Principles of Signals and Systems

Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 39
Fourier Transform - Parseval's Relation, Frequency Response of Continuous Time LTI Systems

Hello, welcome to another module in this massive open online course. So, we are looking at the Fourier transform for continuous aperiodic signals and its several properties. Let us look at another interesting and important property that is a Parseval's relation and this will bring up a lot of other aspects such as the autocorrelation function and so, on alright.

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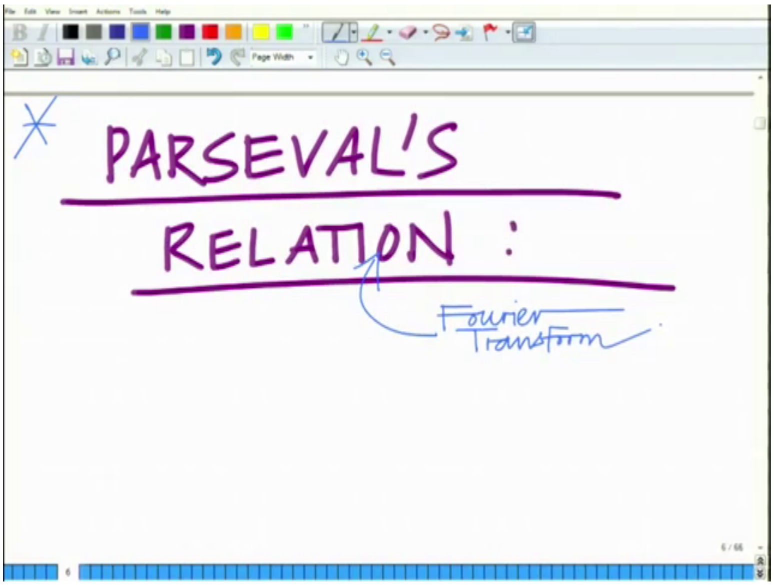

So, let us explore the Parseval's; the Parseval's relation for the Fourier transform. You already seen the Parseval's relation or the Parseval's theorem for the discrete Fourier series we want to explain we want to look at it for the Fourier transform.

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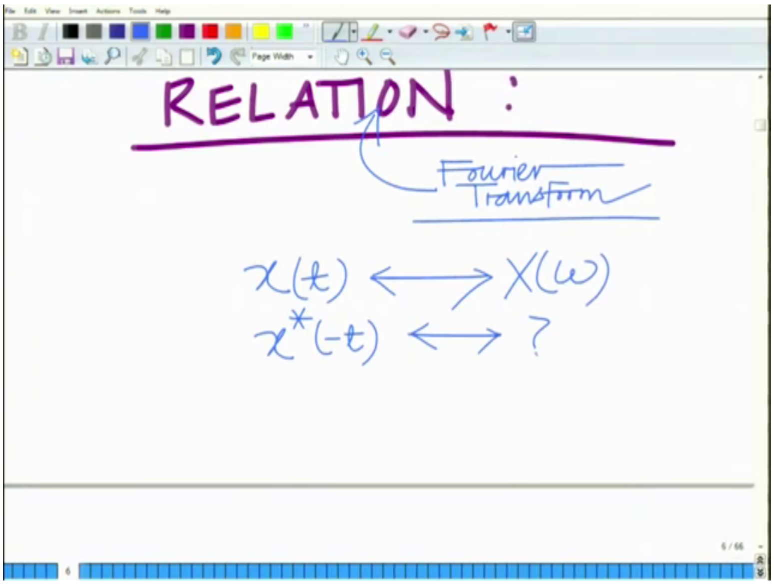

Now, first consider a signal x t which has the Fourier transform X of w. Now we want to look at first what is the Fourier transform of x conjugate minus t what is the Fourier transform of this signal?

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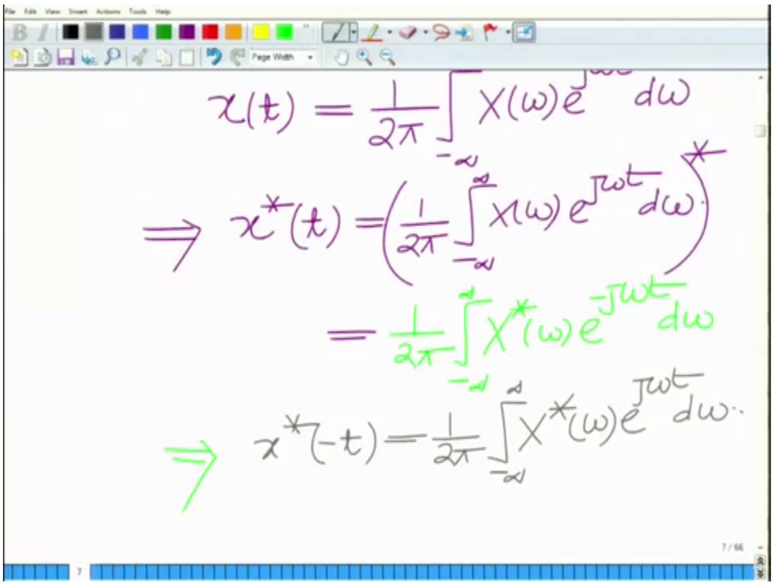

Now, using the properties of the inverse Fourier transform first you can see that x t which is the inversed Fourier transform of capital X of omega this is 1 over 2 pi minus infinity to infinity X omega e raised to J omega d omega which implies that if you take the conjugate the complex conjugate on both sides.

Then you have x conjugate over t equals x conjugate t is 1 over 2 pi x of omega e raised to J omega td omega which if you now take the conjugate operation inside; this will be 1 over 2 pi minus infinity to infinity; x conjugate omega e raised to minus j omega td omega.

