Principles of Signals and Systems

Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology Kanpur

Lecture - 43
Fourier Analysis Examples-Trignometric Fourier Series of Periodic Square Wave, Periodic Impulse Train

Hello welcome to another module in this massive of online course. So, we are looking at example problems various example problems to understand better understand Fourier analysis.

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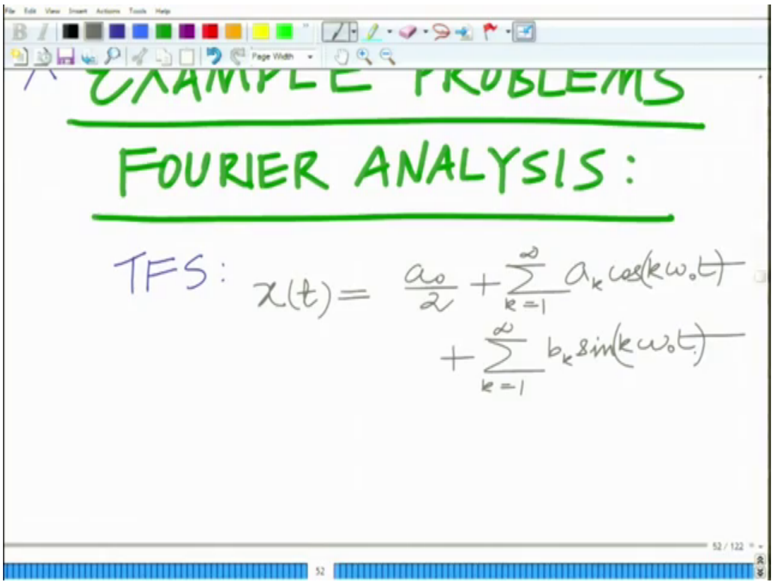

So, let us continue our discussion. So, we want to look at example problems to better understand the concept of Fourier analysis. In particular we have started looking at the Fourier analysis representation of this periodic square wave correct?

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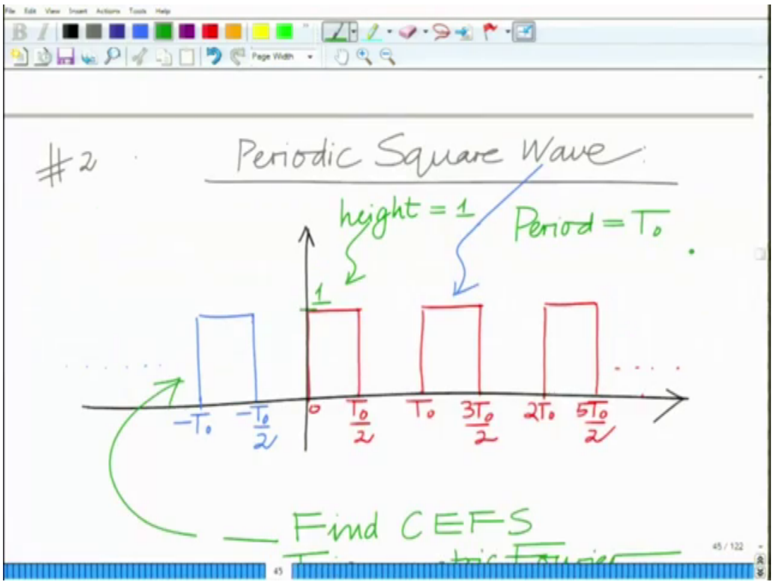

If you remember you might remember we have started looking at the Fourier analysis of this periodic square wave, which has the time period fundamental period that is the of T naught pulses of width T naught over 2 and height 1. The period fundamental period of the signal T naught we have looked the CEFS the complex exponential Fourier series.

We want to look at now look at the TFS that is the trigonometric Fourier series of this signal that is the periodic square wave ok. And the TFS can be obtained as follows to obtain the TFS. So, we are continuing the discussions. So, the signal TFS of the above signal can be obtained as follows, remember the TFS of any signal x t is given as. So, the we have the TFS is given as x t equals a naught over 2 plus summation k equals to 1 to infinity a k cosine k omega naught t plus summation k equals 1 to infinity b k, b k sin k omega naught bk sin k omega naught t, and now we know also the relation for this variance for instance a.

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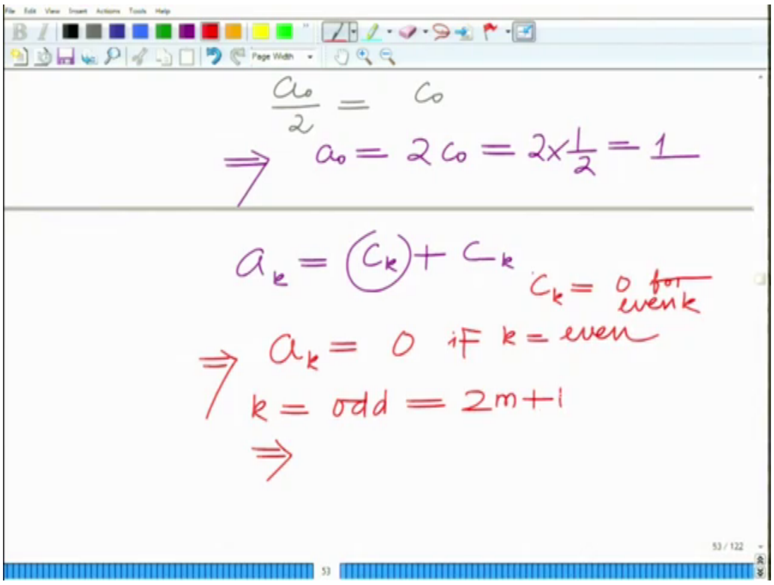

Naught we know a naught by 2 equals C naught that is the properties of TFS which implies a naught equals twice C naught and we valued already valuated C naught, C naught you can see is half a naught is twice C naught which is twice into half equals to 1. So, we have a naught equals 1.

