Principles of Signals and Systems
Prof. Aditya K. Jagannatham
Department of Electric Engineering
Indian Institute of Technology, Kanpur

Lecture - 38
Properties of Fourier Transform - Duality, Differentiation in Time, Convolution

Hello welcome to another module in this massive open online course. So, we are looking at the Fourier analysis or Fourier transform for continuous time and aperiodic signals and we are looking at the properties of the Fourier transform.

So, let us continue looking at the properties we have looked at the time reversal property that is if x t has a Fourier transform X of omega capital X of omega then x of minus t has the Fourier transform capital X of minus omega alright.

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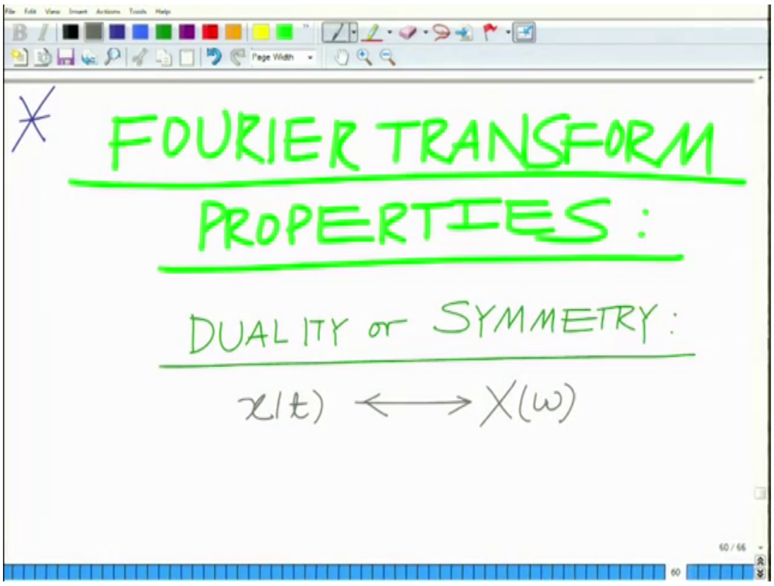

So, let us start continue our discussion on the properties on the on the properties of the Fourier transform. And well let us look at another property which is known as the duality or the what is also known as the symmetry property of the Fourier transform and what we mean by this as follows that is if x of t has the Fourier transform X of omega ok.

Now what can we say about the Fourier transform of X of what can be say about the Fourier transform of the signal capital X of t.

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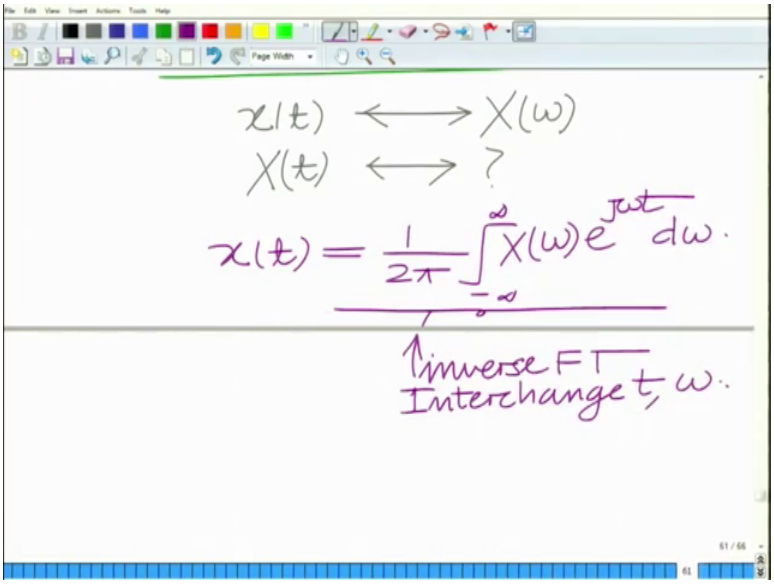

And to understand that let us first start with inverse Fourier transform that is given as 1 over 2 pi integral minus infinity to infinity X of omega is 2 J omega td omega.

Now, what I am going to do in this relation about that is the inverse Fourier transform I am going to interchange the roles of omega and t that is replaced omega by t and replace t by omega. So, remember this is the inverse Fourier transform inverse FT interchange the rules of t and omega.

