Principles of Signals and Systems
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Lecture – 34
Fourier Analysis: Complex Exponential Fourier Series, Trigonometric Fourier Series – Even and Odd Signals

Hello welcome to another module in this massive open online course all right. So, we are looking at the properties of signals and systems and in particular we are trying to interrupt we are focusing on introducing the Fourier analysis or the Fourier transform right as a very viable and a convenient method to understand and analyze the properties various properties of signals and systems all right.

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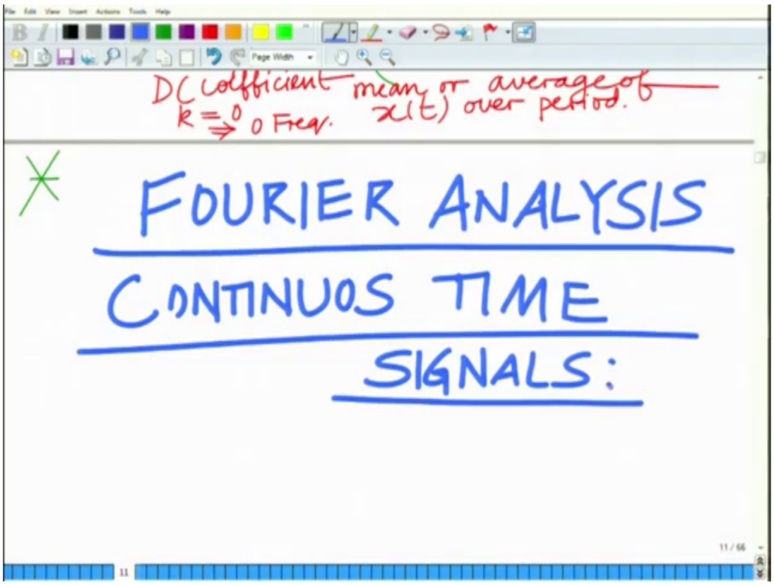

So, let us continue our discussion on this Fourier analysis and of course, we are looking initially at the Fourier analysis for continuous time signals and systems, subsequently we will look at the Fourier analysis for discrete time system signals and systems. So, we are looking at the Fourier analysis and this is for continuous, continuous time signals.

In particular we are looking at the Fourier analysis for a, we are looking at the Fourier series.

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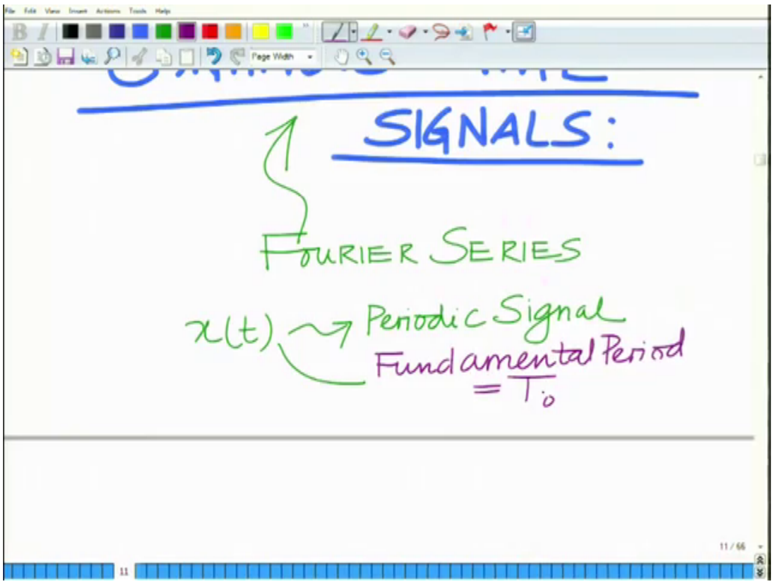

We are looking at we have started with what is known as the Fourier series and the Fourier series representation is as follows. For any periodic signal remember the Fourier series exists is defined for a periodic signal, continuous periodic signal with a fundamental period that is equal to t naught, with the fundamental period equals t naught.

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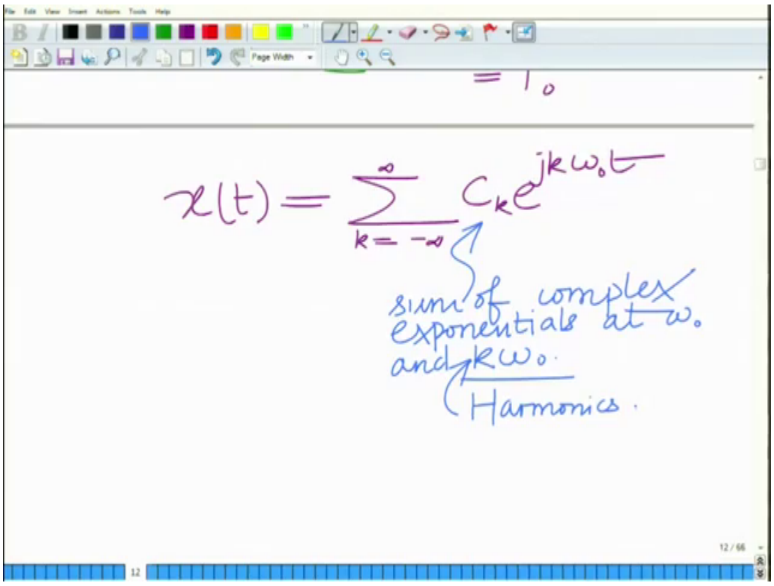

And it can be shown that such a periodic signal with fundamental period t naught can be expressed as the sum of complex exponentials, correct that is what you see and this is important to understand. It can be expressed as sum of an infinite number of complex exponentials at the fundamental frequency omega naught and then its various harmonics, that is frequencies k omega naught. Where, k is an integer multiple of omega correct, k omega naught, where k omega naught is an integer multiple of omega naught there is k is an integer ok.