Further we have the coefficient of cosine omega k omega t given as a k, a k equals we know C k plus C minus k. Now we know C k s are 0. If k is odd or C k equals to 0 for k even k implies. Now a k which is C k plus C minus k is also equals to 0 if k is even because what C k and C minus k and 0 for even k ok. So, implies a k equals 0 if k equals even both C k and C minus k are 0; and if k is odd let us say k equals 2 m plus 1 remember for odd k the expression.

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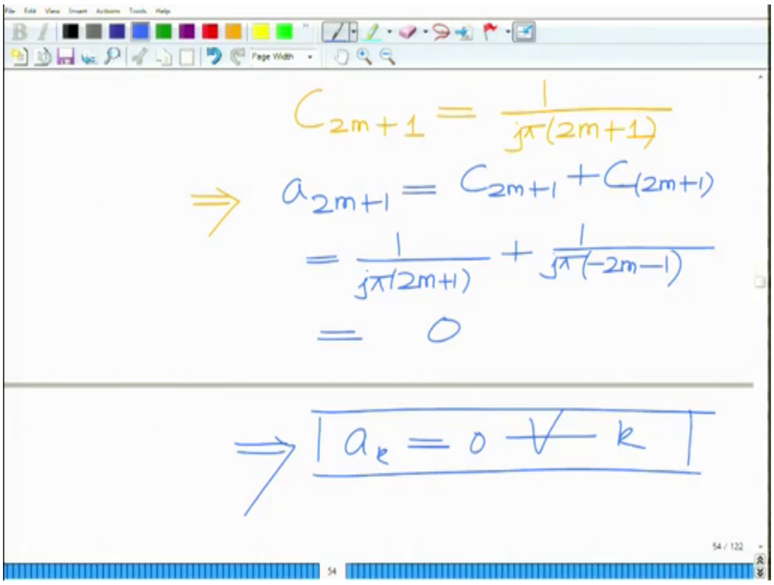

Of C k is c of 2 m plus 1 we have for odd k we have C of 2 m plus 1, we have derived expression for this that is equals to 1 over j pi 2 m plus 1.

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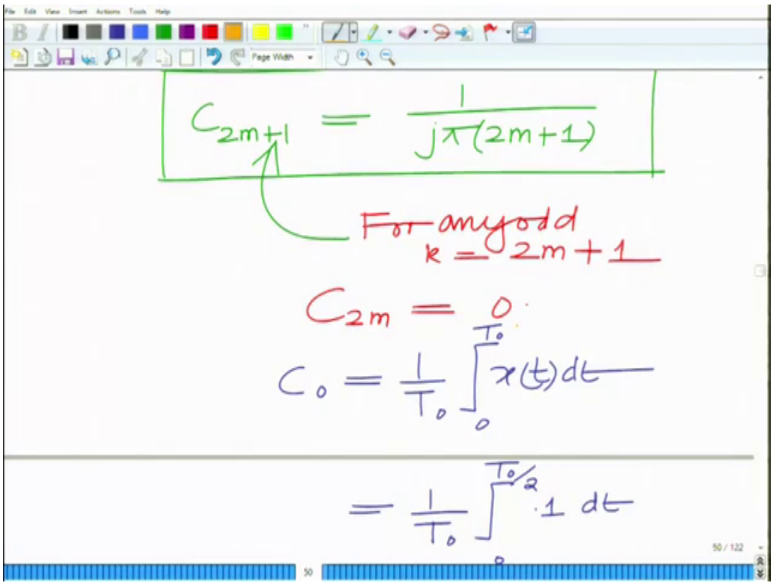

So, C of 2 m plus1 equals 1 over j pi 2 m plus 1 which implies basically that your a k equals or a of 2 m plus 1 since k equals 2 m plus 1, a of 2 m plus 1 equals C of 2 m plus 1 plus C of minus k, that is C of minus 2 m plus 1 which is 1 over j pi 2 m plus 1 plus 1 over j pi minus 2 m minus 1 minus of minus 2 plus 1 that is minus 2 m minus 1 and you can clearly say these 2 cancel each other. So, this is 0.

So, basically what will be able to show that a k is 0, if k equals k is even and a k is 0 as a k is 0 when k is odd as well, which implies a k is 0 or odd this basically implies. So, a k is 0 for a k is either even or odd which implies a k equals 0 for all k ok. So, the coefficient of cosine k omega naught t is 0 for all k ok. Now let us then look at what is b k.