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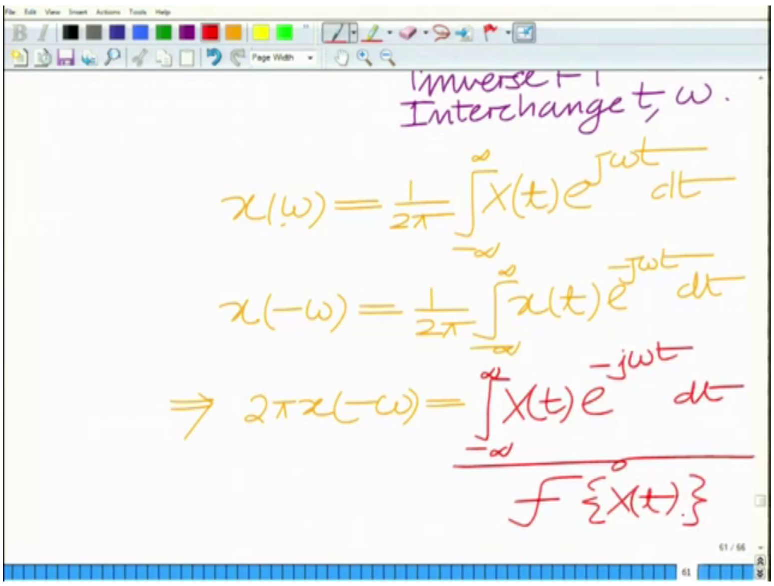

And then what we are going to have is x of omega is 1 over 2 pi minus infinity to infinity replace omega by t xt e raised to J omega t will remain t omega or basically only omega t dt.

And now if you look at x of minus omega x of minus omega will naturally be 1 over 2 pi minus infinity to infinity x of t e raised to minus J omega tdt which basically implies that 2 pi x of minus omega is integral minus infinity to infinity e raised to minus j omega t dt and this is nothing, but the Fourier transform of X t.

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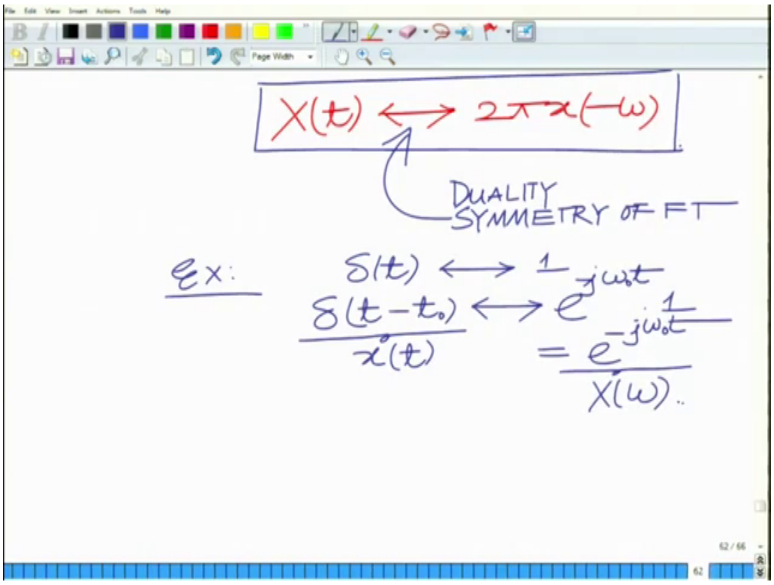

So, what we have is that capital X t has the Fourier transform 2 pi small x of minus omega where small xt has the Fourier transform capital X of omega and this is known as the symmetry or the duality property; this is known as the duality property or also the symmetry of the Fourier transform. So, what we see is that capital X t has a Fourier transform that is 2 pi small x of minus omega ok.

So, let us look at a simple example to understand this for example, just a very simple example we know that delta t has the impulse has the Fourier transform 1 which means delta t minus t naught that is by the time shifting property impulse shifted has the Fourier transform e raised to minus j omega naught t into 1 equals e raised to minus j omega naught t. Let us call this as your x of t and let us call this as your capital X of omega.

