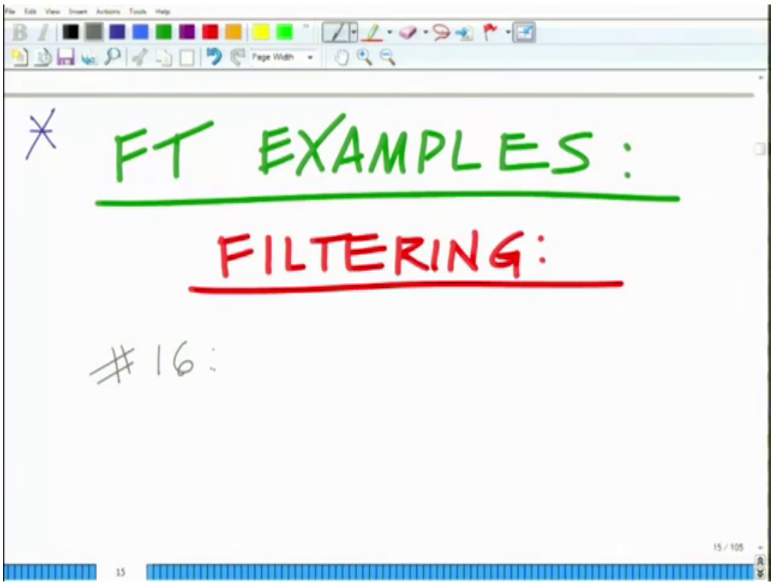
Principles of Signals and Systems
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 50
Fourier Transform Examples: Filtering – Ideal Low Pass Filter

Hello welcome to another module in this massive open online course. So, we are looking at example problems in the Fourier transforms alright. So, let us continue our discussion and the completed discussion on the Hilbert transform let us start looking at another interesting problem that is to understand the process of filtering ok.

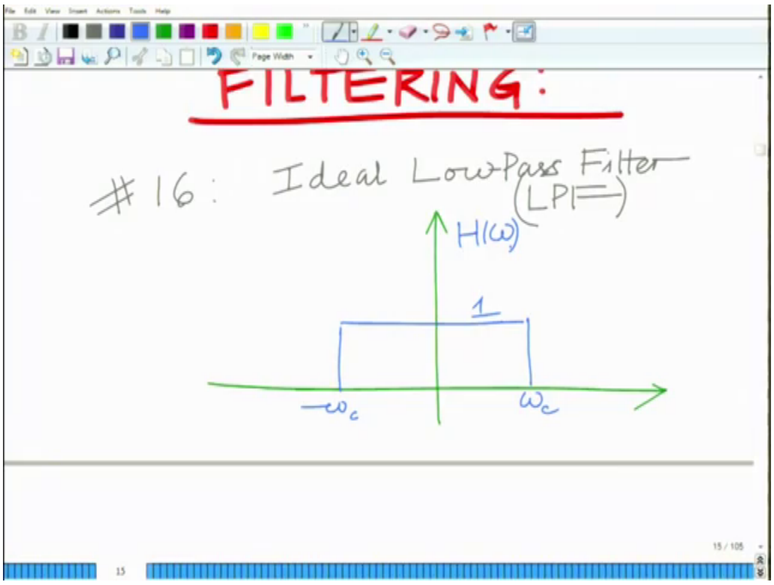
(Refer Slide Time: 00:31)



So, looking at examples or a practical examples for the Fourier transform and in this particular module let us understand the process of filtering.

You will recall that we have discussed ideal filters high pass, low pass, band pass and band stop. So, we were going to look at an application on the theory of filtering ok. So, now, as you might already remember or recalls let us term this as process this is problem number 16.

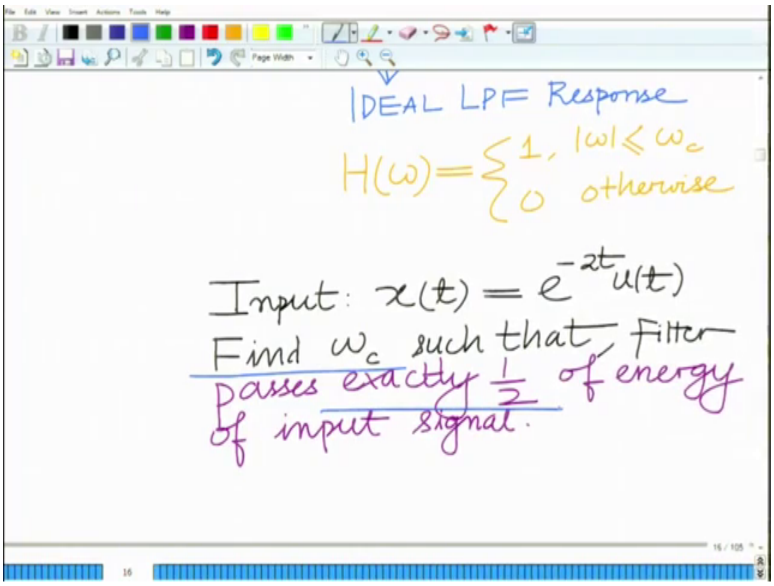
(Refer Slide Time: 01:24)



So, we have an ideal low pass filter. So, consider an ideal low pass filter and in this ideal low pass filter if you might remember the ideal low pass filter the response looks something like this.