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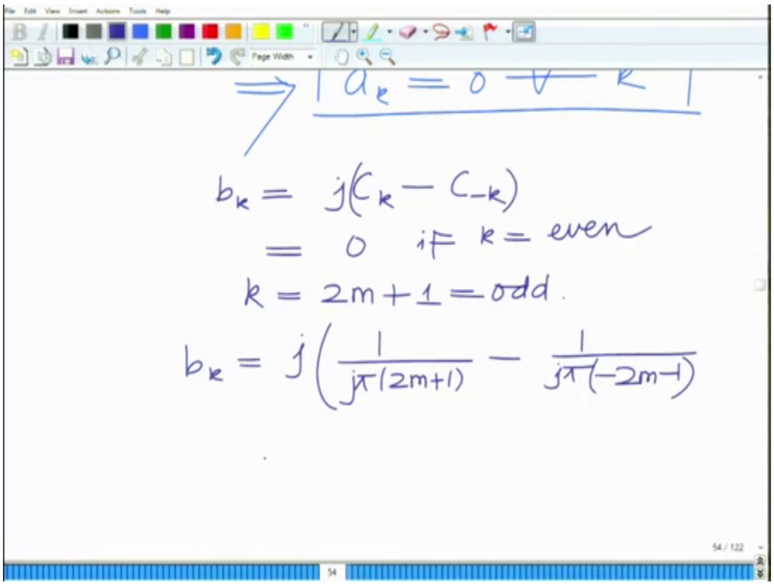

Now, b k is basically you can see we know the relation for b k that is j times C k minus C of minus k, which is once again equal to 0 if k equals even because remember again k is even C k is 0 C minus k is 0. So, b k is also 0 if k equals odd, then b k can be simplified as let us assume k equals some odd quantity to 2 m plus 1, then b k equals j times C k that is 1 over if I remember correctly that 1 over j pi 2 m plus 1 minus 1 over C minus k minus 1 over j pi minus 2 m, minus 1 which is equal to twice j cancels. So, this is equal to twice pi over 2 m plus 1.

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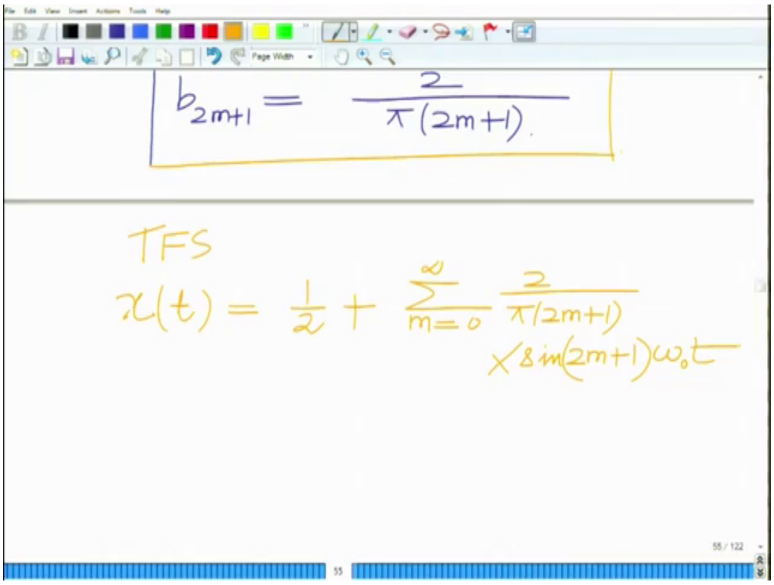

B k equals or this is in fact, b of 2 m plus 1 in case is odd. So, b of 2 m plus 1, b of 2 m plus 1 is given by this and therefore, finally, x of t the TFS is given as follows we have x of t equals a naught by 2 a naught is 1. So, this is half plus b b k summation b k, but here b k is non zero for e odd k. So, this is basically summation m equals 0 to infinity 1 over or 2 over pi 2 m plus 1 times sin of 2 m plus 1 omega naught t and if you take the 2 over pi common in this summation what you have is.

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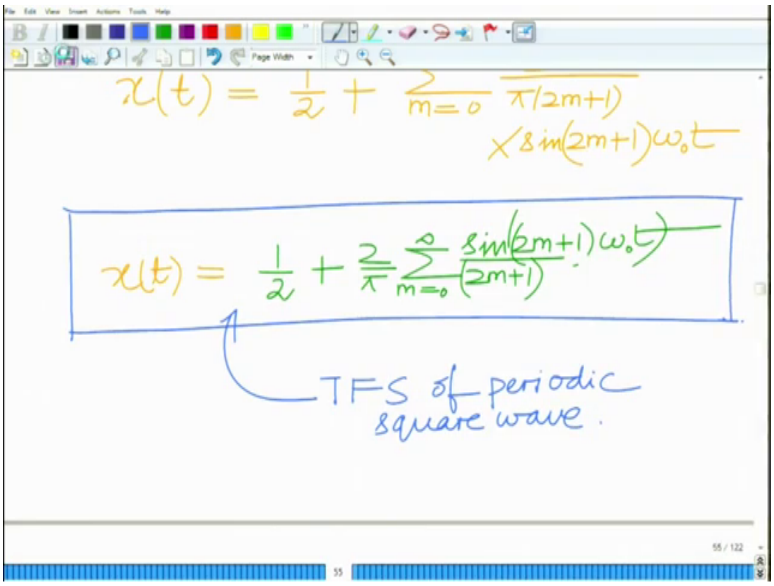

You have x t equals you have x t equals, half plus 2 over pi summation m equals 0 to infinity 1 over 2 m plus 1. In fact, 1 over 2 m plus 1, sin of 2 m plus 1 times omega naught t and this is the this is the trigonometric Fourier series, this is the TFS of the this is the TFS of the trigonometric Fourier series or that the TFS that is the TFS of the periodic square wave and further simplify it is to illustrate you can write it as half 2 over pi equals n equals 0 this is simply sin of omega naught.