And now if you replace t by minus t this implies that x conjugate of minus t is 1 over 2 pi integral minus infinity to infinity; x conjugate omega e raised to J omega td omega.

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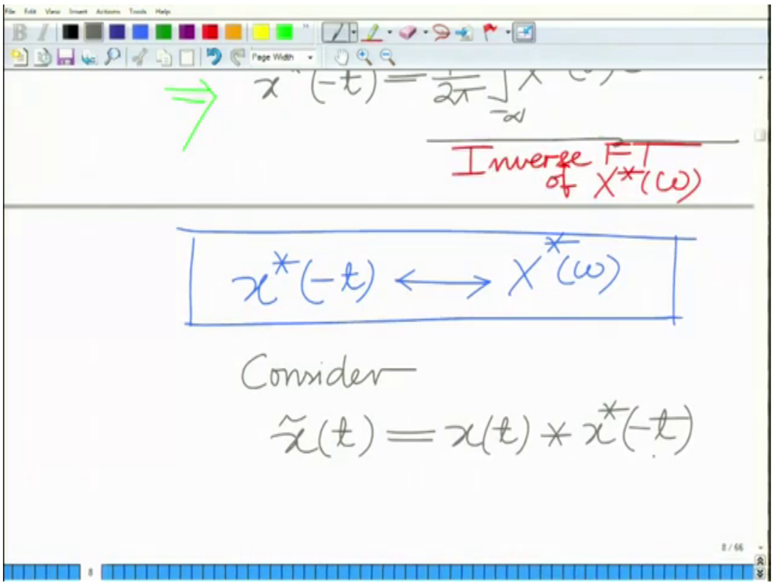

And now you can see this right hand side is nothing, but the inverse Fourier transform of X conjugate omega alright. So, you can see x conjugate minus t is basically the inverse Fourier transform of x conjugate omega.

So, therefore, one can conclude that x conjugate minus t has the Fourier transform capital X conjugate of omega alright. So, this is the first property that we want to start with that is x conjugate minus t has the Fourier transform X conjugate of omega the capital X conjugate of omega.

Now we define the function consider now x tilde t defined as x of t convolved with x conjugate of minus t. So, we are taking x of t convolving it with x conjugate of minus t. So, x tilde t is x of t convolved with x conjugate of minus t.

Now naturally what we have seen is the convolve if you seen an important property the Fourier transform previously that is the convolution in time domain is basically that is the signal which is the convolution of two signals in the time domain has a Fourier transform which is basically the multiplication is obtained by the multiplication of the Fourier transforms of these two signals.

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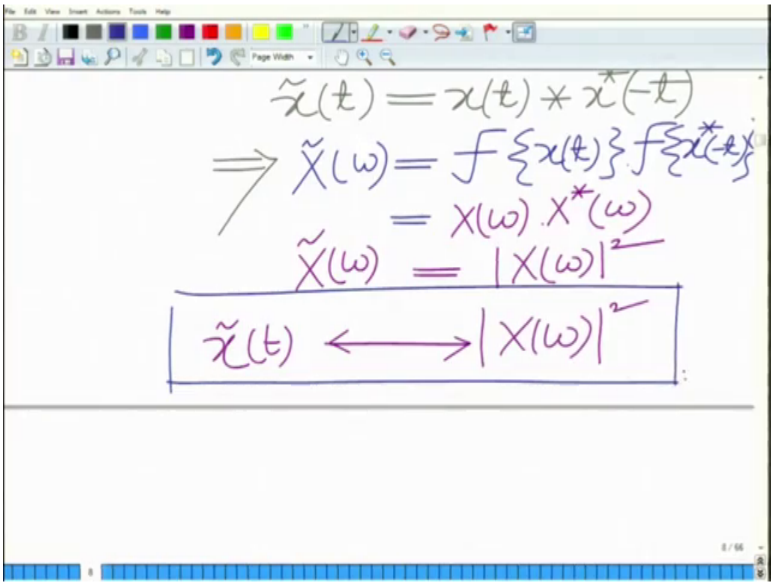

Therefore, now, this implies if you take the Fourier transform of X tilde t you have X tilde omega it is a multiplication of the Fourier transform of x of t times the Fourier transform of x conjugate minus t.

But the Fourier transform of x of t is X of omega and the Fourier transform X conjugate minus t is X conjugate of omega as we have seen above; So, this is X of omega into X conjugate of omega equals that is; so, we a magnitude X omega square.

So, we have X tilde omega equals magnitude X of omega square which means x tilde t which is the convolution of x t with x conjugate minus t this has the Fourier transform magnitude X of omega square.

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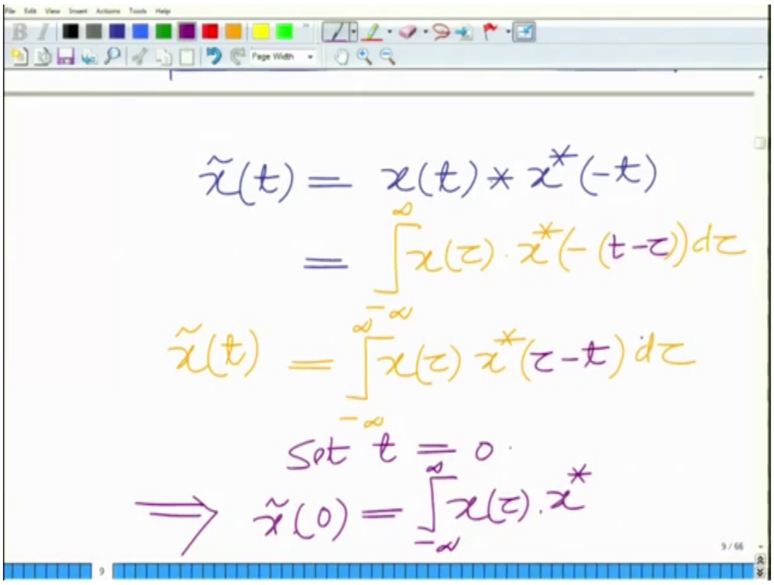

Now, let us look at what is let us again expand x tilde t that is the convolution of x of t with x conjugate of minus t.