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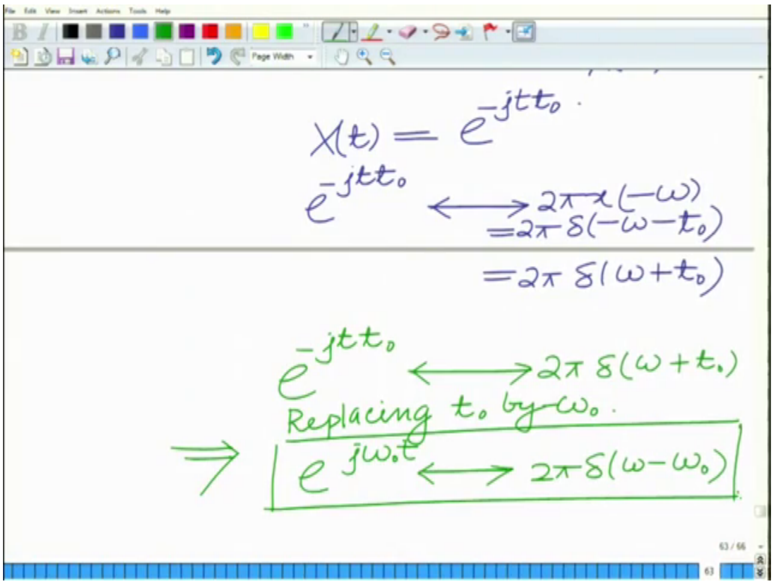

Now, using duality what happens is we have capital X of t equals replace omega by I am sorry this will be omega times t naught replace omega by t. So, this will be minus j tt naught. So, we have e raised to minus j tt naught has a Fourier transform 2 pi x of minus omega where 2 pi delta of x of t is delta x small x of t is delta t.

So, this will be delta of minus omega 2 pi delta of minus omega minus t naught which is nothing, but 2 pi delta omega plus t naught; that is delta minus omega minus t naught is basically the impulse located at omega equals minus t naught. So, we just same thing as is the same thing as delta of omega plus t naught.

So, basically what we have using duality is that e power minus j tt naught has the Fourier transform 2 pi omega plus t naught. Now, replacing t naught by omega naught we have e raised to minus j omega naught t has the or replacing it by minus omega naught. So, we have e raised to j omega naught t has the impulse response delta of omega minus omega naught that is e raised to that is the complex exponential e raised to J omega naught t.

This is an impulse located at omega naught scaled by 2 pi impulse located at omega naught scaled by 2 pi ok. So, what we have shown is that the complex exponential e raised to j omega naught t has the Fourier transform 2 pi delta omega minus omega naught. And this we have demonstrated using the property of duality alright.

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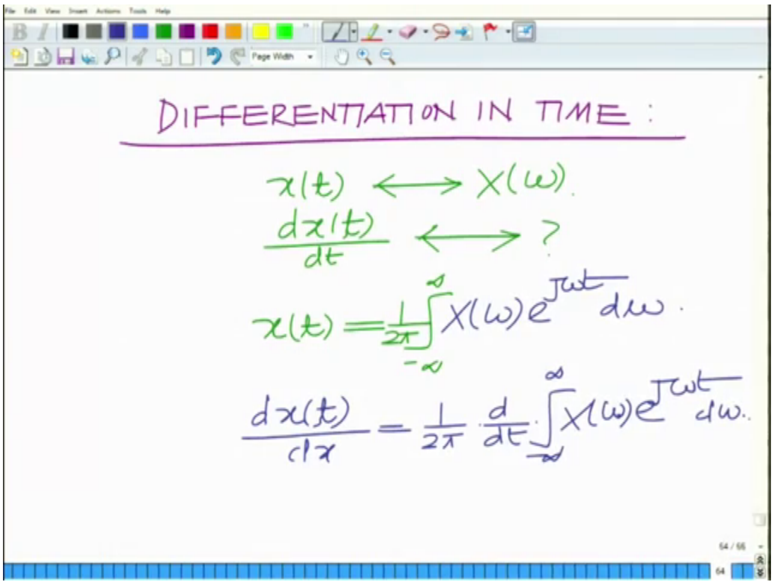

Let us look at some other simple properties for instance we have differentiation in time. Let us say we have xt has the Fourier transform X omega what can we say about dxt over dt what is a Fourier transform of this quantity?