So, I have a cut off frequency omega c and the response magnitude of H F equals 1 or we can also call it H of omega this was just magnitude. So, H of omega equals 1 for omega magnitude omega less than omega c and H of omega equal to 0 otherwise this is an ideal low pass filter ok.

(Refer Slide Time: 02:17)

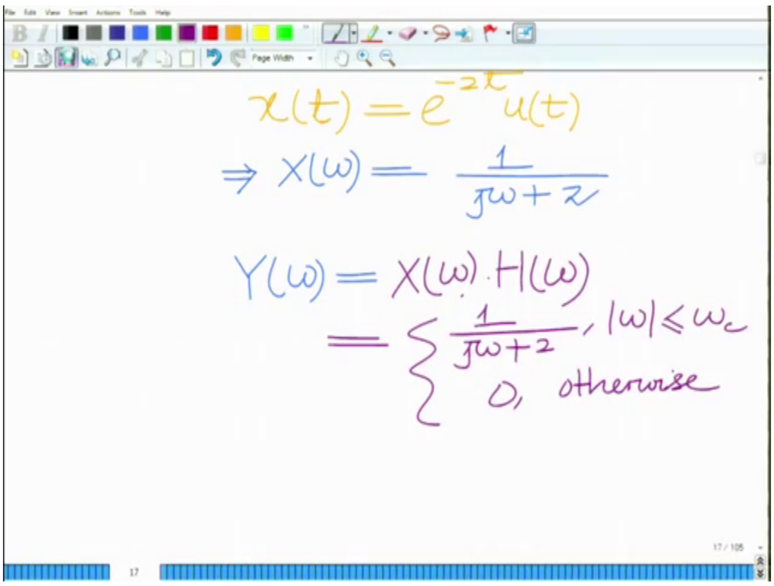


So, this is your ideal LPF response and this can be characterized as H of the response of the LTI system H of omega equals 1 for magnitude of omega less than or equal to omega c and this is 0 otherwise or magnitude of omega greater than omega c this is 0 ok.

Now, what we want to do is for this LPF we want to consider an input signal. So, our input signal to the LPF input equals the input signal equals x of t given as the input is x of t equals e raised to minus twice t u t this is our input signal. Now what we want to do is we want to design a filter. So, this problem is with respect to filter design. So, we want to design a filter which means we want to design or choose a cutoff frequency omega c such that it passes a certain fraction or a certain desired component of the signal.

Now, what is the desired component of the signal? So, what we want to do is we want to find omega c of the low pass filters such that find omega c such that the filter passes exactly half of energy of the of the given input signal ok. So, the point is we want to find omega c such that the LPF that is your ideal LPS passes exactly half or allows exactly there is a energy of the output signal should be half of the energy of the input signal that is e raise to minus 2 t ut ok. So, this is our design problem filter design problem and we can approach this problem as follows.

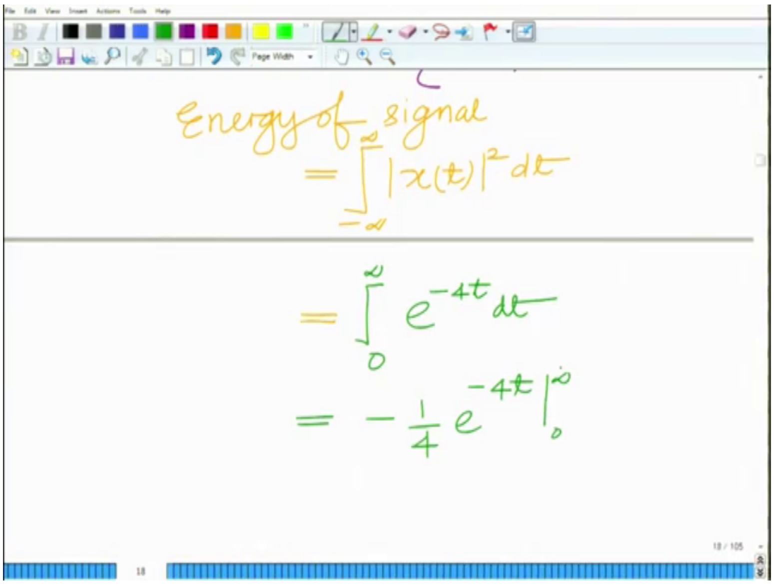
(Refer Slide Time: 05:08)



Now, first let us use the Fourier transform approach. So, first note that we are given this signal x of t equals e raised to minus 2 t ut which implies now taking the Fourier transform no X of omega of this signal is 1 over j omega plus 2. Now, therefore, we know from the theory of the Fourier transform that the output Fourier transform by omega is Fourier transform the input signal X omega times Fourier transform of the filter response that is H of omega. So, y omega is X of omega times H of omega which is now remember H of omega is 1 only for magnitude of omega less than omega c.