Which is basically I can write it as minus infinity to infinity x of tau times x conjugate of minus tau minus t d tau correct; this is the definition of a this follows from the definition of the convolution. So, this is minus infinity to infinity x of tau x conjugate t minus tau d tau which implies setting t equal to 0.

Which implies you have x tilde 0 equals minus infinity to infinity x of tau into x conjugate of I am sorry this is x of this is x of tau into x conjugate of x conjugate of minus I am sorry this is x of tau into x conjugate of minus tau minus minus tau minus t d tau.

So, this will be tau minus t and your set t equal to 0.

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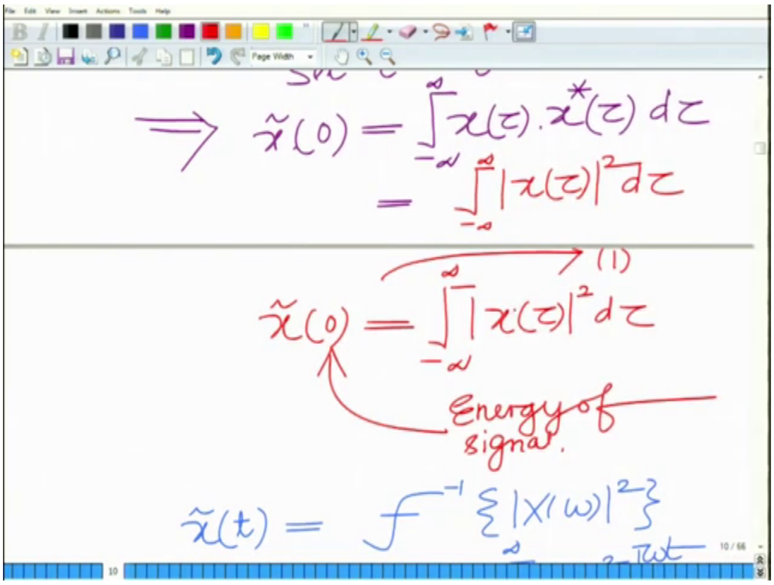

So, this x of tau into x conjugate of tau d tau which is integral minus infinity to infinity magnitude x x tau square d tau which is nothing, but the energy of the signal; so, what you can see is x tilde 0 equals minus infinity to infinity magnitude x tau square d tau that is this is I equal to the this is equal to the energy of the signal.

That is x tilde which is the convolution of x t and x conjugate of minus t if you evaluate it at t equal to 0; that is x tilde 0 this is this is simply gives me the energy of the signal. Now, we also know that x tilde t is the inverse Fourier transform of magnitude x of omega square; that is what we had shown above we. So, from this property that is if you look at this property x tilde t is the inverse Fourier transform magnitude of x of omega square.

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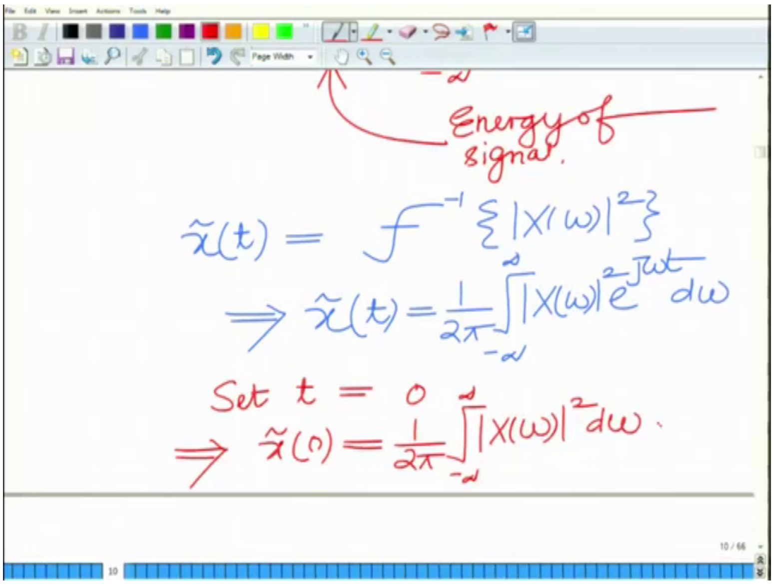

So, we have x tilde t equals IFT or basically the inverse of the Fourier transform of magnitude x omega square; this implies that x tilde t is 1 over 2 pi minus infinity to infinity; magnitude x omega square e raised to J omega td omega.

Now set equal to 0 once again and this implies x tilde 0 equals 1 over 2 pi minus infinity to infinity; the magnitude x omega square d omega. So, we have two relations here; we have x tilde 0 equals minus infinity to infinity magnitude of x tau square d tau.

And we also have x tilde 0 equals let us call this relation 2, you also have x tilde 0 equals 1 over 2 pi integral minus infinity to infinity magnitude x omega square d omega.