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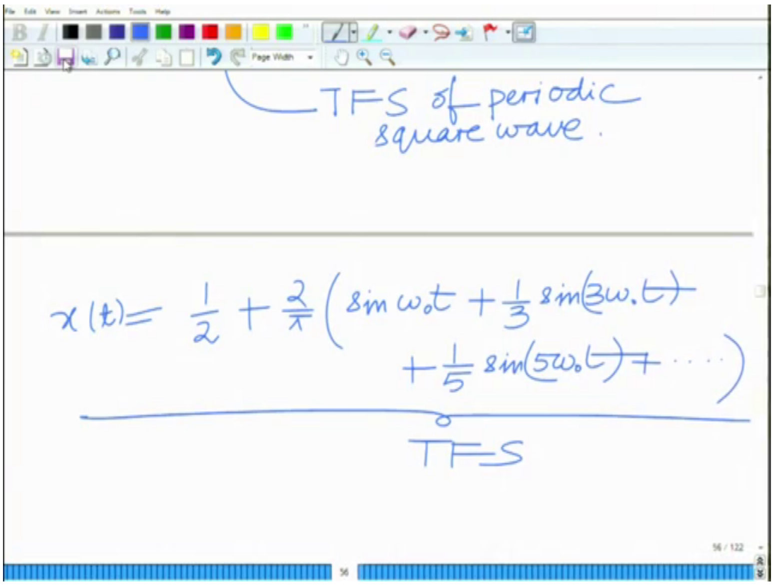

For n equals 1 this is 1 over 2 m plus 1 that is 1 over 3 sin of 3 omega naught t plus 1 over 5 plus sin of 5 omega naught t plus so on and so on. So, this is the again this is your TFS that is the trigonometric Fourier series representation.

So, if you look at example problem, in which you consider the periodic square wave of pulse width T naught over 2, fundamental T naught height 1 and you can derive the both the CEFS as well as the TFS representation for this periodical continuous time signal all right. So, let us proceed to the next problem which has the following things let us consider now a periodic impulse train, and this has a lot of relevance in sampling which we will see later. So, this is problem number problem number 3.

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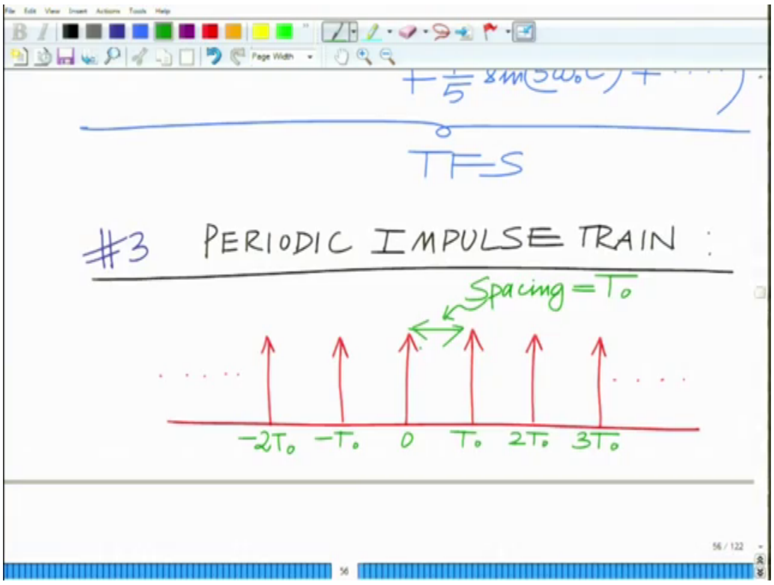

Ah That is the periodic this is the periodic impulse train, which is basically train of this is nothing, but train of impulses. So, at this train of success impulse is the successive impulses at a spacing of t naught. So, impulses at 0 T naught, twice T naught thrice T naught minus T naught minus 2 T naught, this is the impulse at integer multiples of T naught.

So, this is the spacing between the impulses successive impulses is T naught. So, you can say this is the periodic signal ok. So, this is the impulse train this periodic signal comprising of impulses of impulses unit scale impulses based at periodic intervals of T naught right multiples T naught is termed as an impulse train ok.

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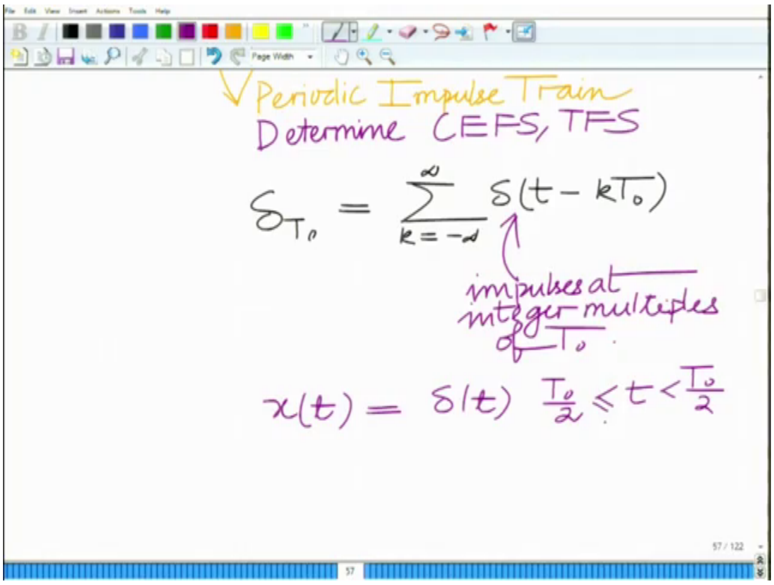

So, this is the periodic signal or you can call it as a periodic impulse train ok. So, this is an impulse train and in fact, this is periodic impulse train and impulse train that is impulse train that is periodic; and we want to determine for this we want to determine the what we want to determine is once again, we want to determine the CEFS and as well as the TFS for this periodic impulse train; and this is fairly simple.