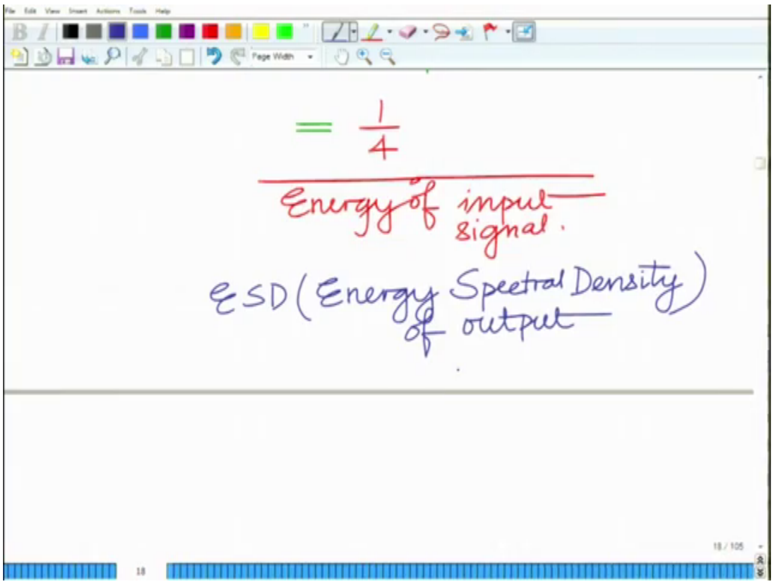
So, it allows only the signal component or the frequency components of the signal in this minus omega to omega c omega seek because the pass through and it suppresses because its response is 0 outside of this band. So, it suppresses all the frequency components which are outside this band. So, what we have is this is basically 1 over j omega over 2 form or omega less than or equal to omega c and this is 0 this is 0.

(Refer Slide Time: 06:45)



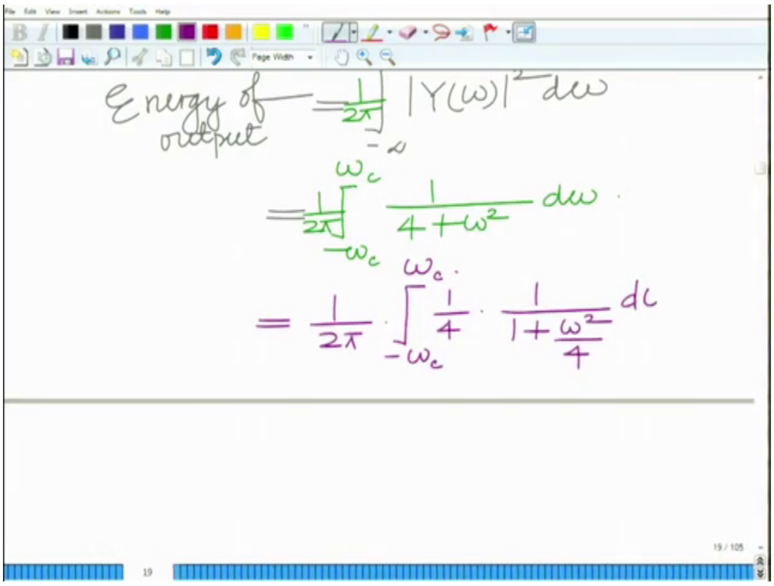
Otherwise now, the energy of the signal this is equal to now first let us find the energy of the input signal that is minus infinity to infinity magnitude x t square minus infinity to infinity magnitude x t square d t which is minus infinity to infinity e raised to or 0 to infinity e raised to minus 4 t d t which is minus one by 4 e raised to minus 4 t between the limits 0 to infinity which evaluates to basically 1 over 4 this is the energy of this is the energy of the input signal.

(Refer Slide Time: 07:42)



Now, we want to lose the concept of what is known as the energy spectral density remember energy spectral density gives us the distribution of energy of the input signal over the frequency spectrum. So, this is a very good example to illustrate the illustrate a practical application of this concept of energy spectral density ok. So, now, what we want to do is we want to consider the ESD over the energy spectral density of output.

(Refer Slide Time: 08:48)

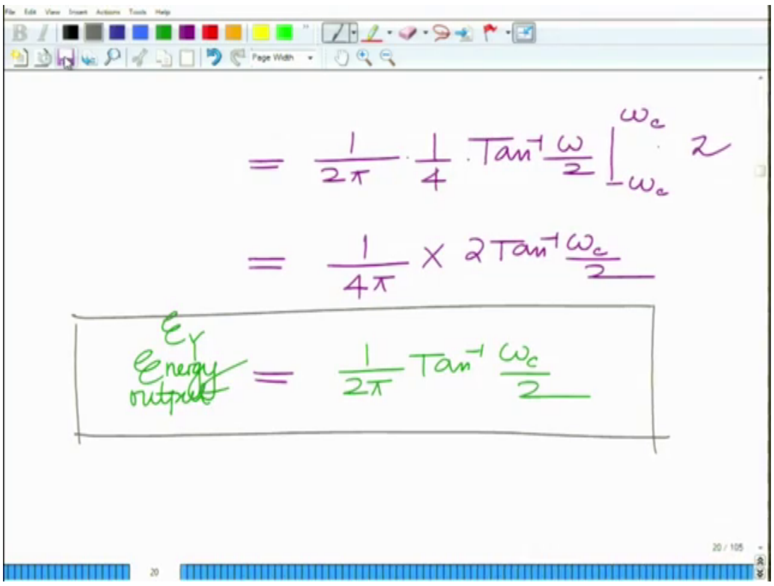


Now, remember the energy spectral density of the output that is magnitude Y of omega square remember we have seen what is Y of omega; we get Y of omega from this expression. So, magnitude Y of omega square is basically 1 over 4 plus omega square for magnitude omega less than or equal to omega c and 0 otherwise. So, this is the energy spectral density of the output signal. So, this is your this is the ESD the energy spectral density of the output signal, this is 1 over 4 plus omega square correct for omega the frequencies that lie in the band omega c minus omega c 2 plus omega c ok.