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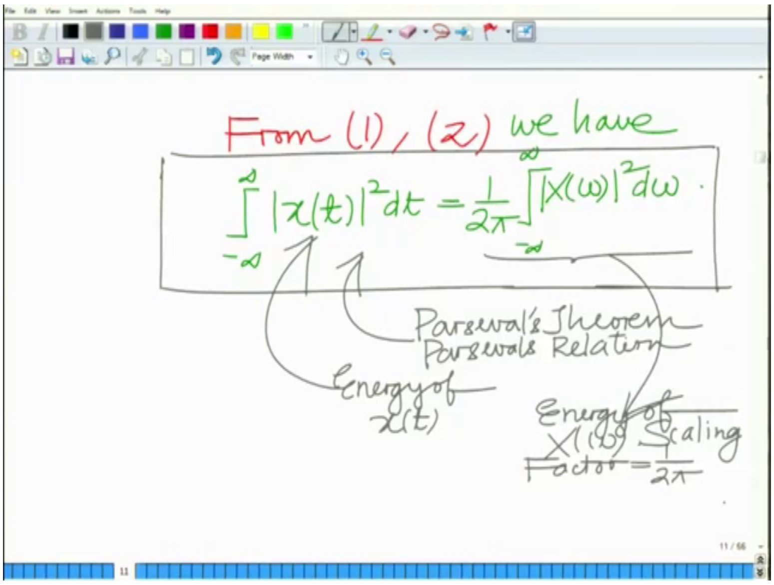

So, from 1 and 2 we have basically the energy of the signal minus infinity to infinity magnitude x tau square d tau or magnitude x t square dt equals 1 over 2 pi integral minus infinity to infinity magnitude X omega square d of d omega. And this is basically that is the energy in the time domain equals the energy in the frequency domain and this. So, this is basically your Parseval's this is basically the Parseval's theorem or the Parseval's relation.

This is the energy of x t this is the energy now you can see this is the energy of x omega except for a scaling factor of 1 over 2 pi. Notice that there is a scaling factor of 2 pi; so, this is the energy of x omega. So, the we can say energy of x t is basically the energy of x omega that is energy in the time domain equals the energy the frequency domain that is what the Parseval's framework is about.

Except that note that there is a scaling factor of 1 over 2 pi and this quantity magnitude X omega square this quantity now if you look at this quantity.

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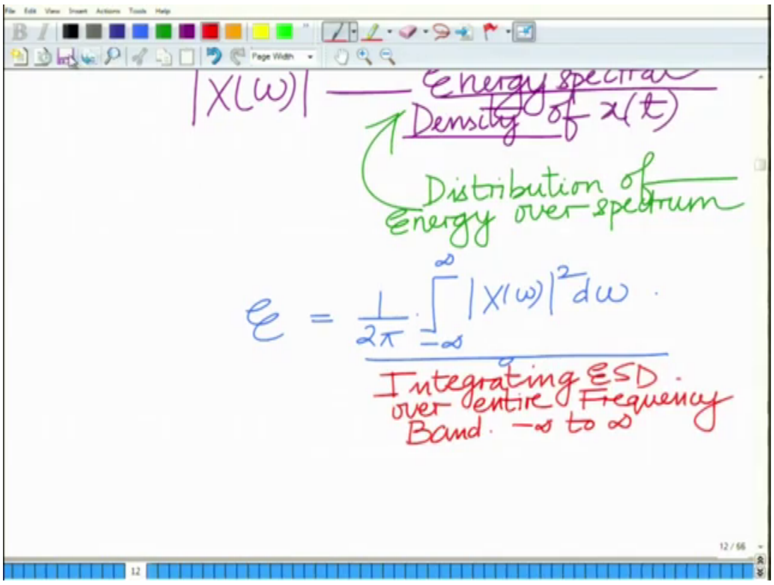

This quantity magnitude X omega square this has and very nice interpretation this is termed as the energy spectral density of x t. This is termed as the energy spectral density that is the density of the that is you can say this is the distribution of the energy or the density is remember density of an object is the distribution of the mass over the volume.

So, this energy spectral density is the distribution of the energy over the spectrum. That is how is the energy of this signal x t distributed in the various frequency bands that is given by this energy spectral density magnitude X of omega square.

And in fact, integrating this energy spectral density or the entire frequency band therefore, naturally gives right; therefore, naturally gives naturally gives the energy of the signal similar to when you integrate the density multiplied by a volume alright and integrate the density that gives the mass of the object; integrating take this taking this energy spectral density multiplying it by an infinitesimally small spectral band and integrating over the spectrum gives the energy of the signal.

And therefore, what you can see of course, there is a scaling factor of 2 pi. So, this is 1 over 2 pi minus infinity to infinity magnitude. So, this is basically integrating the Energy Spectral Density ESD over the entire frequency band minus infinity to infinity alright; So integrating the energy spectral density over the entire frequency band that gives the energy of the signal.

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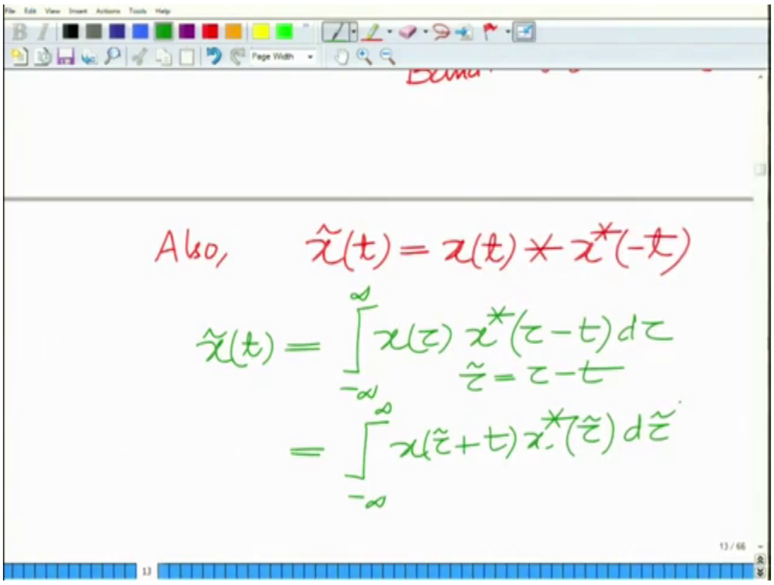

Now, also observe the following thing; we have x tilde t equals x t convolved with x conjugate of minus t. Now, therefore we can write it as x tilde t equals we already written this as x tilde t equals minus infinity to infinity; x of tau x conjugate of tau minus t d t or this can also be written as minus infinity to infinity you can say tau tilde equals tau minus t.