So, this sum can be expressed as summation k equal to minus infinity to infinity C k e is to j k omega naught t correct, sum of complex exponentials and remember these are the harmonics that is when you have k omega naught these are known as the harmonics and remember omega naught is the fundamental angular frequency.

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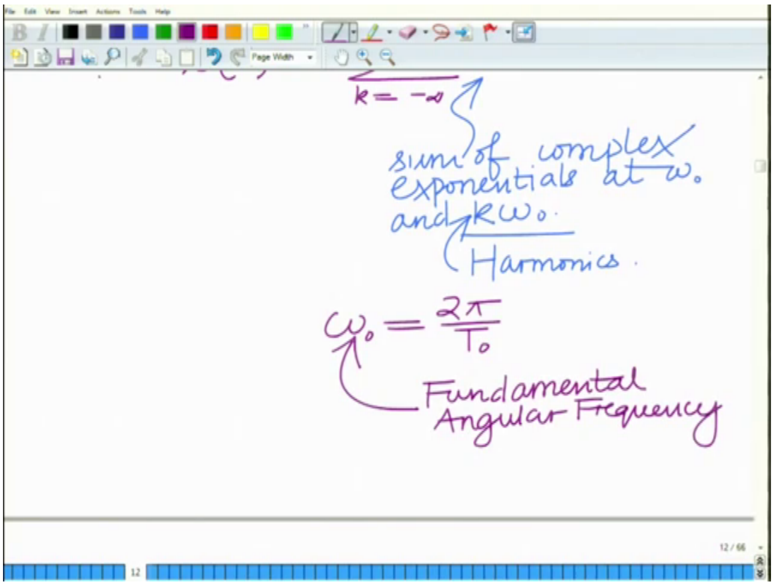

Omega naught equals 2 pi by t naught, this is important to remember this is the fundamental; this is the fundamental angular frequency and the coefficient C k in the Fourier series.

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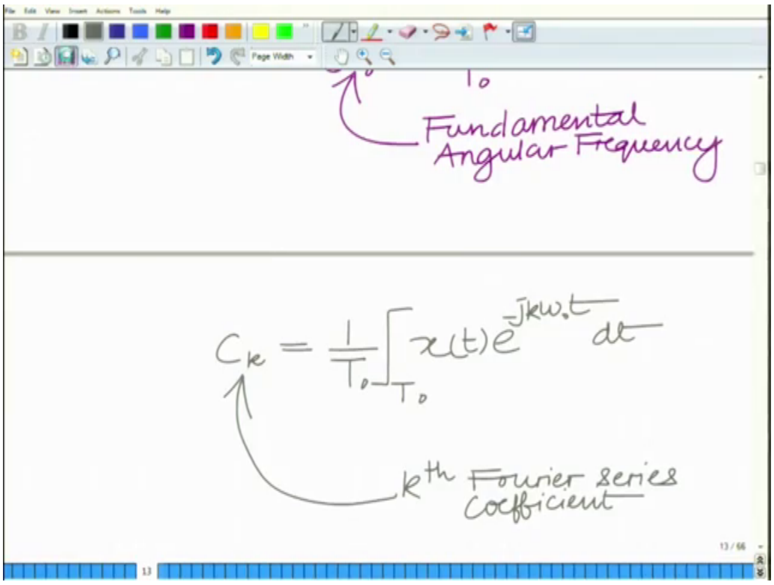

The coefficient C k are given as one over t naught integral over any period T naught x t e raise to minus J k omega naught t dt, this is the coefficient kth Fourier series coefficient ok.

This is your kth, this is your kth Fourier series coefficient and remember we have derived this yesterday the formula to derive the kth Fourier series coefficient of x t we have looked at that and how to derive that formula correct or how to derive this expression that we have looked at yesterday. Today let us continue our discussion on this Fourier series coefficient, now let assume that x t.

Now, let us look at some of the properties of this Fourier series representation.

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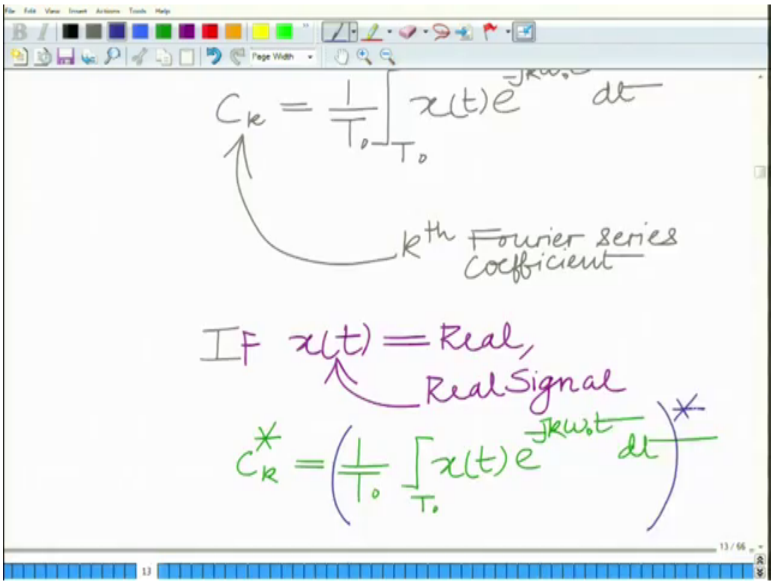

Now, if x t, now one of the properties an important property x t is real; that means, imaginary part of x that is x t is a real function or a real signal if x t is a real signal then we have, remember we have this already C k equals integral over T naught x t e raised to minus J k omega naught t dt. Now, if I take C k conjugate here, that will be the conjugate on the right hand side and now if I bring the conjugate operation inside.