So, let us call this as delta T naught of t that is periodic impulse train with facing T naught you can see this is nothing, but summation k equals minus infinity to infinity delta impulses at every integer multiple of T naught ok. This is basically denotes an impulse at multiple impulses at integer multiples of T naught.

Now, if you look at any particular period in minus T naught over 2. For instance minus T naught over 2 to T naught over 2 this is simply delta t in t less than minus T naught over 2 less than equal to less than T naught over 2. So, this is simply the delta t will look at one continuous intervals of duration T naught and therefore, the coefficient of the Fourier series remember that is C k is given as C k can be evaluated as follows.

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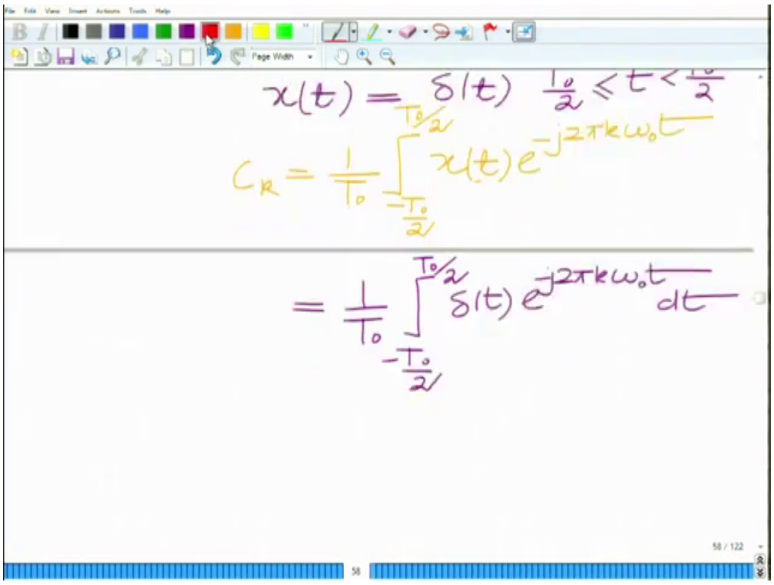

So, you have C k equals 1 over T naught integral minus T naught over 2 to T naught over 2 particularly convenient to choose minus T naught over 2 to T naught 2 because the impulse is contained well contained in this interval. So, that is x of t e raise to minus j 2 pi k omega naught t which is basically in this interval this is simply the delta function, delta t e raise to minus j 2 pi k omega naught t d t property of the delta.

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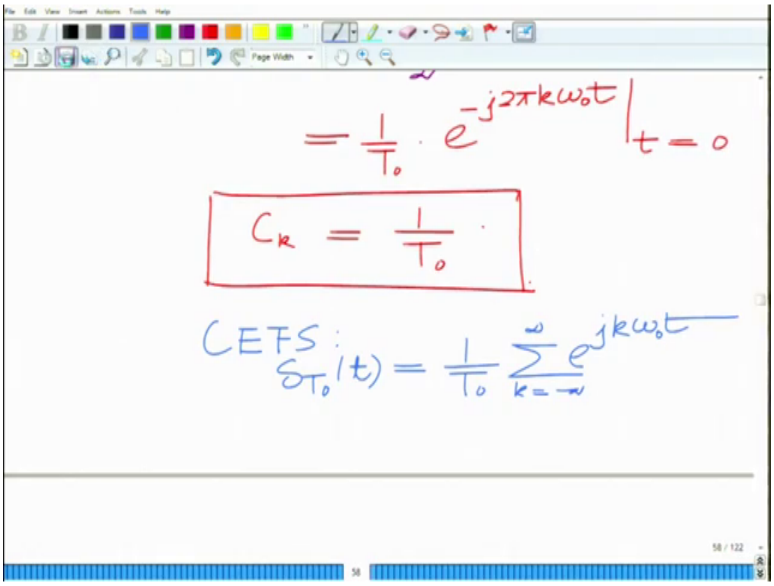

Function this is simply the value of the function that is e raise to minus j 2 pi k omega naught t evaluated at t equal to 0 and this is simply 1 over T naught.

This is e raise to minus j 2 pi k omega naught t evaluated at t equal to 0. So, each of the Fourier series coefficients, the coefficients of the complex exponential Fourier series that is the quotient of the CEFS all the coefficients are 1 over T naught. So, each C k is 1 over T naught ok. So, you can see that each C k is basically T naught therefore, the complex the CEFS is given as that is your delta T naught t equals 1 over T naught summation k equals minus infinity to infinity, e raise to j e raise to j k omega naught t and the trigonometric Fourier series the TFS of this can be obtained as follows.