So, this is x of tau tilde plus t times x conjugate of tau tilde d tau tilde and this is termed as the autocorrelation look at this is explores. So, what you are doing is you are multiplying x x tau tilde correct x tau tilde or x conjugate tau tilde with a shifted version of this x conjugate tau tilde plus tau tilde plus t.

That is a advanced that is a time advanced version of this by t and you are integrating from minus infinity to infinity. So, this is a measure of the similarity between these two signals x conjugate of tau tilde and its advanced version that is x of tau tilde plus t and this is termed as the autocorrelation of the signal x t.

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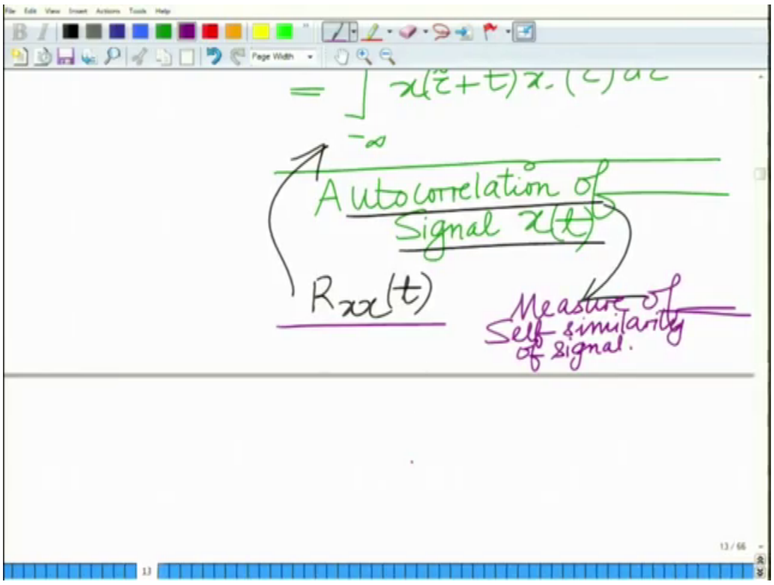

And this is this is termed as the autocorrelation of this signal x of this is termed as the autocorrelation of the signal x of t and this is denoted by R xx of t R xx of t that is autocorrelation of the signal.

And this is the measure of the similarity self similarity measure of the self similarity; there is a measure of the self similarity alright is a measure of the correlation. Also you can say is a measure of the correlation between the signal values separated by a distance of separated by the interval of t right.

How similar is the signal to itself when you shift it or when you advance it by t that is what it means intuitively. And therefore, what you can now see is that this autocorrelation.

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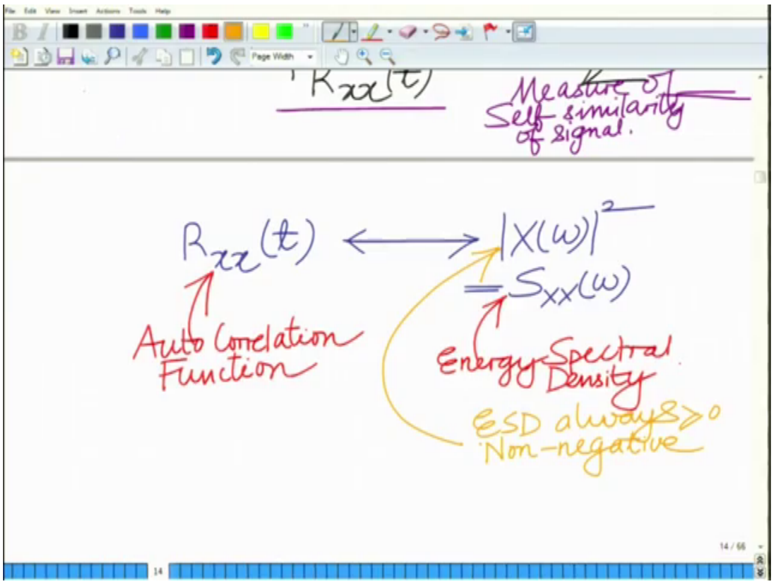

Therefore, what we have shown is that this x tilde t which we have now calling this as the autocorrelation R xx t we have shown that this as the Fourier transform X the magnitude X omega square which we also denote now by this notation S xx omega that is the energy spectral density.

So, the energy spectral density is the Fourier transform of the is the Fourier transform of the autocorrelation function. And also note that this its magnitude X of omega square. So, therefore, the ESD is always greater than or equal to 0 always greater than or equal to 0 implies that this is a real quantity and more importantly this is non-negative; the energy spectral density is nonnegative.

And the energy spectral density as I have shown as I have already said basically shows the or it gives us an idea of the distribution of the energy of the signal in the different frequency bands. So, if you want to find the energy of the signal in the particular frequency band; I have to integrate this energy spectral density or the band of interest.