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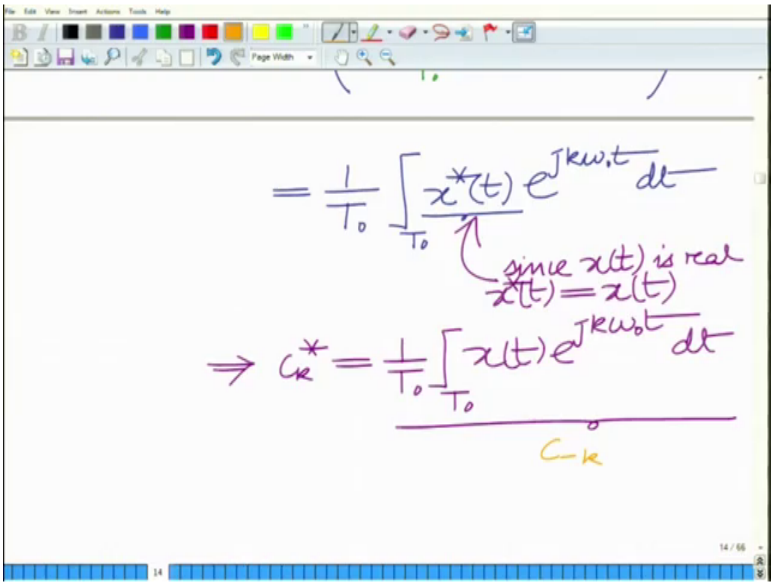

Now; obviously, t naught is real that is the time.

So, ill simply have one over T naught integral over T naught, x conjugate T e raised to minus j k omega naught T conjugate of that is e raised to J k omega naught t d t. Now, since x is real x t is real x t conjugate equals x t, x conjugate equal to x t which implies ck conjugate equals 1 over T naught, ck conjugate equals 1 over T naught integral over any period T naught x t, since x t conjugate equals x t e raised to J k omega naught T d t but.

Now, if you look at this, this is nothing, but c of minus k the Fourier series coefficient at the integer minus k that is one over T naught integral or t naught x t e raised to plus j k omega naught T d t that is nothing, but the Fourier series coefficients c minus k. So, this implies for a real signal x t, C conjugate of k equals c minus of k.

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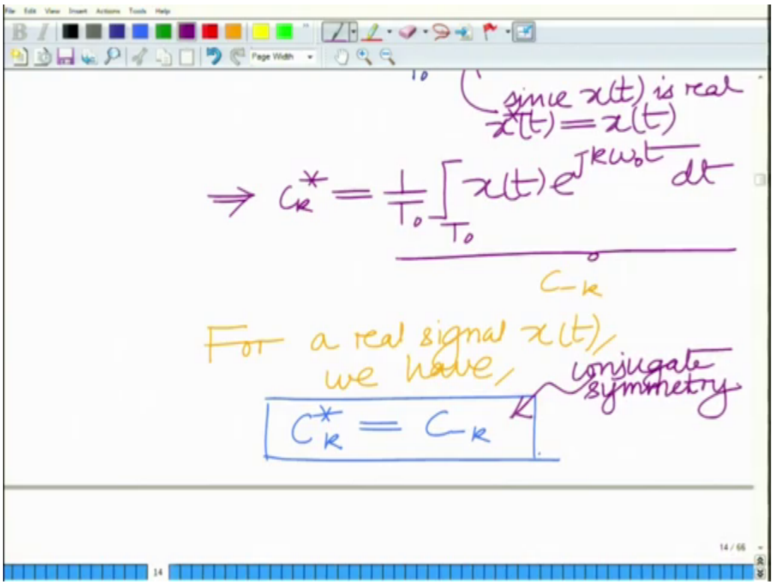

That is the Fourier series coefficients exhibit conjugate symmetry and this is an important property for real signal x t we have c conjugate k equals c minus of k, that is the exhibit conjugate symmetry, that is the exhibit conjugate symmetry for a real signal x t ok.

So, that is an interesting and that is also an important property that is, which means that.

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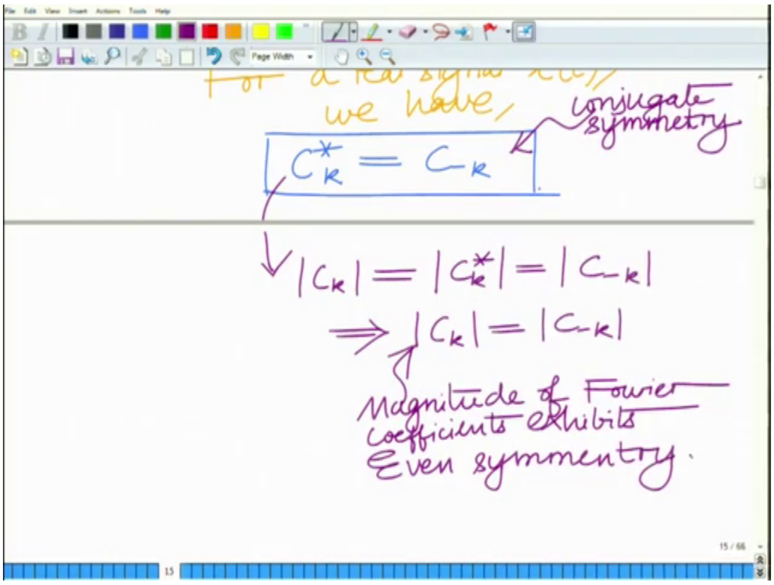

Now, if you look at this magnitude of ck that is if we look at the magnitude spectrum magnitude of ck correct because magnitude of C k equals magnitude of ck conjugate which is equal to since ck conjugate equals c minus k. So, that is will be magnitude of c minus which is implies that the magnitude spectrum will have even symmetry. So, implies magnitude of C k that is the magnitude of Fourier is, magnitude of Fourier coefficients exhibits and similarly if you look at the phase spectrum the angle of ck equals minus angle of ck conjugate and then ck conjugate equals C minus k.