And we can see this again let us start with the inverse Fourier transform this is minus infinity. In fact, 1 over 2 pi minus infinity to infinity X of omega e raised to J omega td omega dxt over dt equals 1 over 2 pi d over dt of minus infinity to infinity integral minus infinity to infinity x omega e raised to J omega t d omega; now interchanging the order of; so now moving the differentiation in inside the integral.

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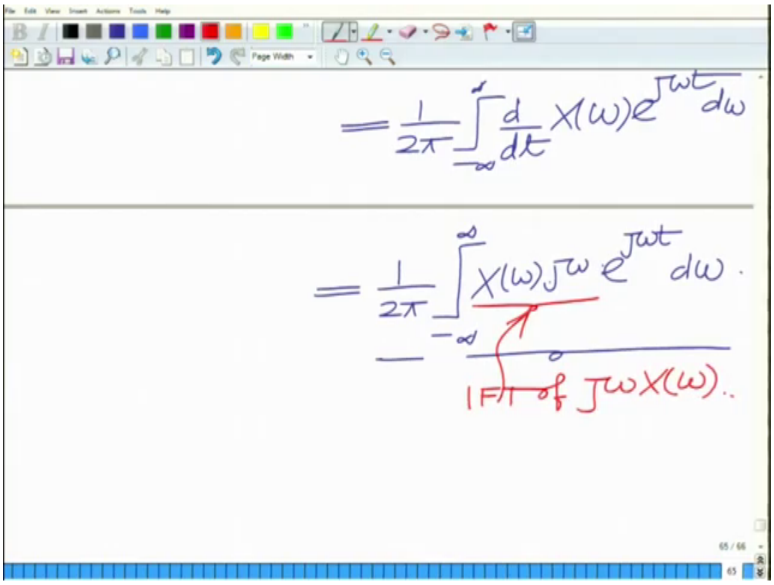

We have this is equal to minus infinity to infinity d over dt X e raised to J omega td omega which is nothing, but 1 over 2 pi minus infinity to infinity X omega derivative of e raised to J omega t with respect to dt is e raised to J omega times e raised to J omega t e times e raised to J omega t d omega.

And now you can see this quantity is the IFT this is IFT in the Inverse Fourier Transform of J omega X omega.

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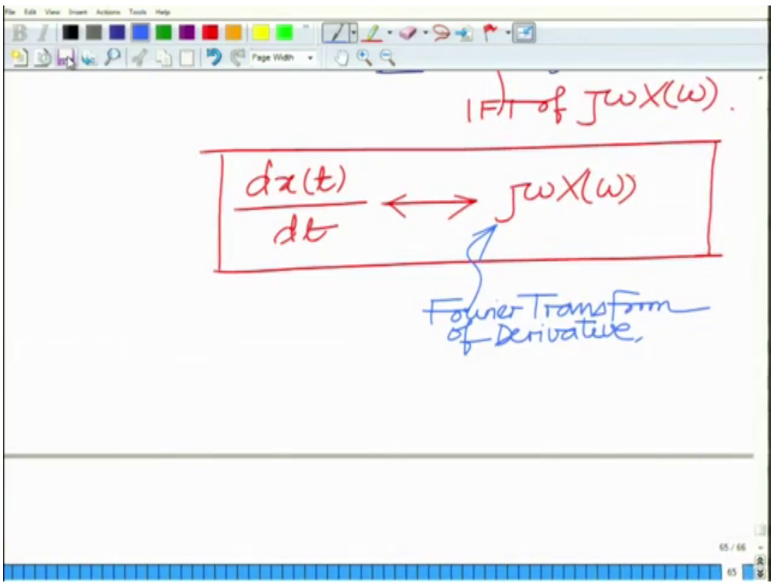

Therefore, this implies that your the derivative time derivative dxt over dt has the Fourier transform X of omega. This is the Fourier transform of the derivative; this is the Fourier transform of the derivative.

Therefore, what we observed is that differentiation in time is analog is to basically equivalent to multiplying by J omega in the frequency domain. So, if you differentiate in time the corresponding Fourier transform X omega corresponding Fourier transforms obtained by multiplying by J omega ok.