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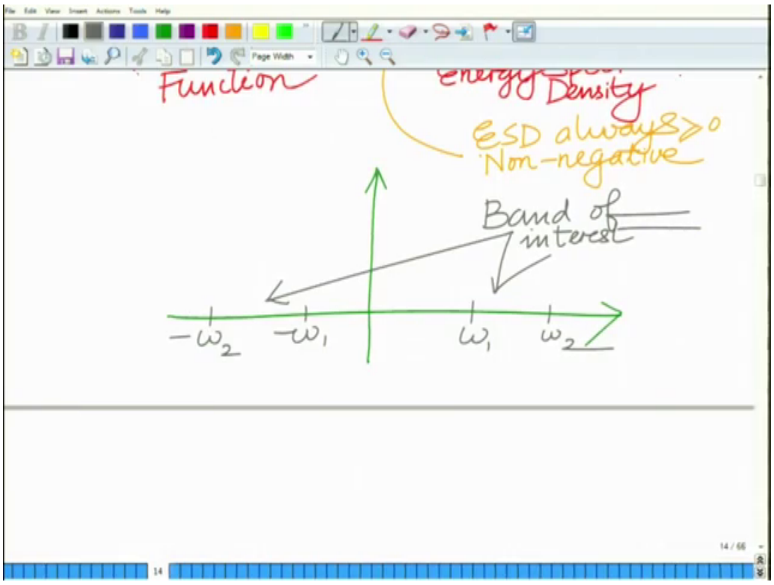

So, if I have a particular band; I am interested in finding the energy in. So, this is let us say the band omega 1 2 omega 2 I am interesting I am interested in finding. So, this is my band of interest this is my band of interest.

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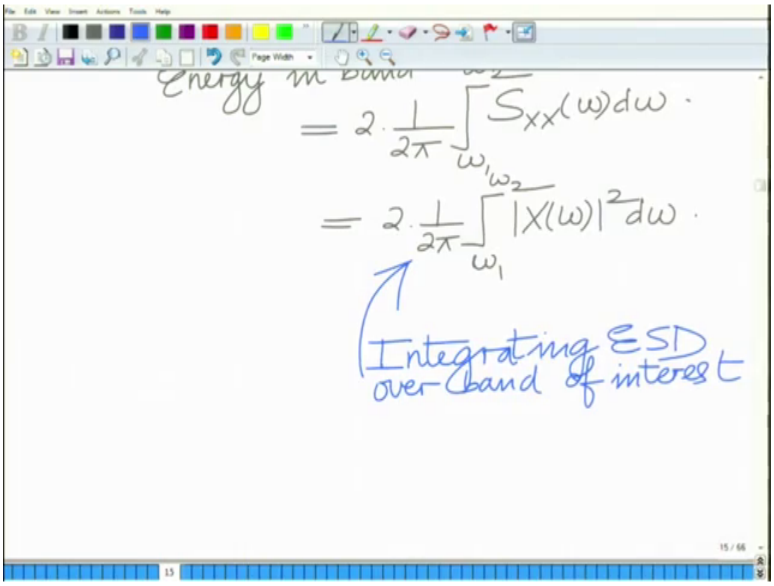

So, energy in the band equals twice because it is symmetric 1 over 2 pi from omega 1 to omega 2 the energy spectral density over that band which is basically nothing, but twice 1 over 2 pi omega 1 to omega 2 magnitude X of magnitude X omega square d omega.

So, basically what we are doing is basically we are integrating the energy spectral density over the band of; we are integrating the energy spectral density over the over the region of interest or over the spectral band of interest ok. So that is the utility of the energy spectral density.

So, basically what we have done is we have defined the autocorrelation function the autocorrelation function is the convolution of x the signal x t with x conjugate minus t that has the Fourier transform which is the energy spectral density which; characterizes the distribution of the energy of the signal across the various spectral bands ok.

Similar to the density which characterizes the distribution of the mass of an object over its volume alright; Now, consider the frequency response.

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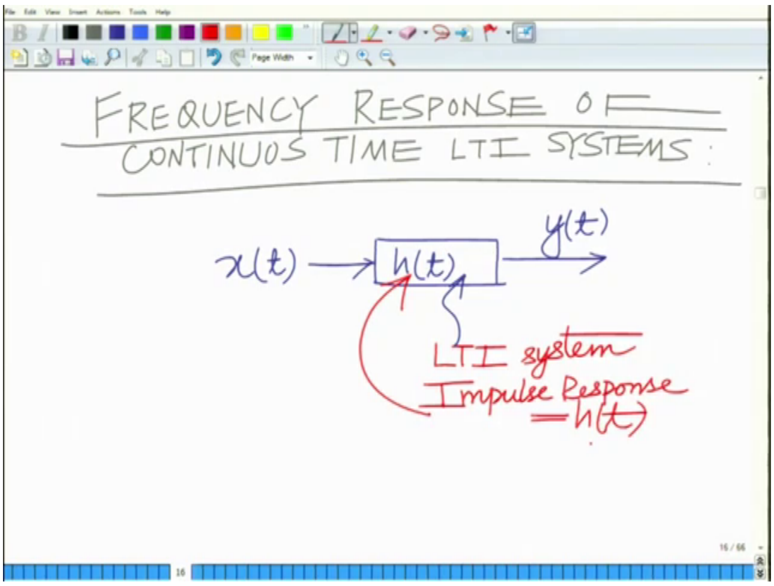

Now, we want to next we want to explore, we want to explore the frequency response of. So, we want to explore the frequency response of continuous time LTI system; since you are dealing with continuous signals and this case can be obtained.

So, let us say we have a signal x t which is input to an LTI system with impulse response h t.

So, this is my schematic of an LTI system. And the impulse response of this LTI system is h t impulse response equals h t and therefore, we know that when the signal x t is input of an impulse a system LTI system with impulse response h t, then the output y t is nothing, but the convolution of the input with the impulse response.