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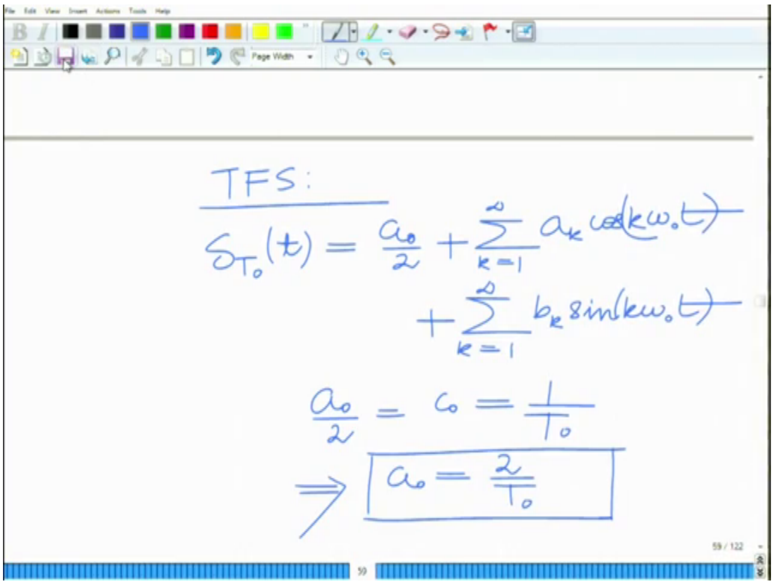

The TFS is given in the straight forward fashion remember delta T naught of t, I can write this as summation as a naught over 2 plus summation k equals 1 to infinity a k cosine k omega naught t, plus summation k equals 1 to infinity, b k sin k omega naught t. Now a naught equals a naught by 2 equals C naught which is basically remember C naught is k equals 0, that is 1 over T naught all C k are 1 over t not which implies a naught equals 2 over. So, this implies that a naught equals 2 over T naught.

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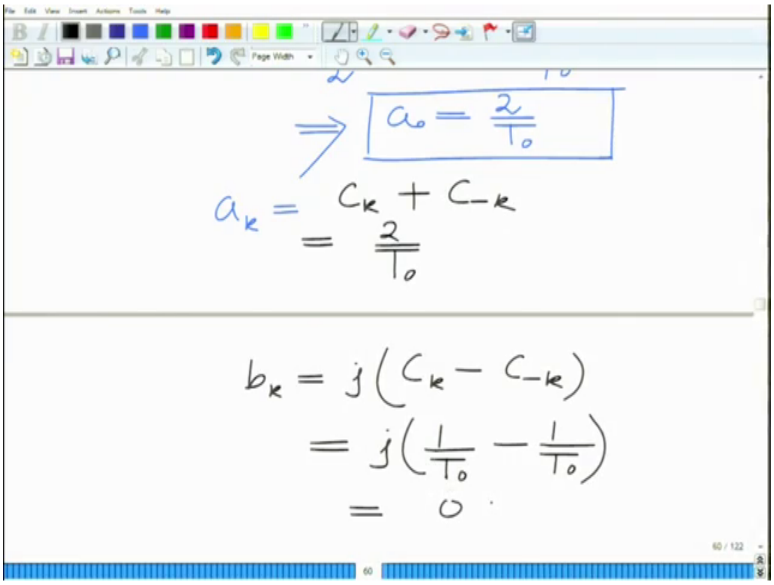

Further a k remember a k is C k plus C minus k both C k and C minus k are t naught. So, a k for any k is 1 over T naught plus 1 over T naught that is twice T naught ok. So, this is C k plus C minus k that is twice T naught 2 over T naught and b k equals j, C k minus C minus k which by the same logic is 1 over T naught minus 1 over T naught equals 0 that is j times minus c minus c.

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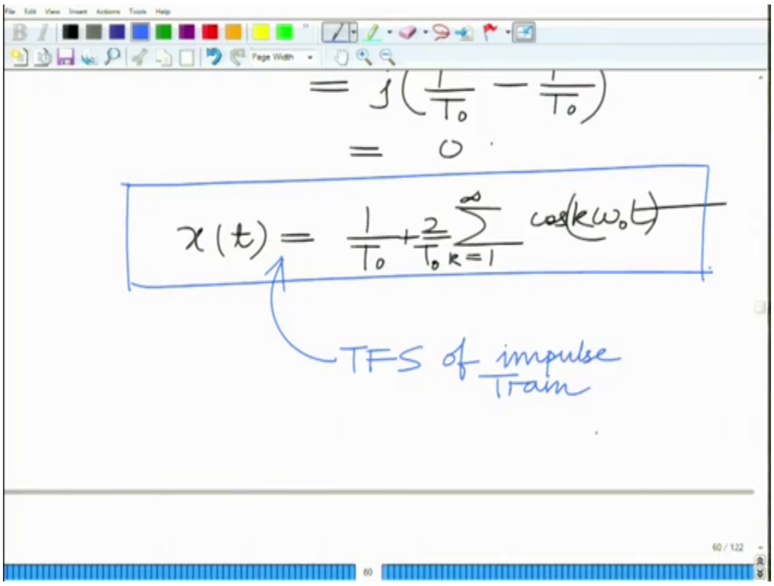

So, x t putting it all together the TFS is x t equals T naught by 2, that is twice over T naught by 2 that is 1 over T naught plus summation k equals 1 to infinity a k, that is twice over T naught. So, twice over T naught is common to all. So, write it between summation. So, that is twice 2 over T naught cosine k omega naught t.

This is the TFS of impulse train. This is the TFS of the impulse train that is trigonometric Fourier series of the impulse train which is 1 over T naught plus twice over T naught summation 2 over T naught cosine k omega naught t basically that plays the trigonometric both the CEFS and the TFS of the impulse train and we have already assumed it to it plays a very important rule, in the sampling of a continuous time signal and we are going to look at like later in the later modules.