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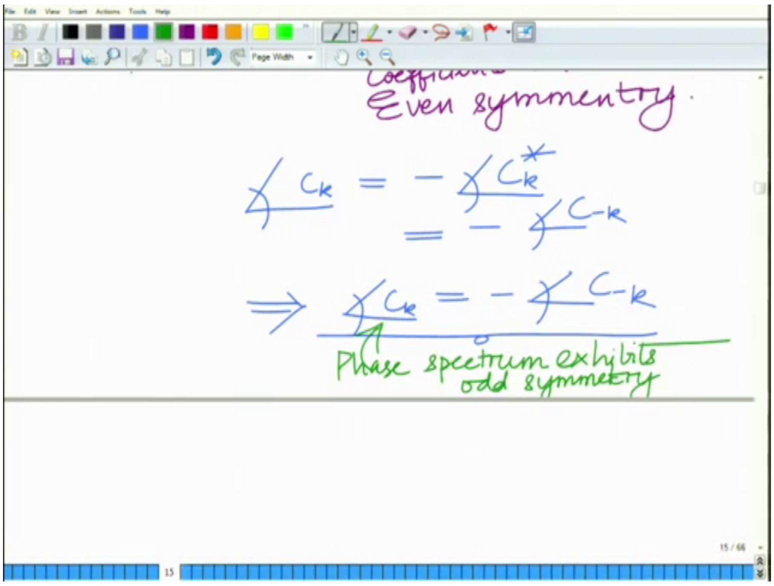

So, this is minus of angle of c minus k which implies the phase angle of ck equals minus of angle of. So, the phase spectrum that is if you look at, what that is, that is what we mean by that is the phase of the spectral coefficients, remember we said we can think of the Fourier series as the spectral domain representation all right. So, if you look at the phase spectrum of the phase of these different spectral coefficients that exhibits an odd symmetry. So, the phase spectrum, the phase spectrum exhibits an odd symmetry. So, that is an interesting property.

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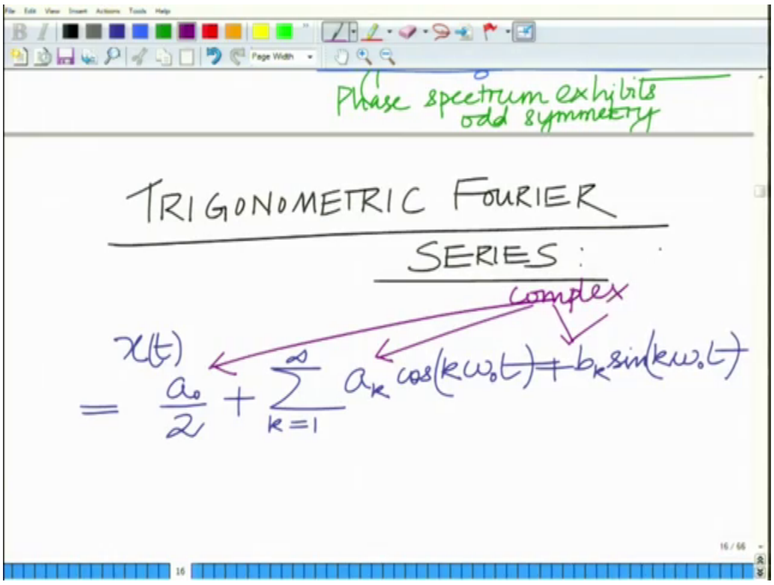

Let us now move to a different representation although related this is known as the trigonometric Fourier series, the trigonometric and this is simply as follows that is if I have the signal x t which is a periodic signal that can be represented as a naught over 2 plus summation k equals to 1 to infinity a k cosine k omega naught t plus bk. Sin k omega naught T this is the trigonometry these are; obviously, these are trigonometric functions that is the cosine and sin these are not complex exponentials.

Therefore this is known as the trigonometric Fourier series representation, it is convenient we are going to look at its properties and these coefficients once again these can be complex. The a k s, these coefficients can be complex for a general signal x t these can be complex and what is a k?

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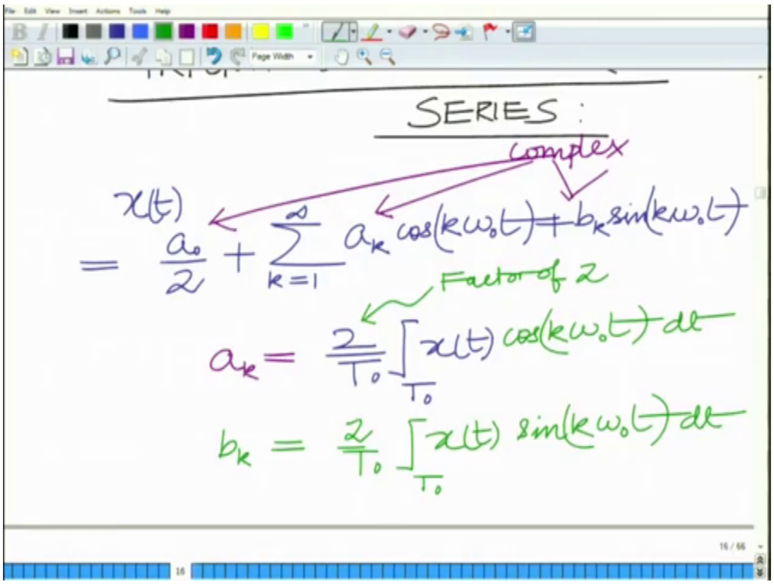

And now you canalso derive this similar to how we derived the expression for ck, you will realize that once again the cosines and the sines are orthogonal correct, that is if you multiply cosine k omega naught t by any sine l omega naught t or any cosine l omega naught t and integrate over one period that is t naught the product vanishes all right.

