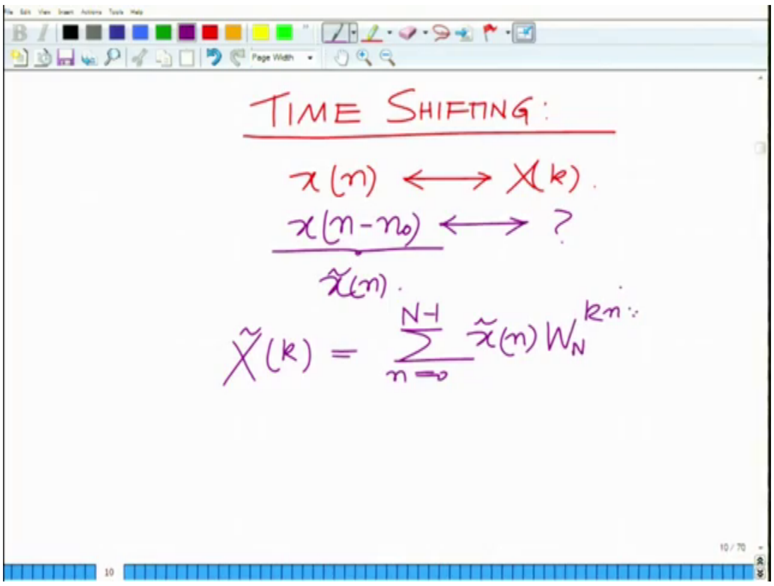

Now, what about time shifting? x n has the DTFT x of k what can we say about x of n minus n. What can we say about this quantity x of n minus n naught? Now, let n minus n naught mod n equals m, all right. So, what we have is so let n minus. So, we have basically let us denote this by x tilde or let us denote this by x tilde n. So, X tilde k summation n equal to 0 to N minus 1 x tilde n e raised to minus j or basically you can just write it in terms of W N, W N to the power k n.

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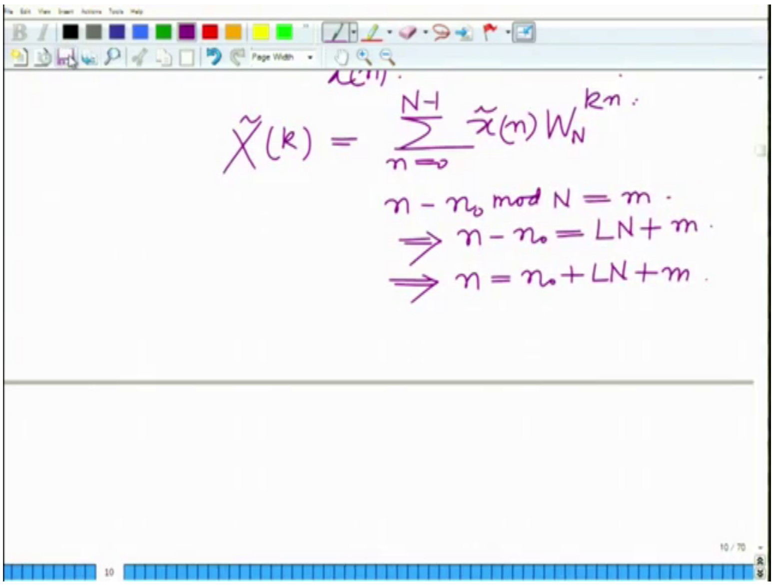

Now, let n naught minus n or n minus n naught mod N equals m this implies n equals n plus. So, with this implies n naught minus n minus n naught is basically or n minus n naught is basically sum is basically some constant L times N plus m ok. So, this is some multiple of this is some n minus n naught is some multiple of n plus m ok. So, which implies n equals n naught plus LN plus m.

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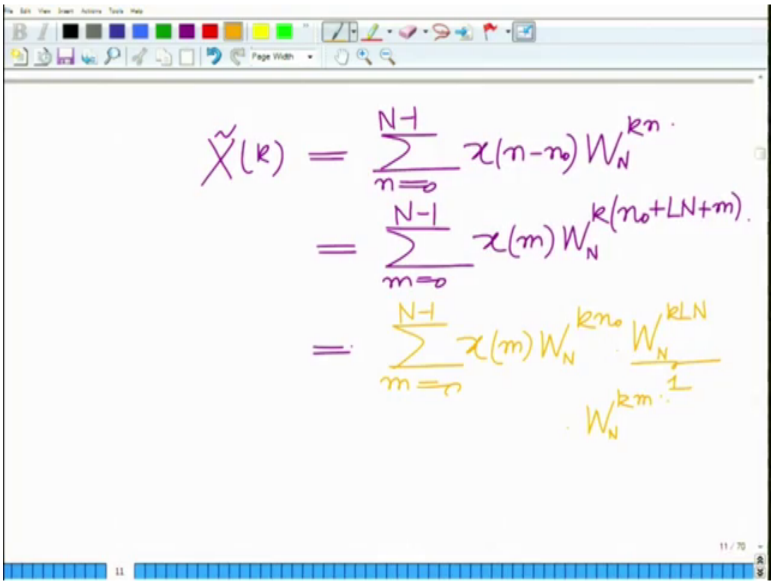