And therefore, the energy of this output signal is nothing, but the integral of the energy spectral density over the entire frequency band that is from minus infinity to infinity, but. So, to be nonzero from minus omega c to omega c therefore, it is the integral of the energy spectral density from minus omega c to omega c. So, therefore, energy of output now we have a very interesting application of the energy spectral density infinity to infinity magnitude Y of omega squared d of omega which is minus omega c to omega c.

. In fact, this is 1 over 2 pi I have to have a factor of 1 over 2 pi this is a constant factor scaling factor 1 over 2 pi times 1 over 4 plus omega square d omega which is minus omega c to omega c 1 over 4 into 1 plus omega square now I am going to simplify this integral 1 plus omega square by 4 d omega.

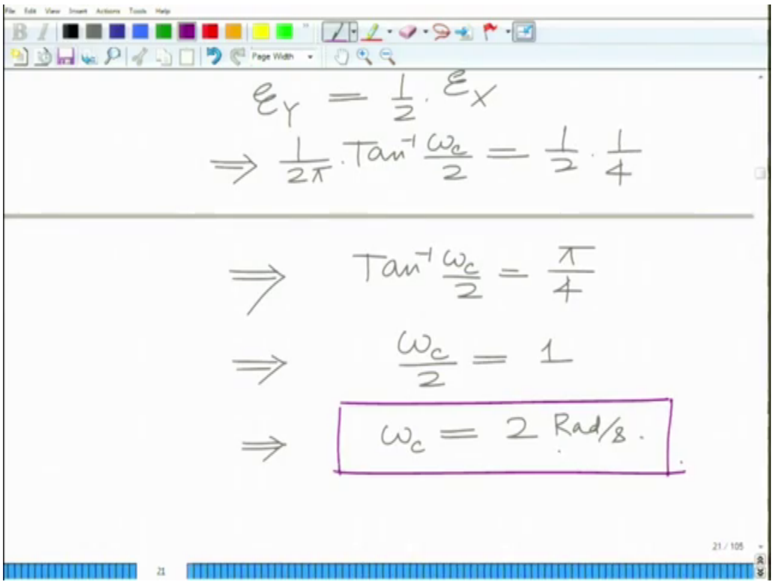
(Refer Slide Time: 11:24)



Now, if you look at this is nothing, but one over 2 pi into 1 over 4 into tan inverse omega over 2 evaluated between the limits minus omega c to omega c times 2. So, this is basically you have 1 over 4 pi tan inverse or 1 over 4 pi 1 over 4 pi into twice tan inverse omega c by 2 which is equal to now you can see this is 1 over 2 pi tan inverse omega by 2. So, this is the energy output.

You can also call this as E Y this is the energy of the this is the energy of the output signal this is 1 over 2 pi tan inverse omega c over 2 this is a function of this is naturally going to be a function of the cutoff frequency omega c of the ideal low pass filter ok. And now all we have to do is basically equate this to half of the input. So, we know that we want to design the filter with a cut off frequency omega c such that the output energy is basically half of the input energy and the input energy is 1 over 4. So, half of the input energy is 1 over 8.

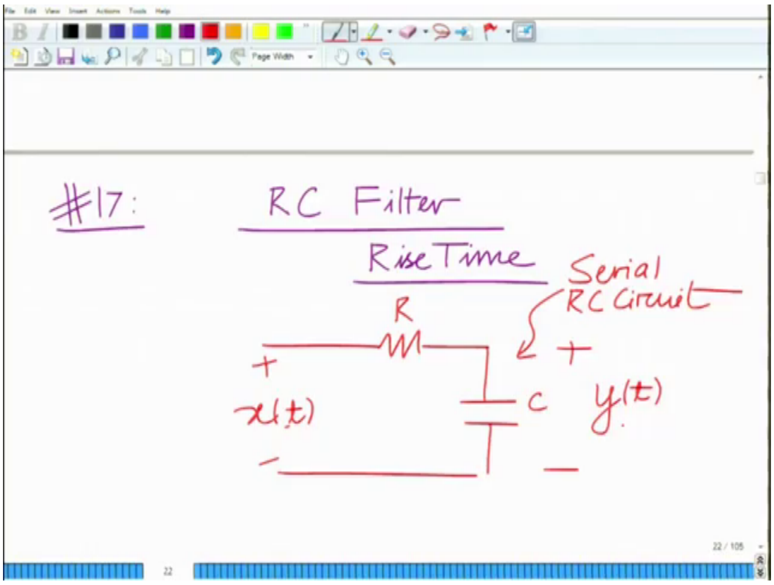
(Refer Slide Time: 13:15)



So, basically what we have is E Y equals half of E X which is input energy which means 1 over 2 pi tan inverse omega c over 2, this is equal to half into 1 by 4 which implies basically tan inverse omega c by 2 equals pi by 4 which means omega c by 2 equals tan of pi by 4 which is 1 which means omega c equals 2 radians. So, that is 2 radians per second omega c equals to radians per second.