So, you can see that these basis functions the cosines and sines is a chosen for a particular reason because these are orthogonal. Correct, similar to the complex exponentials at different that is a different harmonics that is k omega naught and l omega naught orthogonal these are also orthogonal and you can based on that property you can readily derive what are the expressions for these coefficients of the trigonometric Fourier series ok.

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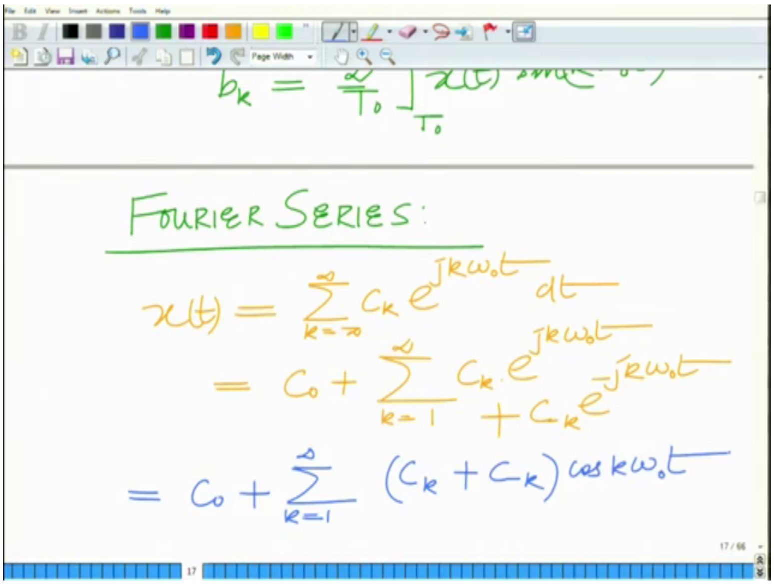

So, that is a k equals 2 over t naught summation x, integral x t you know cosine k omega naught T. Notice that there is a factor of 2 here and this you will see in the derivation, one should not forget this and b k equals summation twice over t naught integral over t naught x t sin k omega naught t d t and now if you compare with the Fourier series. Now, let us look at these 2 things side by side now compared with the Fourier series that is the general Fourier series with the complex exponential.

If you compare that you can see we have the Fourier series is summation k equals minus infinity to infinity ck e raised to j k omega naught t d t which I can write as summation for lets first isolate the term corresponding to k equal to 0. So, that is 0 plus summation k equal to no for each k that is non 0 you have 2 terms, one is ck e raised to j k omega naught t plus the other is c minus k e raised to minus J k omega naught and this second further simplified as follows c naught.

I can further simplify this as follows, c naught plus summation k equal to 1 to infinity that is you have well ck e raised to j k omega naught t plus c minus k e raised to minus J omega naught t. So, that will give you well that will give you ck plus c minus k because you can write e raise to j k omega naught t as cosine k omega naught t plus J times C k minus c minus k sin k omega naught t ok.

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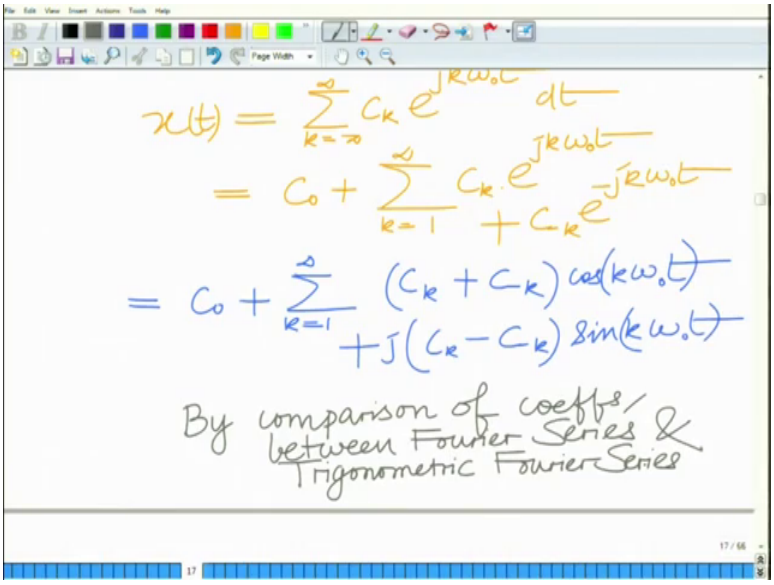

And therefore, what you can see now by simple comparison of coefficients you can see by comparing, by comparison of coefficients between the Fourier series and the trigonometric Fourier series. Comparison of the coefficients between the Fourier series and the trigonometric Fourier series you can see that.