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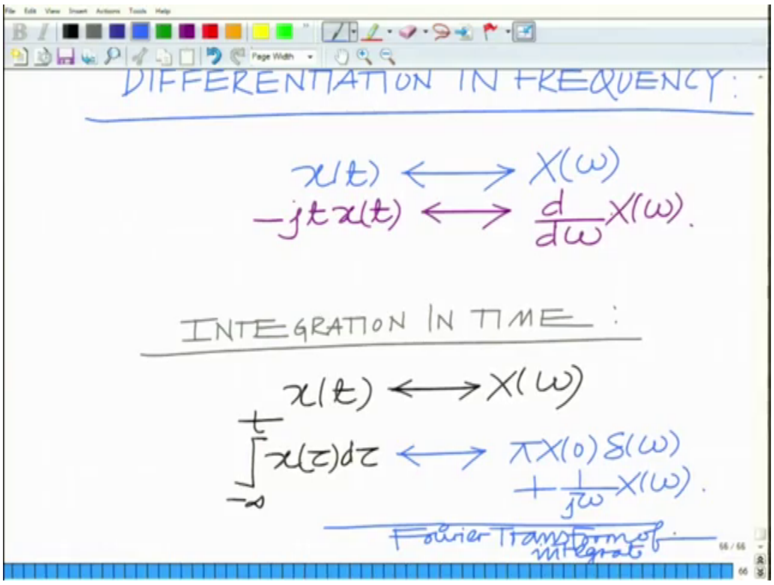

Now, similarly we also have differentiation in frequency which; obviously, you can guess will be an analogue or a dual of this that is differentiation in frequency; now when you differentiate in frequency it can show that if xt signal small xt has Fourier transform X of omega.

Then minus j t xt has the Fourier transform d over d omega times X over d or d omega of X omega that is the derivative of X omega alright the derivative of x is raised if we differentiate in the frequency with respect to omega; the corresponding time domain signal is multiplied by minus j two t. So, minus j t small x t has the Fourier transform that is the derivative d of d over d omega of X of capital X of omega and you can prove this similar to the proof for differentiation in time ok.

Now, let us come on to one of the other properties that is integration in time. The integration in time which is basically if you look at this that is if you consider the signal; let us say xt has Fourier transform X of omega. Then you can show that the integral of this integral minus infinity to t x tau d tau this has the Fourier transform pi x of 0 into delta omega plus 1 over j omega into X of omega; this is the Fourier transform when you integrate the signal in time. This is the Fourier transform venue of the integral in Fourier transform of the integral in time.

Now, we come to another important property that is the convolution property of the Fourier transform.

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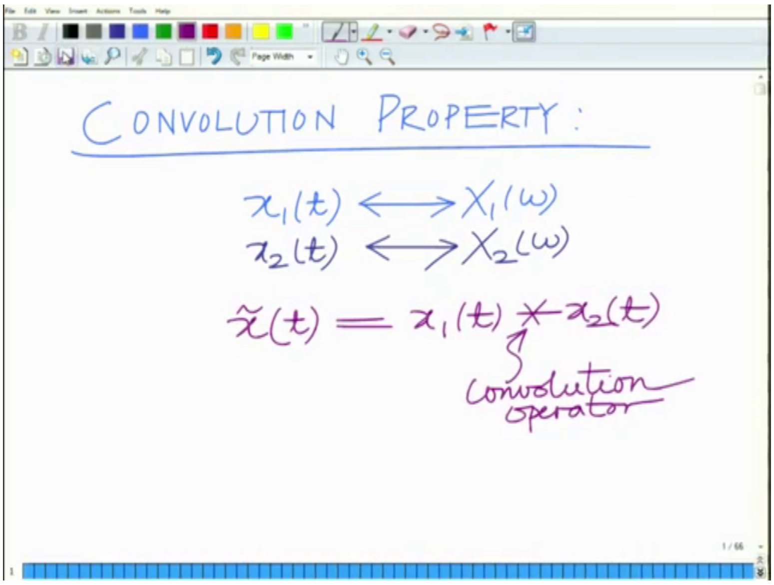

Now the convolution property of the Fourier transform; this is as follows that is we have x 1 t which has the Fourier transform X 1 omega. And we have the we have x 2 t which has the Fourier transform X 2 omega. Now what can we say about well x tilde t which is the convolution of x 1 t into x 2 this convolution of x 1 t and x 2 t.