So, that is basically. So, basically this gives us a filter. So, this gives us a filter such that input energy alright since the energy of the output signal corresponding to the input signal e 2 e raise to minus 2 t ut there is exponential signal the output energy is basically half the energy of the inputs ok. So, the cutoff frequency omega c is basically 2 radians per second alright.

(Refer Slide Time: 14:42)



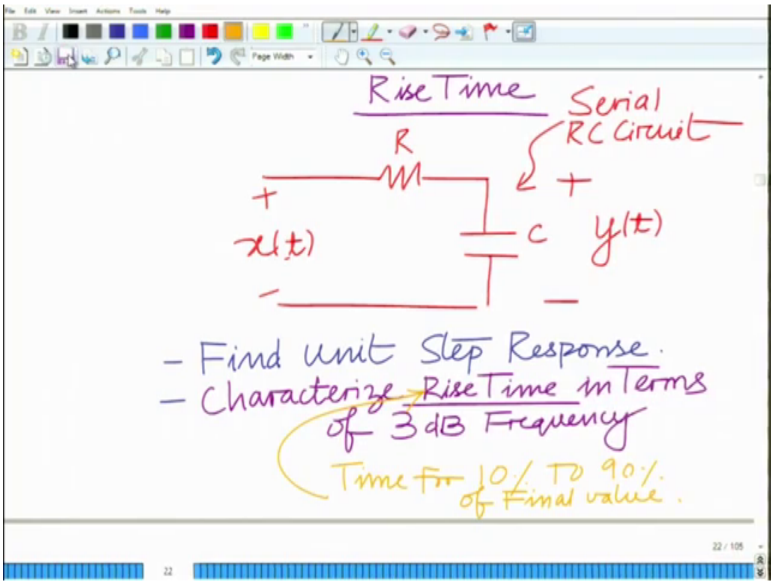
So, let us proceed to the next problem that is problem number 17 and this is with respect to an RC filter and this blocks talks about the rise time of an RC filter. And this deals with the concept of the rise time of an RC filter and the problem is as follows. So, let us say we have a serial RC circuit; we have a serial RC circuit, the output across the capacitor the input is the signal xt and the output across the capacitor the output voltage across the capacitor is yt.

So, this is your serial RC circuit and now the problem is to find the unit step response and explicitly find the rise time that is the time that takes from the step response to go from 10 percent to 90 percent of its final value characterize the rise time. So, for a unit step response we have an output for this RC circuit ok.

Now what we want to do is we want to characterize the rise time what is the relationship of the rise to specifically what is the relationship of the rise time of this RC circuit there is a time it takes to rise from 10 percent to 90 percent of the final value alright in terms of the 3 dB frequency of this RC circuit ok. So, that is what you want to know.

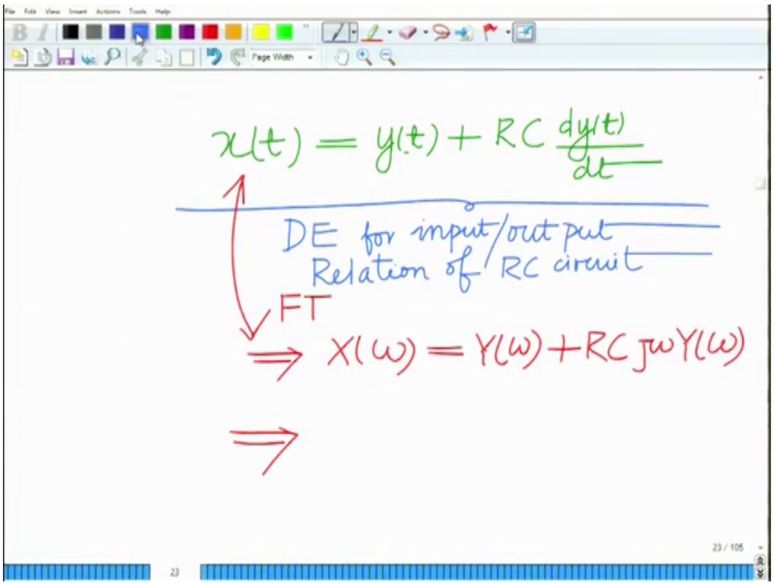
So, we want to characterize the unit step response. So, want to find the unit step response that is the first step.

(Refer Slide Time: 16:57)



And then we want to characterize the rise time in terms of 3 dB; now rise time is basically time taken to go from 10 percent to 90 percent. So, time taken to go from 200 to from the k by time taken for the capacitor output to go from 10 percent to 90 percent of its final value that is termed as a rise type of this serial RC circuit ok.