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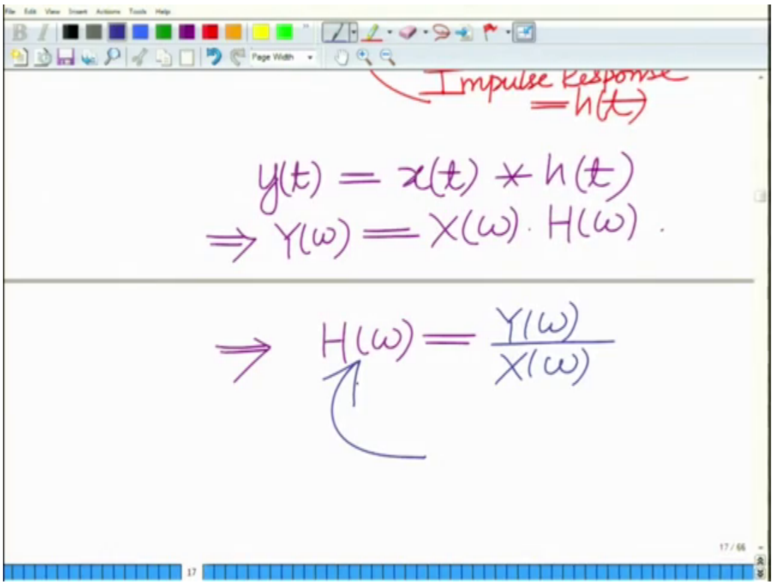

Therefore, we have y t equals x t convolved with x t. And naturally we also know that the convolution in time domain is multi leads to the multiplication on the corresponding frequency responses.

So, if you take the Fourier when you take the Fourier transform Y of omega that is the multiplication of the corresponding Fourier transforms if X omega times h H omega which implies that you have the response of the impulse function equals Y omega divided by X omega and this is termed as the transfer function frequency response of the system.

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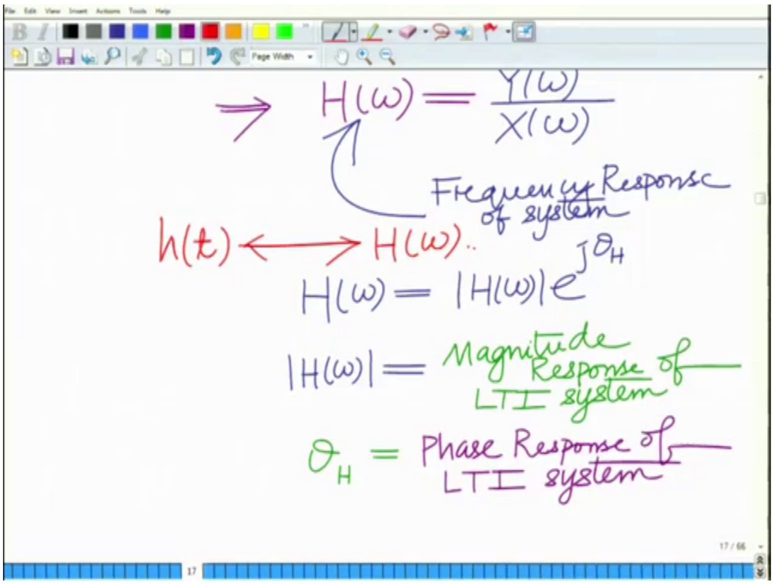

As you have seen in the context of a Laplace transform also. So, this is termed as the this is termed as the frequency response of the system. I can also characterize the magnitude and phase responses. So, you can write this in polar coordinates as magnitude of H of J H of omega times e raised to J theta H, the magnitude the component the quantity magnitude H omega.

This is termed as the magnitude response of the system magnitude response of the LTI system. And the quantity theta h this is termed as the phase response; this is termed as the magnitude response of the LTI system; theta which is termed as the phase response.

So, we have 8 of omega which is the transformer which is the frequency response of the LTI system that is nothing, but the Fourier transform the impulse response ok; the magnitude of that magnitude of H of omega is the magnitude response and theta H that is the phase of H of omega is the phase response.

And remember h of t the h of omega the frequency response is nothing, but the Fourier transform of the impulse response.

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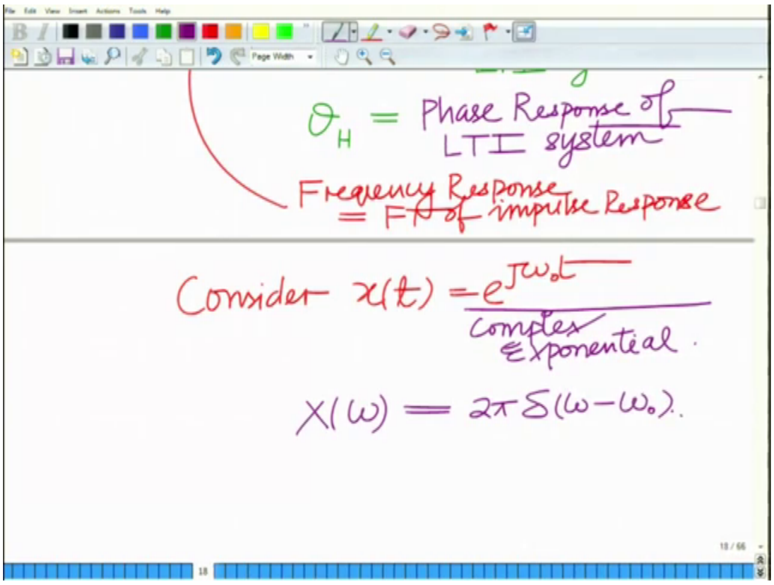

So, frequency response equals Fourier transform is the Fourier transform the impulse response ok. Now in addition consider now the input signal complex exponential; this remember we all know that this is a, this is a complex exponential the Fourier transform of this is X of omega that is the shifted impulse 2 pi delta omega minus omega or there is impulse at omega naught scaled by 2 pi.