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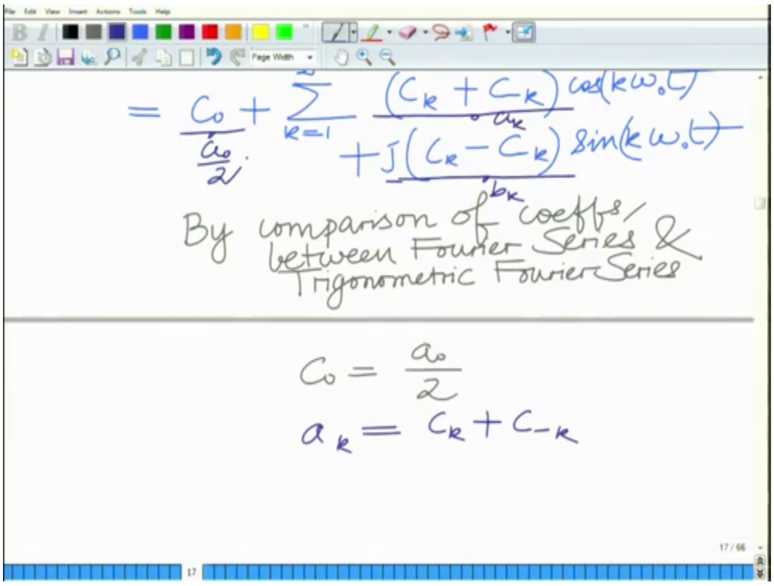

You know first of all you can see that c naught the dc coefficient equals corresponding to 0 frequency equals a naught by 2 plus and you can see a k the coefficient of cosine k omega naught T, that is that is equal to ck plus c minus k look at this, this is your a k and this is your b k that is coefficient of sin k and this is your a naught by 2.

So, c naught equals a naught by 2 a k equals ck plus c minus k and b k equals J times C k minus c minus k.

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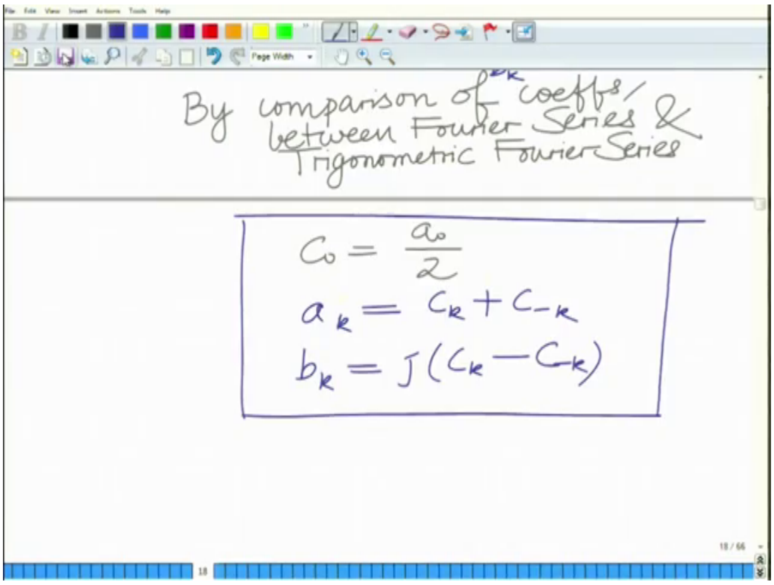

That is the relation between the Fourier series coefficients of the relations between the coefficients, there is a relationship between the coefficients of the Fourier series and the trigonometric Fourier series all right. Both of them are equivalent from one through the coefficients of 1 you can get the other ok.

And now; obviously, you can simplify this other way also that is finding the coefficients of the Fourier series from the trigonometric Fourier series.

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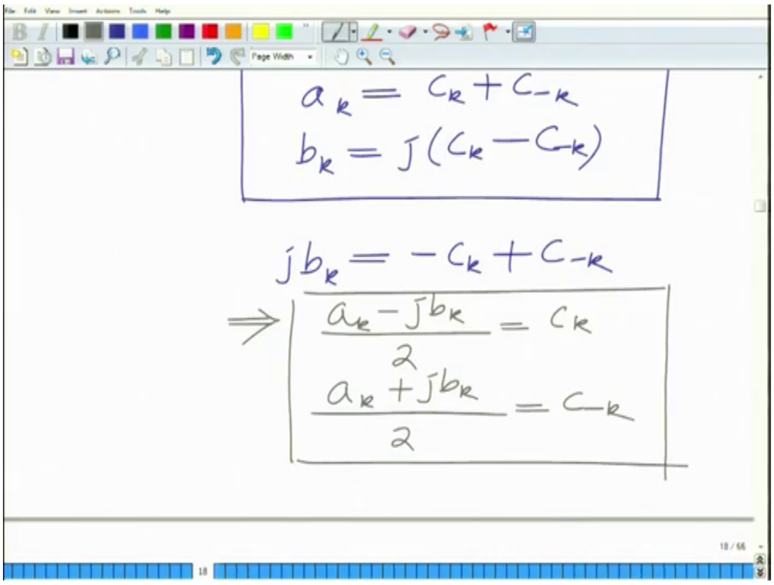

So, this gives the trigonometric Fourier series in terms of the Fourier series, now Fourier series in terms of the trigonometric Fourier series you can see that J times b k equals J into J that is minus 1, minus of ck plus c minus k which implies that a k plus J times b k or a k minus J times b k divided by 2 equal to ck and a k plus J times b k divided by 2 equals c of minus k ok.

So, these are the coefficients relationship of the Fourier series coefficients, in terms of the coefficients of that trigonometric series.