(Refer Slide Time: 18:00)

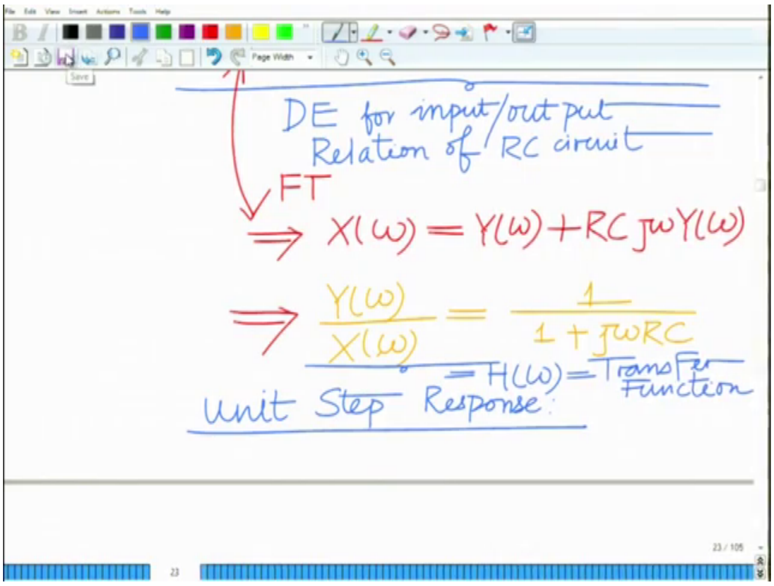


So, now let us approach this problem now we have seen this several times before that is the input voltage xt is the voltage across the resistance is a voltage across the capacitor plus the voltage across the resistance which is I times R R times I, but I is C dv by dt where dv is the voltage across the capacitor. So, this is C dy by dt this is the differential equation that this is basically differential equation for input slash output relation of RC circuit ok.

And therefore, this implies that X of omega now taking the Fourier transform once you take the Fourier transform we have X of omega equals Y of omega plus RC dy t by dt as omega a Fourier transform j omega y omega because if the Fourier transform of y t is Y omega the Fourier transform which the derivative is j omega times y omega. Now we have a simple equation in terms of the Fourier transform which we can solve to obtain the transfer function.

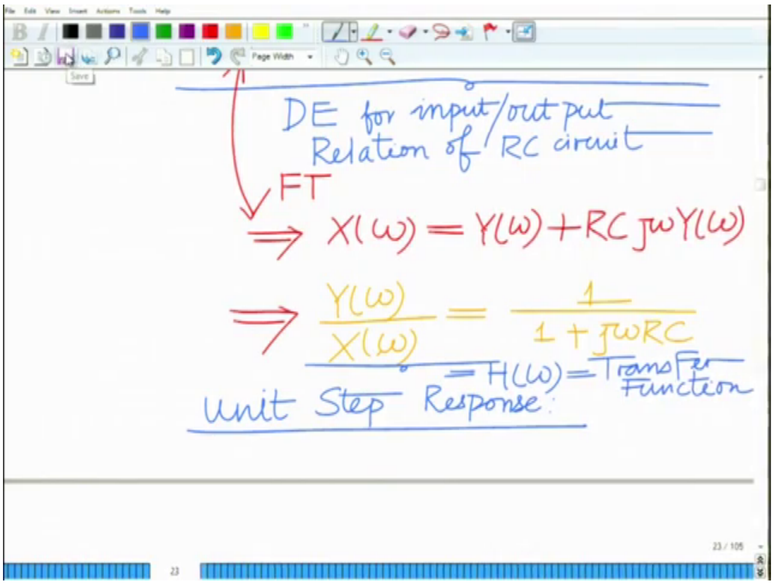
So, this implies that basically what we have is Y omega divided by j omega.

(Refer Slide Time: 19:41)



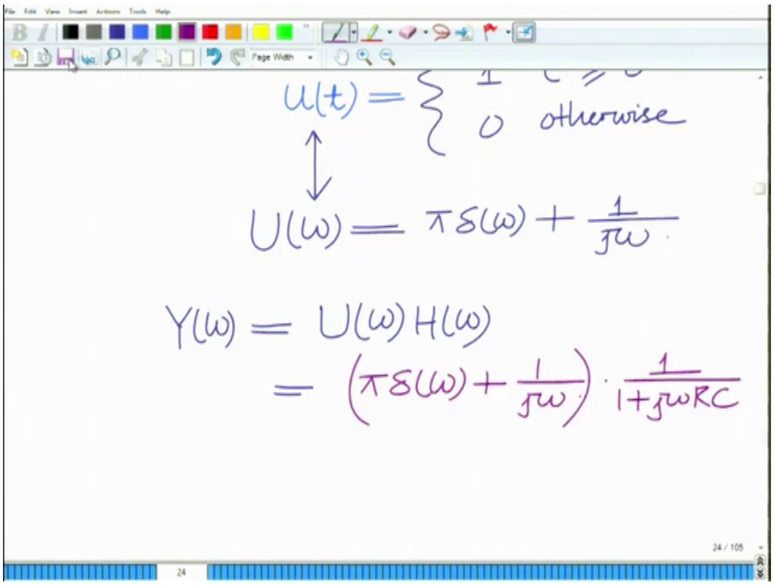
Or Y omega divided by X omega this is basically 1 over 1 plus j omega RC this is 1 over 1 plus j omega RC. Now the unit step response can be found as follows now remember this is equal to H of omega this is the transfer function this is the transfer function and now unit step response.