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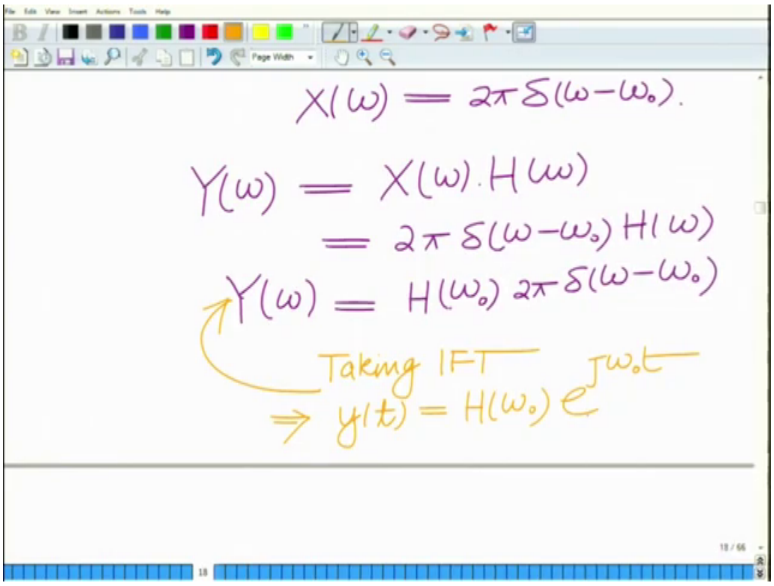

And therefore, the response to this input is Y omega equals X omega into H of omega where H is the impulse response.

That is nothing, but 2 pi delta omega minus omega naught into H of omega and you can see this is nothing, but because delta omega is 0 omega minus omega naught is 0 everywhere except where omega equals omega naught. So, this is simply H of omega naught 2 pi delta omega minus omega naught.

And now if you take the inverse Fourier transform you can see that now taking the IFT you can simply see Y of omega is a scaled version of the impulse shifted by omega naught. So, therefore, this implies y of t is simply H of omega naught times the impulse response of 2 pi delta omega minus omega naught that is e raised to j omega naught.

So, what you are seeing is that if you input a complex exponential e raised to J omega naught, the output is another complex exponential at the same frequency simply scaled right amplified or attenuated by H of omega naught alright. So, what we are seeing is an interesting property where the output is a simply a scaled version of the inputs; such a function which when input to your system alright the output is simply a scaled version of the input this is termed as an Eigen function of the system.

And therefore, what you can see is that this complex exponential e raised to J omega naught t is an Eigen function of this LTI system of any LTI system for that matter and its corresponding Eigen value is h omega naught that is Eigen; that is the Eigen value corresponding to the complex exponential e raised to J omega naught t that is angular frequency with angular frequency omega naught is h of j h of omega naught.

That is the impulse response that is the frequency response of the LTI system evaluated at omega naught.

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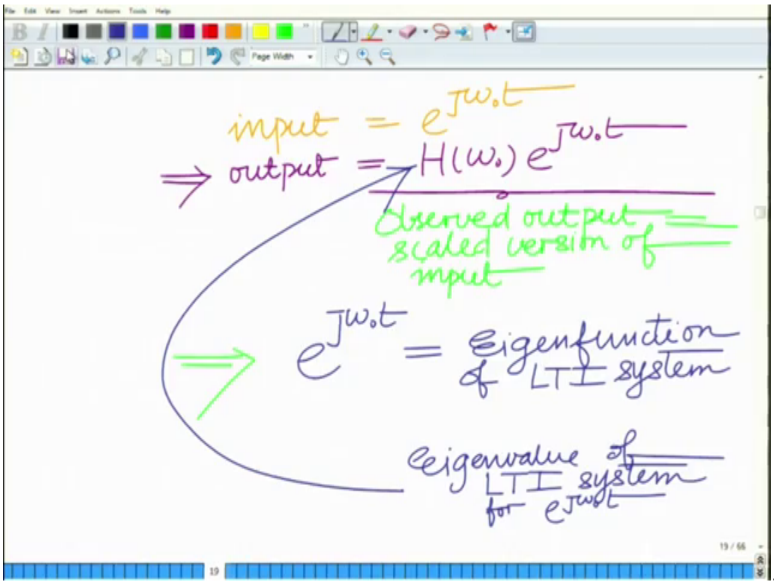

So this implies; so, we have input equals e raised to J omega naught t and this implies output equals H of omega naught, now observe output is a scaled version of input; output equals output is simply a scaled version of the input. This implies e raised to J omega naught t this is termed as a Eigen function of the this is termed as an Eigen function of the LTI system.

Because if you have input x t the output is simply k times x t and this k it is this H of omega naught this is the this is the Eigen value of LTI system for corresponding to e raised to corresponding to this Eigen signal e raised to J omega; corresponding to this Eigen function e raised to j omega naught alright.

So, this basically clarifies that this complex exponentials e raised to J omega naught t; these are the Eigen functions of the LTI system and the corresponding Eigen values are h omega naught that is the frequency response evaluated at the frequency angular frequency alright. So, in this module we have seen interesting aspects of the Fourier transform.

We have exploded further, we have defined the autocorrelation function of a signal the energy spectral density, we have seen then concept we have understood the concept of the energy spectral density if the Parseval's relation for a continuous aperiodic signal. And we have also seen the Eigen functions of LTI systems alright. So, we will stop here and continue in the subsequent modules.

Thank you very much.