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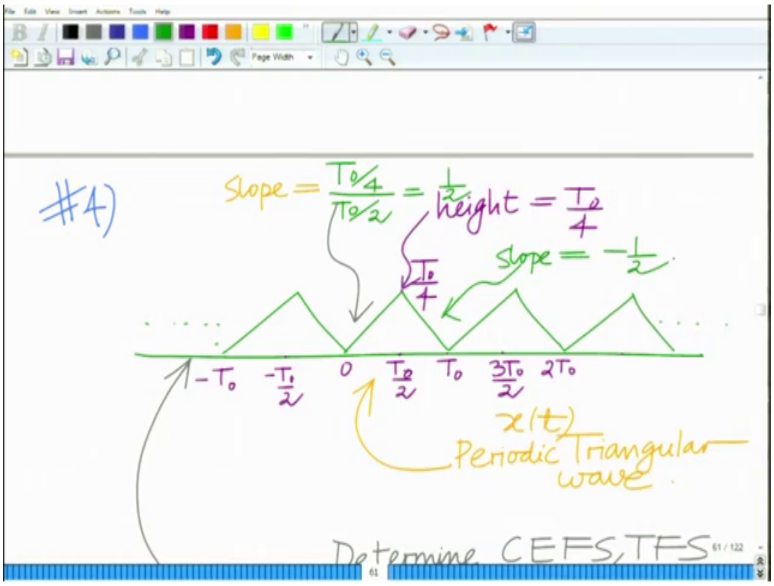

Let us look at another problem, let us now look problem number four let us now look at a triangle. So, we have consider various periodic signal that is we have consider a square wave consider a impulse train, let us now consider the triangular wave which is slightly more difficult to analyze. So, here I have a periodic triangular wave of period T naught. So, this is 0 T naught over 2, T naught, 3 T naught over 2, twice T naught and so on minus T naught over 2 minus T naught.

And what I can see here and let the height of this be equal to T naught over 4 or just to choose something convenient has. So, this is height equals T naught over four and this is your periodic triangular wave ok. If you see the periodic square waves before. So, this is the periodic triangular wave. This is the periodic triangular wave and what we want to do again once again for this signal, is we want to derive determine we want to determine we want to determine the CEFS and the TFS for this periodic triangular wave. Off course one can employ the straight forward approach here that is the write down the CFS and derive the expression for each coefficient c k, but we are going to use a nice trick here.

First what we are going to do is we are going to differentiate, then we differentiate it you will see that we will get something very interesting. Now if you look at this triangular wave now the slope in the raising part of this is the height of this triangle divided by the width. So, the slope of this if you look at the slope of this let us call this signal x t ok. So, periodic triangular wave x t.

So, the slope of this interestingly is the height T naught over 4 divided by the width of the raising part it is T naught over 2, which is basically equals which basically equals half. So, the slope of this is half. So, the slope of the raising part is half slope of the following part is naturally this is symmetric. So, this slope is minus half.

So, what we have is if you differentiate it that is you consider x prime t.

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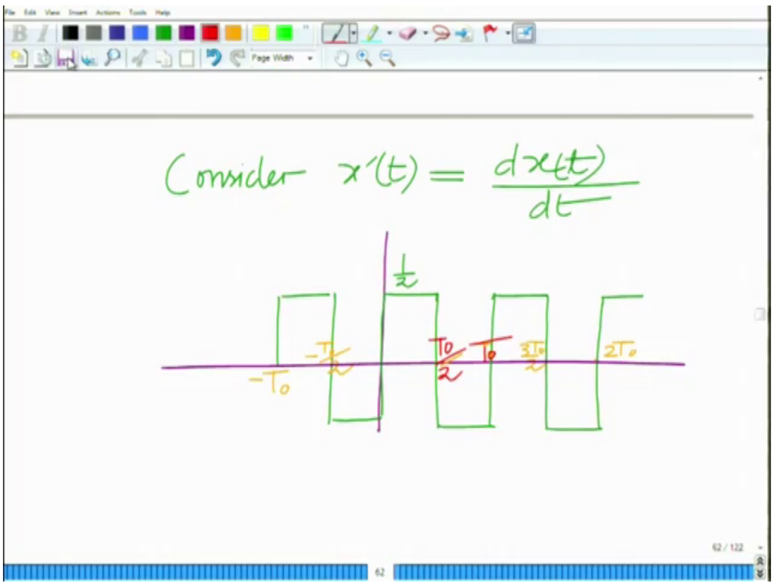

So, consider. So, consider x prime t equals d x t, d x t over d t that is the derivative. Now x prime t if you can draw it x prime t will look something like this that is at times 0 it starts at half because the increasing slope is half. So, its half for T naught by 2, and then it is minus half over T naught by 2 half minus half and then it keeps continuing ok. So,. So, if you call this height as half. So, this is T naught by 2, this is T naught, 3 T naught by 2, this is twice T naught, minus T naught by 2 this is minus T naught ok. So, this we can see let me just write this as let me write clearly. So, this is T naught by 2 this is T naught and so on. So, we get a periodic square wave. In fact, this is basically possibly this is basically goes has a height peak of half and negative peak of minus half.

This is basically this point is minus half. So, this is your x prime t.