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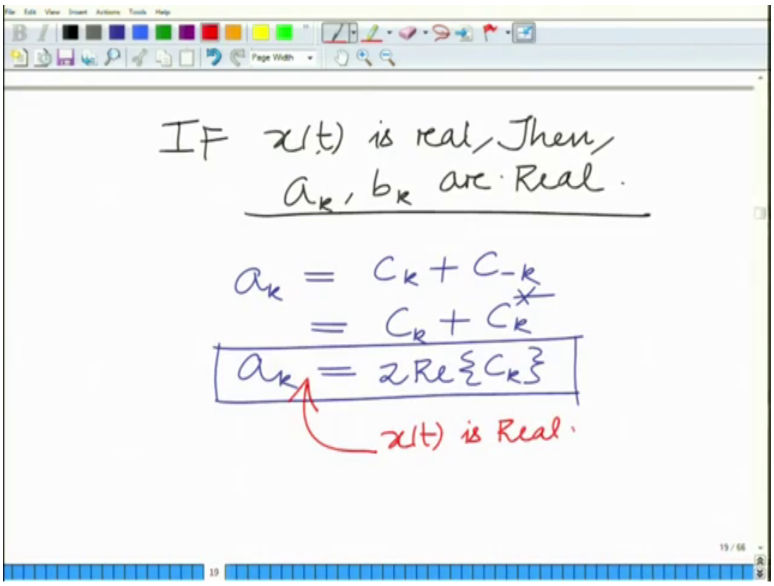

Now, once again if x t and b t are real or if x t is real then the coefficients of the trigonometric series a k and b k are real. If x t is real then a k and b k the different ak s and b k is that is the coefficients in the trigonometric Fourier series these are real quantities all right and also we have remember, you can look at it this way a k look at the relationship between a k and ck a k equals remember ck plus c minus k, but if x t is real then we have C k equals c minus k conjugate which means we have ck plus c minus k ck conjugate.

So, this is ck conjugate. So, this is ck plus ck conjugate equals twice the real part of so ck plus ck conjugate is twice a real part of ck, but remember this only if x t is real only when x t equal x t equals.

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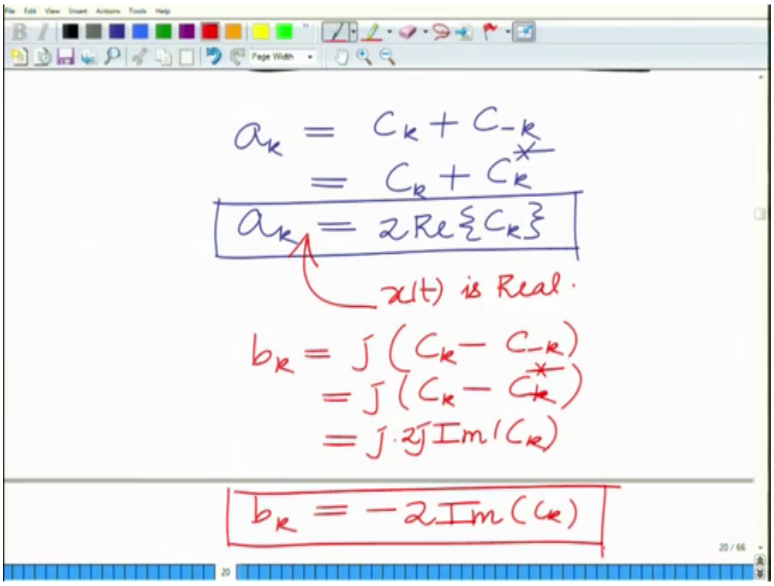

Now, in addition now remember b k equals J times ck minus c minus k equals J times ck minus, remember c minus k C k conjugate. So, ck minus ck conjugate equals J into J times the imaginary part off or twice the imaginary or twice J times the imaginary part of C k which is minus twice the imaginary part of c k. So, b k equals minus twice the imaginary part of C k for real arguments again for a real signal x t, for a real signal x t.

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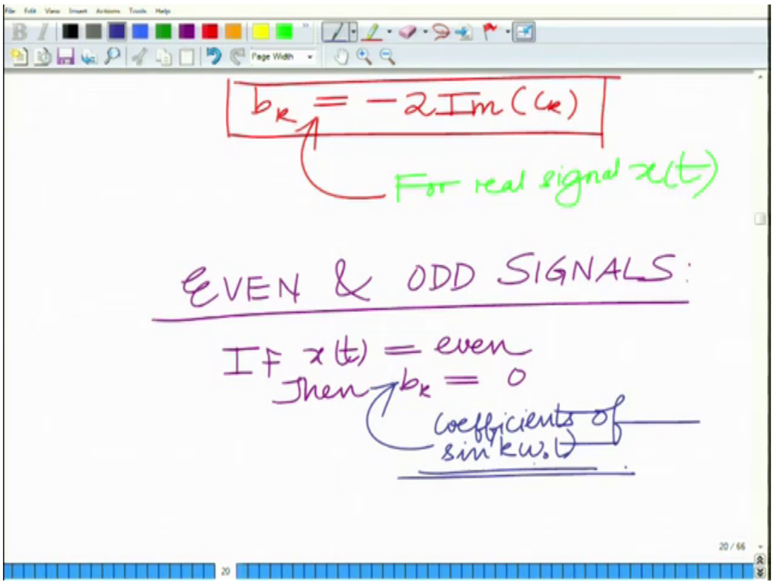

Now, what about even and odd signals let us look at some other interesting property that is for even and odd signals.

Now if or even and odd signals, if x t is even remember then in the trigonometric Fourier series it is easy to infer that you know b k is equal to 0 remember these b ks are coefficients of the sin functions and remember sin of k omega naught t this is an odd function, correct.