So, remember this is the convolution operator. So, we have x 1 t which has the Fourier transform capital X 1 omega x 2 t small X 2 t which is a Fourier transform capital X 2 omega. Now we want to see what is the Fourier transform of the convolution of these two signals that is x tilde t which is the convolution of x 1 t with x 2 t.

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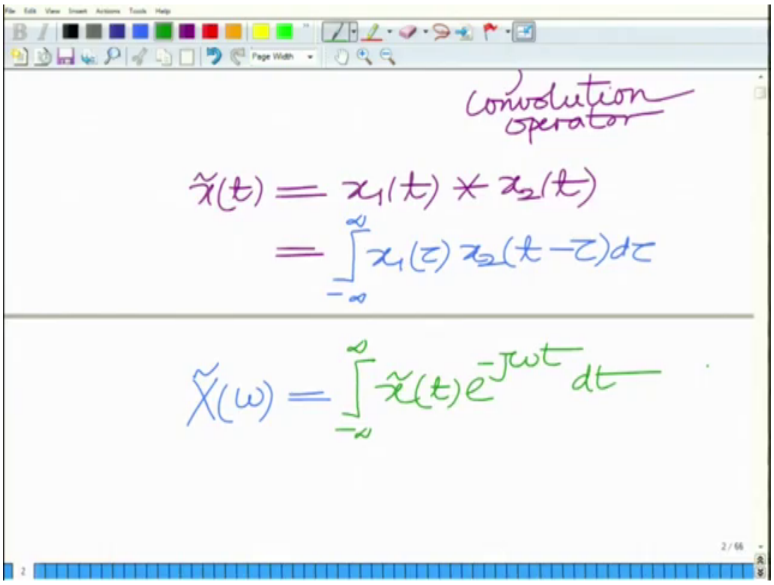

And this can be derived as follows or this can be shown as follows that is you have x tilde t equals convolution of x 1 t with x 2 t which is basically I can write this as minus infinity to infinity x 1 tau x 2 t minus tau d tau. And therefore, the Fourier transform of this X tilde omega this will be minus infinity to infinity well x tilde t e raised to minus J omega t dt.

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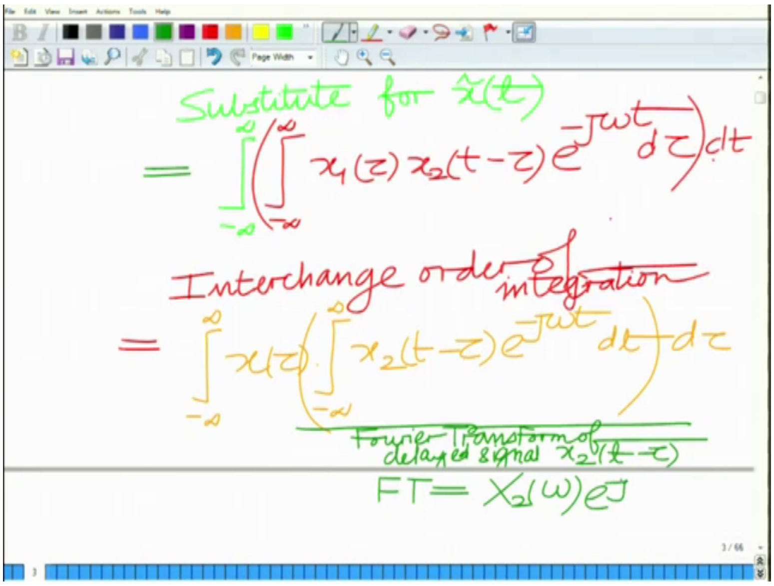

Now, substitute for x tilde from above that is you substitute expression substitute for x tilde t this will be integral minus infinity to infinity integral minus infinity to infinity;

well I am first substituting for x tilde t. Now x tilde t will be x 1 tau, x 2 t minus tau e raised to minus J omega t minus J omega t d tau inner integral is with respect to d tau outer integral is with respect to t.

Now, what we do is now we interchange the order of integration. Now if you interchange the order of integration, what you will observe is this becomes the outer integral with will become with respect to tau. So, this will become and it does not depend on t; so, you have x 1 tau and the inner integral will be minus infinity to infinity x 2 t minus tau e raised to minus J omega t dt times d tau that is outer integral has now become with respect to tau.