So, we have X tilde k, again X tilde k now, if you write X tilde k, if you write X tilde k this will be summation n equal to 0 to N minus 1 x tilde n or basically x of n minus n naught x of n minus n naught, remember x of n minus n naught W N to the power of k n.

Now, remember x of n minus n naught has to be interpreted as n minus n naught modulo n which is m. So, this will be summation again m also goes from 0 to N minus 1 x of m W N to the power n raised to the power of k and n is a naught plus LN plus m and this is equal to this is equal to summation m equal to 0 to N minus 1 x m W N to the power of k n naught into W N to the power of k LN which is 1 into W N to the power of k m ok. And now, W N to the power of k n naught comes outside because it does not depend on m. So, that is W N to the power of k n naught summation m equal to 0 to N minus 1 x m W N to the power of k m which is basically this quantity is basically x k.

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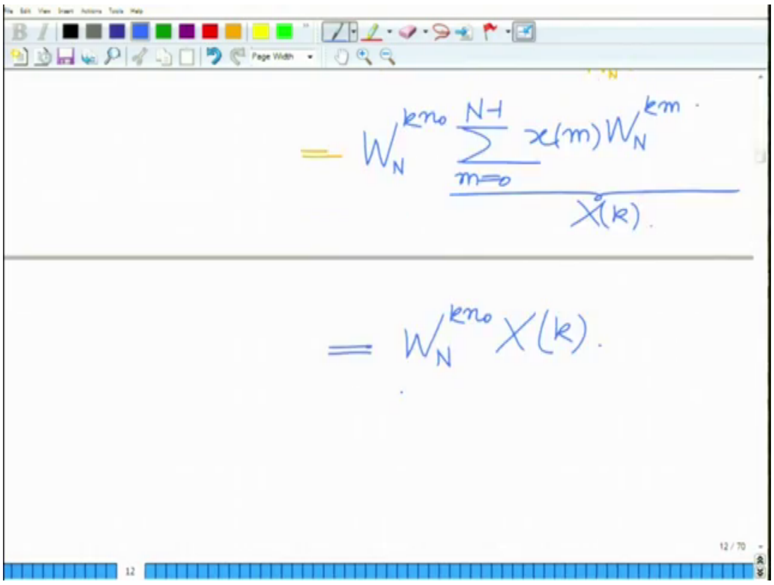

So, this is W time W N times W N times k n naught into x which is similar to basically the modulation property or basically the time shift in time leads to modulation in the frequency domain time shift leads to modulation in the frequency. So, that is basically W N. So, what you have is basically x of n minus n naught has the DFT which is W N to the power of k n naught which is X k ok, all right.

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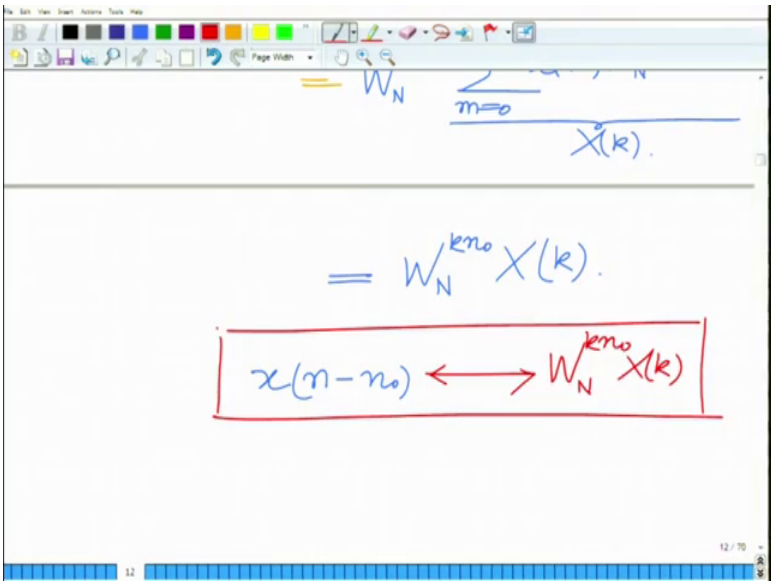

So, that basically covers the time shift property, alright. So, in this module we have introduced the DFT looked at its definitions some of its properties. So, we will stop here and continue in the subsequent modules.

Thank you very much.