(Refer Slide Time: 20:36)



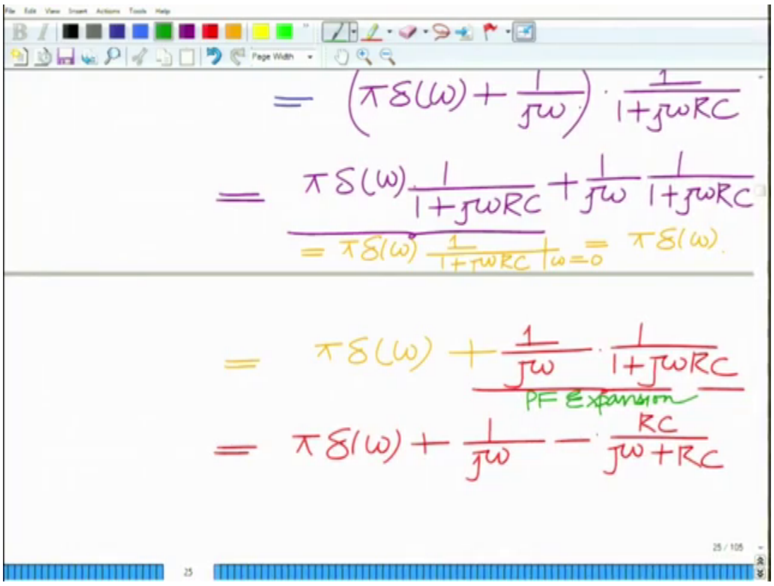
Now, consider the unit step function u t which is equal to 1 t greater than equal to 0 and 0 otherwise this is your unit step function or this is a unit step signal. And if you take the Fourier transform of this U of omega we already know this is pi of delta omega plus 1 over j omega there is a unit step response ok.

(Refer Slide Time: 21:15)



And therefore, we have y omega the output corresponding to unit the unit step response is basically the Fourier transform the unit step response there is U omega times the transfer function H omega that gives us the unit step response that is the output of this system corresponding to the unit step input signal ok. So, this is basically U of omega times H of omega which is pi delta omega plus 1 over j omega times 1 over 1 plus j omega RC which is now pi of delta omega into 1 plus j omega RC plus 1 over j omega into 1 plus j omega RC; now, look at this pi of delta omega into 1 plus 1 over 1 plus j omega RC.

(Refer Slide Time: 22:07)

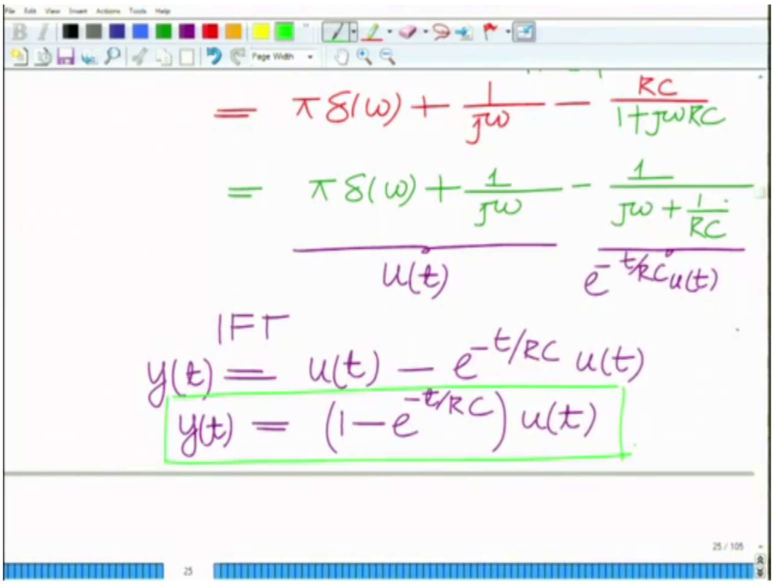


Now we know pi of delta x into F of x is basically pi of delta x into F of 0 correct. So, similarly pi of delta omega in 1 over 1 plus j omega RC this is simply if you look at this is simply equal to pi of delta omega into 1 over 1 plus j omega RC evaluated at omega equal to 0 which is simply pi of delta omega.

So, you look at this you can see convince yourself that this is the first term is simply pi of delta omega plus the second term which is 1 over j omega into 1 over 1 plus j omega RC is just pi of delta omega plus 1 over j omega minus 1 over j omega plus now you I can use the partial fraction expansion for this.

So, what I am going to do is I am going to use the partial fraction expansion 1 over j omega into 1 over 1 plus j omega RC. So, I am using the partial fraction expansion for this. So, this is 1 over j omega minus RC over j omega.

(Refer Slide Time: 24:04)



Plus j omega plus RC which is basically pi of delta omega plus 1 over j omega minus 1 over. Now I will bring the RC in the numerator to the denominator that will give me 1 plus j omega divided by RC ok.