Now, if you look at this is the Fourier transform you can see that this is the integral minus infinity to infinity x 2 t minus tau e raised to minus J omega t dt this is nothing, but the Fourier transform of the delayed signal x 2 t minus tau that is x 2 t delayed by tau. Therefore, the Fourier transform this will be capital X 2 omega correct into e raised to minus J omega tau. So, this is the Fourier transform of the delayed signal x 2 t minus tau and the FT is Fourier transforms of X 2 omega e raised to minus J omega tau.

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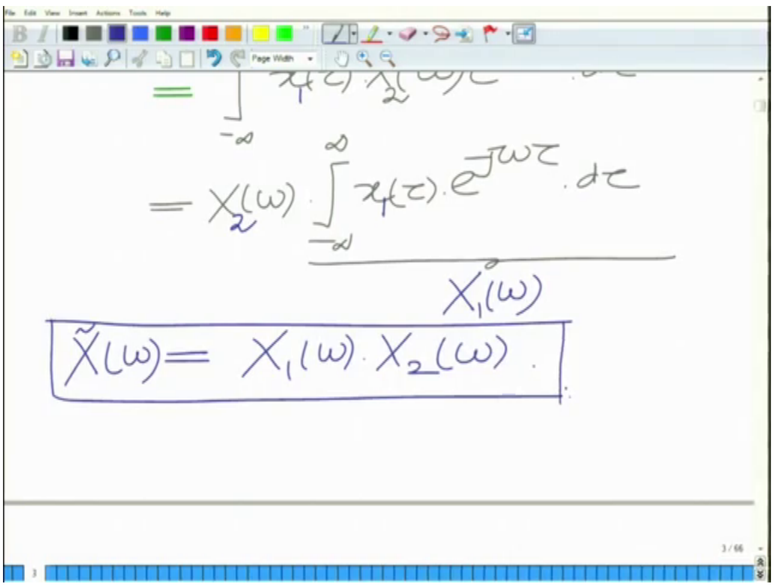

Which is equal to therefore, minus infinity to infinity x tau substitute X omega or X 2 omega e raised to minus J tau d tau now X 2 omega does not depend on tau. So, that comes out of the integral minus infinity to infinity x tau e raised to minus J omega tau. And now you can see this quantity integral x tau e raised to minus J omega tau d tau in fact this is nothing, but x 1 I am missing an x 1 over here that I have replaced over here that is x 1 tau.

So, this will be X 1 of omega and in fact, this is X 2 of omega. So, at this point you have X 1 omega times X 2 omega and that is your x tilde of omega which is the Fourier transform of the convolution of two signals.

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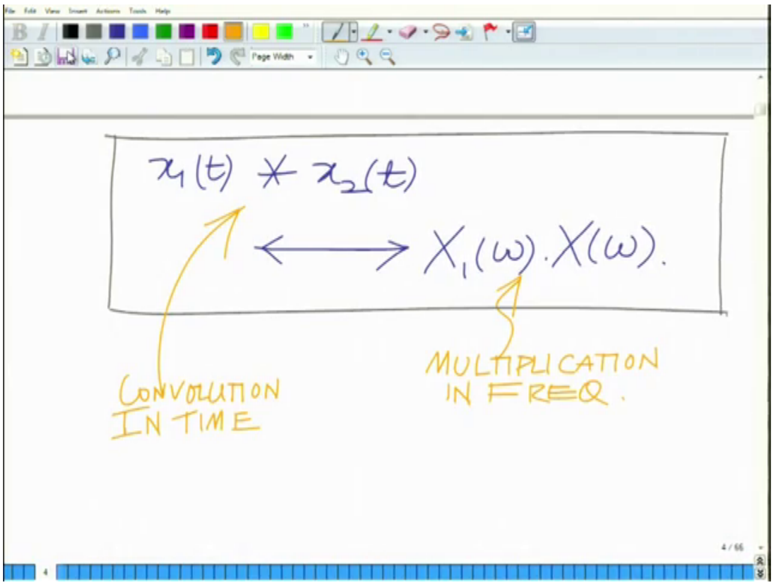

So, therefore, finally, what we have is that x 1 t convolved with x 2 t has the Fourier transform X 1 omega times X 2 omega. And therefore, what we can see is that convolution in time leads to multiplication in the frequency domain. And this is something that we have seen many times before at this point which is very interesting which one of the most interesting properties.