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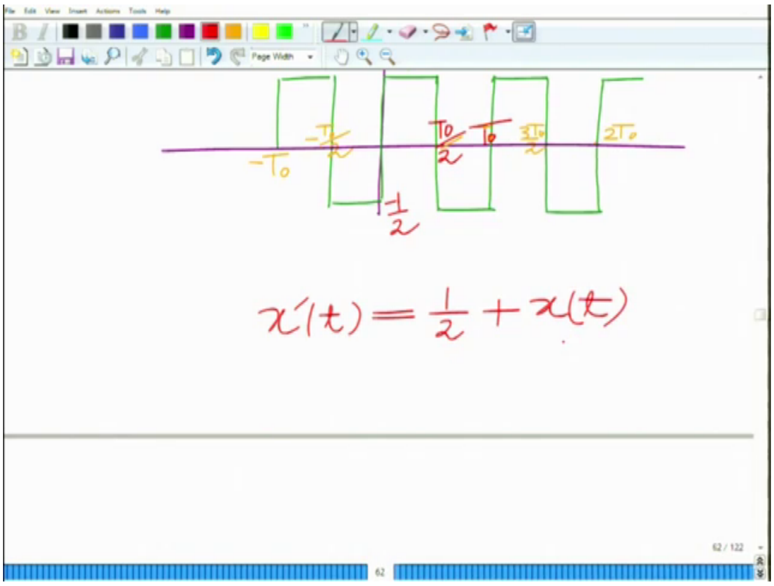

You can say this is very close to the periodic square wave that we have seen so far ok. So, this is your x prime t. Now let u consider x till the t; x till the t which is prime t which is x prime t plus half. So, shift this basically on the y axis by half. So, push it up add half x prime t equals half plus x t you can consider x prime t half plus x t you will observe the.

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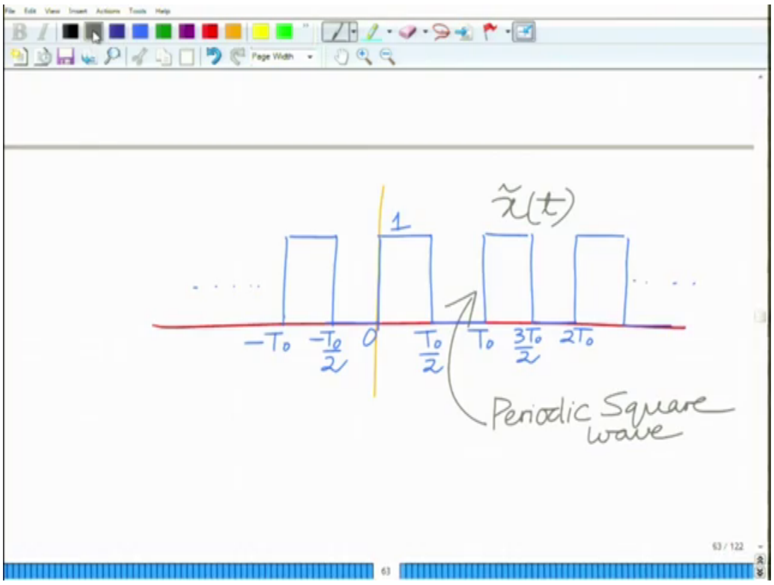

Something interesting what you get is you will get a periodic square wave now of height of height, that is half from height let it will be 1 and this is something that we have seen before. So, this is 1, this is T naught by 2, T naught 3, T naught by 2 twice T naught this is 0 minus T naught by 2, minus T naught and so on.

And this is basically your periodic square wave that we have seen before. pPriodic square wave that we have seen before and this is basically your this is what we are calling as x till the t, this is what we calling as x quantity x till the t.

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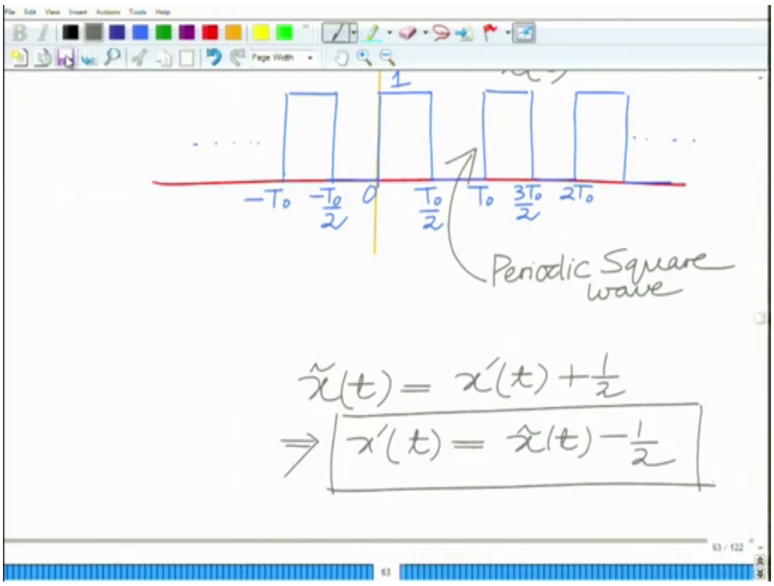

So, what we have shown is that the periodic square wave that we have derived before is x till the t equals x prime t plus half, which implies that x prime t that is d x by d t x prime t equals x till the t minus half ok. Here x till the t is x prime t equals x till the t minus half alright.

So, we basically started with this triangular wave, using the differentiation we considering the derivative we have reduced to the periodic square waves something that we have seen earlier. And we use this property now to derive the both the CEFS as well as the TFS of the periodic square wave, for the periodic triangular wave alright. So, let us stop here and continue with this problem in the next module.

Thank you very much.