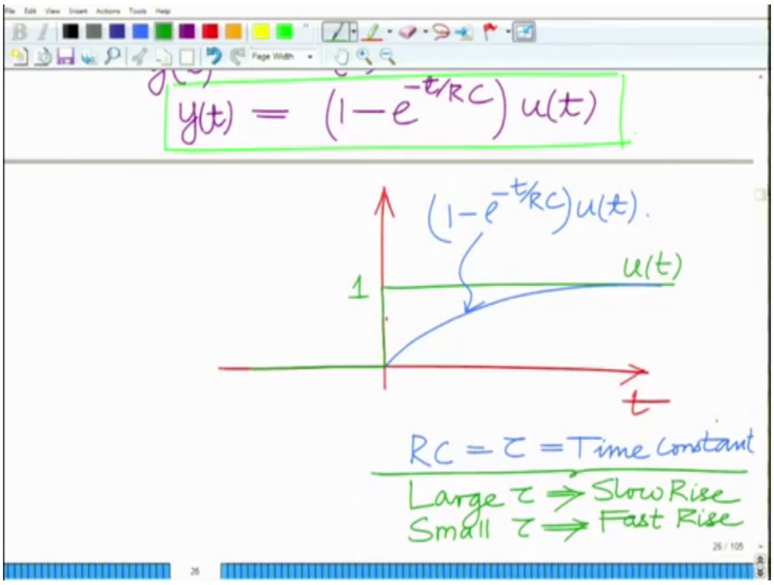
Now, I can now also write this quantity as 1 over 1 plus j omega divided by omega naught I am going to explain this in a little bit. So, omega naught equals 1 over R 1 over j omega over j omega over this is j omega RC j omega j omega RC sorry this seems to be some mistake this is j omega RC, RC over RC over 1 1 plus j omega RC 1 plus RC over 1 plus j omega RC 1 over j omega plus 1 over RC j omega plus 1 over RC j omega plus 1 over ok.

Let me just keep this as it is for a moment. So, this is 1 over j omega plus 1 over RC now what this is now if you look at this now if you take the inverse Fourier transform; now look at this is nothing, but your signal the unit step u t this is pi delta omega plus 1 over j omega u t. Now this is 1 over this is of the form 1 over 1 over j omega plus a. So, the inverse Fourier transform of this is e raise to minus a t u t where a is this constant 1 over RC ok.

So, the inverse Fourier transform of this is e raise to minus t over RC times u t and therefore, the net taking the inverse Fourier transform. So, that you take the inverse Fourier transform and what you have is the output yt equals ut minus e raised to minus t over RC u t which is 1 minus e raised to minus t over RC ut e raised to minus t over RC ut ok. So, this is what we have.

And if you plot it it is going to look something like this with something you must have seen before.

(Refer Slide Time: 27:18)



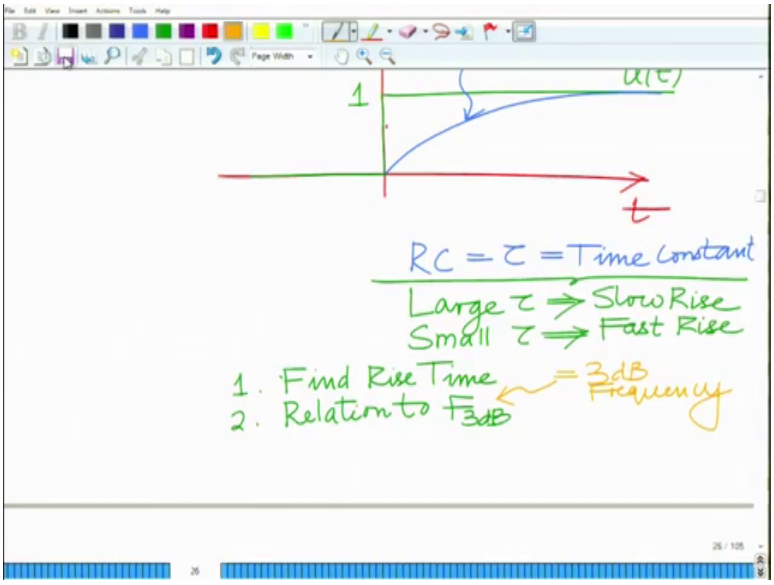
So, I have the time axis this is the unit response, this is the unit step the ut and what happens is if you look at the response of the RC circuit it starts from 0 its 1 minus e raised to minus t over RC. So, at t equal to 0 this is 1 minus 1 that is 0 at t equal to infinity this is e raised to minus t over RC at t equal to infinity e raised to minus t or RC 0. So, 1 minus e raised to minus t over RC is 1.

So, it starts from 0 and at t equal to infinity approaches one and ok. So, this is basically your 1 minus e raised to minus t over RC into ut ok. So, and this is known as the time constant this RC is known as the time constant larger the time constant implies slower the rise ok.

If the time constant is larger it means the rise is more sluggish ok. So, all these things I think are fairly large time constant and so, large time constant implies slow rise small time constant. So, very small time constant implies fast rise. So, for a large time constant it implies very slow rise ok. So, the rise is very sluggish alright and alright.

So, the time constant is and now what we want to find is we want to find the rise time.

(Refer Slide Time: 29:34)



And find its relation to the 2 dB frequency. So, what we want to do is in and we will do this in the next module find rise time. So, we need to do two things find the rise time 2; relation to the 3 dB frequency where F 3 dB; this is the find its relation to basically the 3 dB frequency.

So, we want to find the rise time and find its relation to the 3 dB frequency of the circuit alright. So, we will stop this module here and look at the other look at other aspects in the subsequent modules.

Thank you very much.