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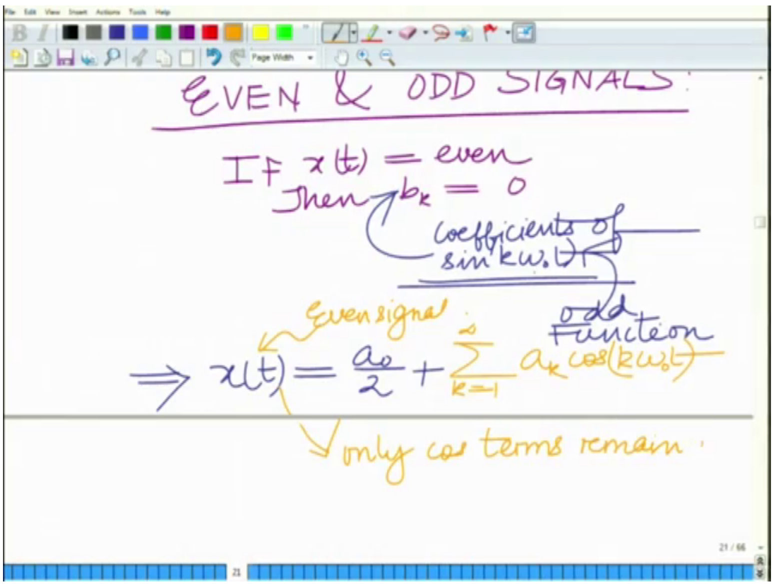

Sin of minus of x is minus of sin of x. So, sin function is an odd function. So, if x t is even naturally you would expect the coefficients of the sinusoid the pure sin, that is sin k omega naught t that is the b ks to be 0.

Again this can be formally proved also it is not very difficult to prove this and you can show that therefore, which implies the trigonometric Fourier series is given as x t equals a naught by 2 plus summation k equal to 1 to infinity a k cosine k times omega naught t that is because the b ks are 0. So, the sin terms vanish this is for a even signal only the cosine terms remain, you can see only the cosine terms remain.

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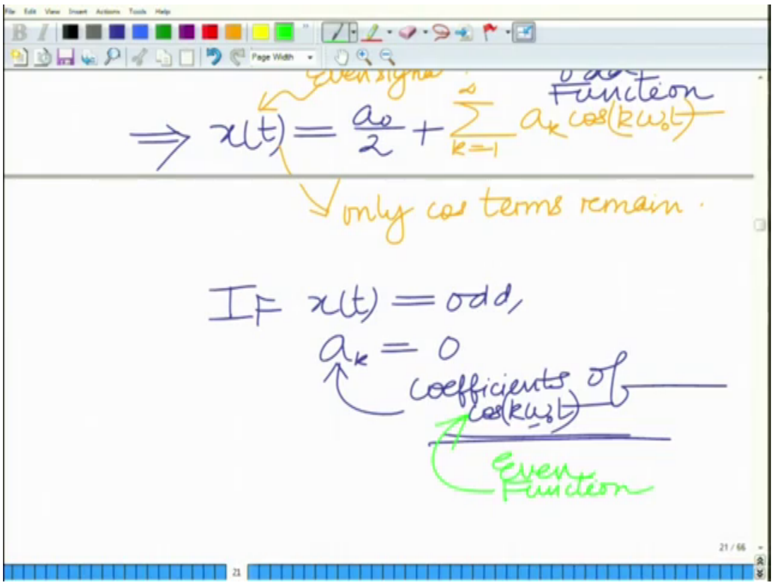

However if x t is odd then we have the a ks are 0 remember, a ks these are coefficients of cosine k omega naught t and cosine k omega naught t remember this is a even function and as is the dc component. So, when x t is odd the coefficients of the even components in the Fourier in the trigonometric Fourier series, that is the cosine right the cosine functions these vanish.

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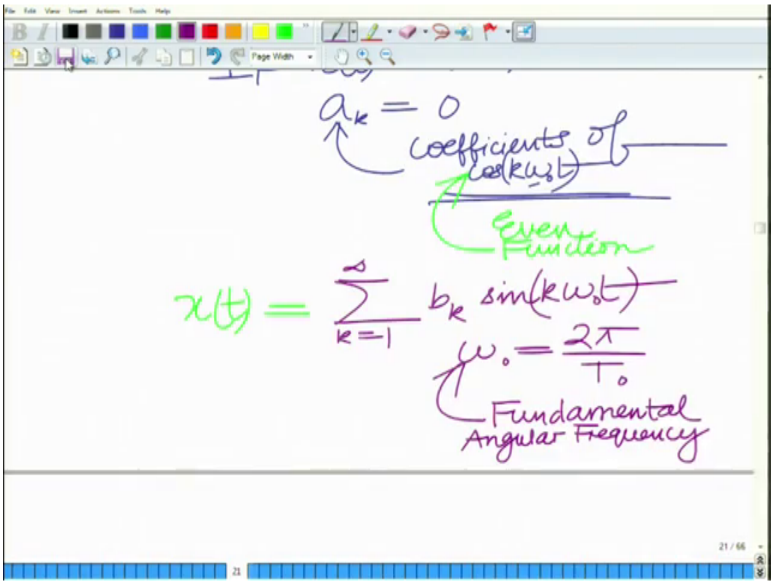

So, therefore, naturally the a ks are all the a ks are 0 and therefore, the Fourier series or the trigonometric Fourier series can be expressed as x t equal.

So, it is a very convenient way to basically express the trigonometric Fourier series it has very convenient way to express, the express the expansion of the series of a this is simply summation k equal to 1 to infinity b k sin k omega naught and remember omega naught as always is 2 pi by T naught this is the fundamental, this is the fundamental angular frequency. So, that completes the discussion on the trigonometric Fourier series.

So, what we are going to do. So, what we have seen in this module is basically I think we can stop here. So, what we have seen in this module is we have seen the we have started with the discussion of the Fourier series, started looking at the trigonometric Fourier series related these 2 expansions. These are equivalent one from the trigonometric Fourier series co presentation one can get the Fourier series from the Fourier series, one can get the trigonometric Fourier series and we have seen the various properties of the coefficients. So, will stop here and continue in the subsequent modules.

Thank you very much.