So, on the left you have convolution and time and on the right you have multiplication in multiplication in the frequency domain. So, when you convolve two signals in the time domain the corresponding Fourier transforms get multiplied in the frequency two. Alright in this is one of the most interesting and important properties which makes the Fourier transform very useful and analysis with the Fourier transform very tractable or very convenient alright.

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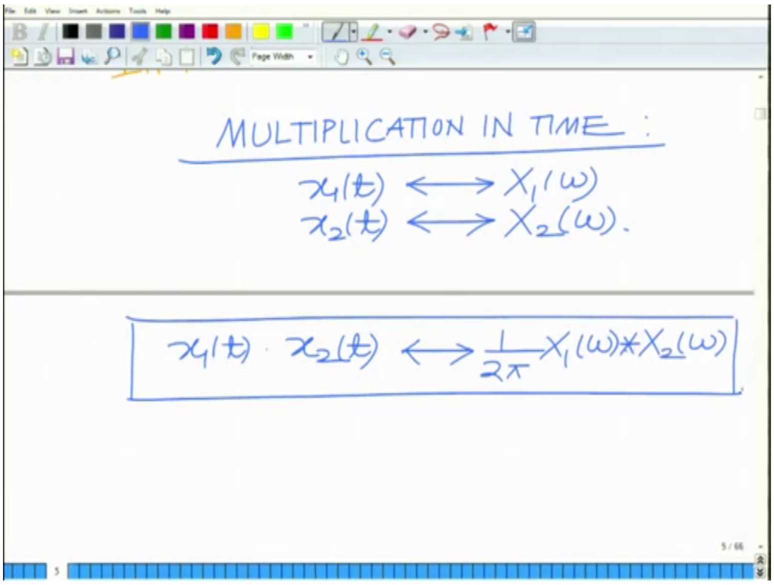

And similarly one last property which is the dual of this that is the multiplication that is what happens when you multiply in time; when you multiply in time that is similar to previous scenario you have x 1 t which has a Fourier transform capital X 1 omega x 2 t which has the Fourier transform X 2 omega.

Then x 1 t into x 2 t that is the multiplication of these two signals has the Fourier transform 1 over 2 pi X 1 into X 1 into I am sorry 1 over 2 pi X 1 omega convolved with X 2 omega ok. So, you have multiplication in time naturally leads to convolution and frequency.

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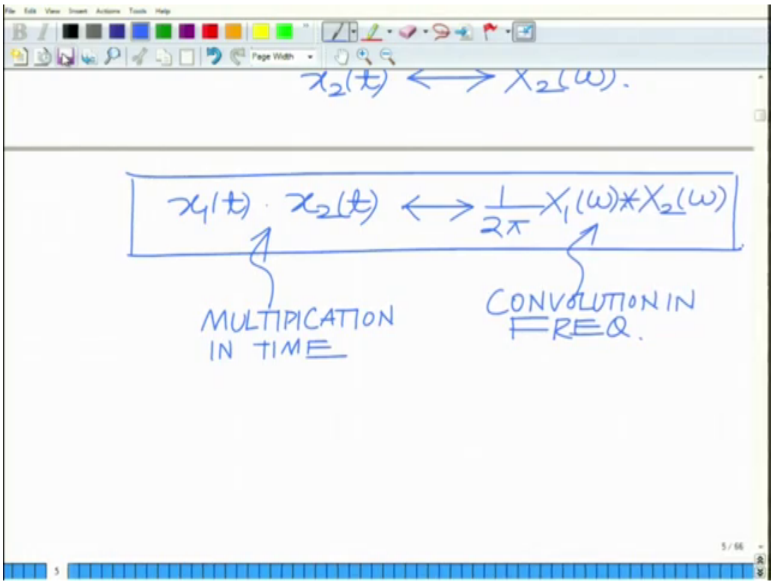

Multiplication in time, that leads to convolution in frequency, in the frequency domain alright.

So, what we have seen in this module is we have continued our discussion of the properties of the Fourier transform. We have looked at the differentiation in the time and frequency domains, also integration in the time domain and we have also looked at the convolution property of the Fourier transform alright; So, other aspects in the subsequent modules.

Thank you very much.