Digital Communication

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Module 01
Lecture 27

Performance Analysis of Binary Digital Modulation: Signal and Noise Statistic In Coherent and Non-coherent Receivers

That we have talking about de-demodulators for binary digital modulations schemes and we have looked at both coherent as well as non-coherent de-modulators and what we have in summary antipodal schemes like BPSK require coherent receivers, orthogonal binary schemes like on off keying or frequency shift keying we can use either coherent or non-coherent receivers right. But we like now to do is to gain some insight about how these receivers actually perform. These are the best known receivers in Gaussian noise right, these are the optimum receivers in Gaussian noise, optimum coherent receivers and the optimum non-coherent receivers, the question now is, what how did they perform?

We like to be able to predict that is able to say what will be the error rate that will come out of additional receivers or a particular binary modulation scheme for a corresponding optimum receiver, for a given signal to noise ratio that mean we available and it is important right. So we like to do some analysis of performance before we proceed further with a de-modulation of more general digital modulation scheme like M-ary schemes ok and then after we had developed this insight will go over to the Mth schemes.

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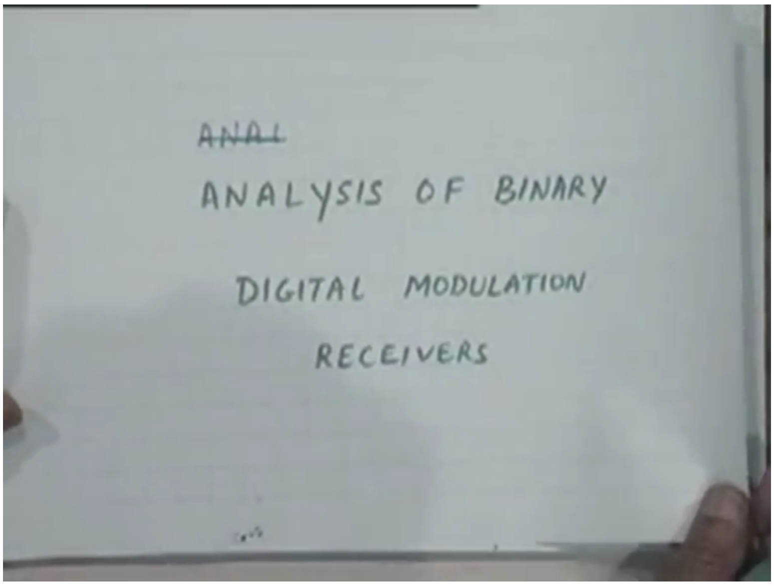

So the tittle for today's topic we essentially that is Analysis is that visible? I think is a bit faint, of binary digital modulations schemes right. Now to do this analysis ultimately we what we want to do is able to predict the error rate or the probability of error that is the probability of mistaking a transmitted zero for a received one or vice-versa right. The probability of error as a function of the two most important parameters that you will have to deal with our time namely the signal and the noise right. Actually the signal parameter signal to noise ratio as we may seem quick to define it.

Now before we can do that we have to be therefore able to express the (())(4:11) properties of the output of the matched filter because ultimately if you remember the decision statistic is the output that we sample and then pass through a threshold right, on the basis of which we decide whether or not a 0 or 1 has been transmitted. So the most important parameter whose properties we must investigate is that decision statistic that is at sample value which is being compared with a threshold for a decision. Depending on how that behaves what are its (()) (04:48) properties what kind of probability distribution function does it have right.

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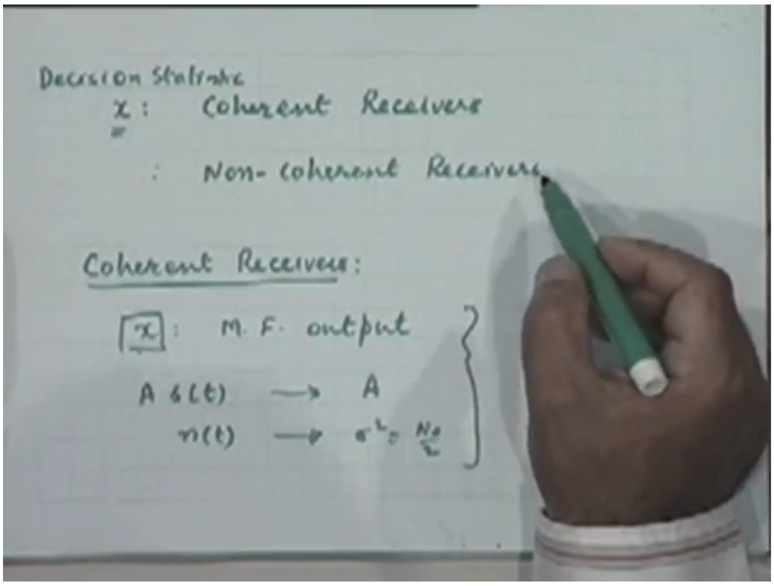

We will be able to do this prediction that we want to do. So must therefore be able to appreciate the nature of the decision statistic that we denote it by X for each of the cases that we are dealing with namely for coherent receivers on the one hand and this is your decision statistic and also for non-coherent receivers on the other hand. So that is what we must do first to be able to proceed further analysis right, where X is a sample output which you are comparing with the threshold for taking the final decision regarding what was transmitted or what has been received ok. Now for the case of coherent receivers, consider the case of coherent receivers first what is X in that case?

X is nothing but a matched filter output right and we already know the properties of matched filter output, both when a signal is presented to it as well as when a noise is presented to it right. We have seen we have studied that adequately when a signal is presented to it which is to which the signal to which the filter is matched the output at the sampling instant is at the peak value suppose the signal is A s t, you have seen that the output will be A assuming offcourse that S t is normalize appropriately for which unit energy (())(6:42)

On the other hand when noise is presented to a White Gaussian noise would a power spectral density of function of X0 by 2 we have seen at the output at the sampling instant or for that matter at any instant is a random variable, where a variance equal to N 0 by 2 right (())(7:08). So therefore for the case of coherent receivers we have a fairly good understanding imperfect to perfect understanding of what the output statistics are, it stands from the following these observations that we just need.

The first observation is, if the input is a Gaussian noise process to a filter the output we know will be a Gaussian noise process who's variance we know right. If it is presented with noise alone it will be a Gaussian noise process with this variance N 0 mean because the input is zero mean. On the other hand if it is coming along with some signal either A s t or minus A s t or whatever the output process will now be again white output will again be a Gaussian noise process with the same variance but its mean will be appropriately shifted right, will be either A or whatever this A or minus A depending on if you are considering lets say antipodal schemes right.

So in the case of coherent receivers we fairly well understand this statistics of this decision statistics X with which will have to work which is compare with the threshold and therefore will be able to do the analysis very easily as well as see very soon right, because we know it is Gaussian we know its variance we know its mean we can write down the expression for it probably density function at the output right because all the parameters that we need to know are known to us. However the same cannot be said about non-coherent receivers.

Well it can be said but will have to a bit of analysis of that and therefore what I will do first is just like we have a reasonably good understanding of the statistics of X in the case of coherent receivers we like to also have a similar understanding of the statistics of X for the case of non-coherent receivers before I proceed with the analysis, because I want to loop both the analysis very one after another.

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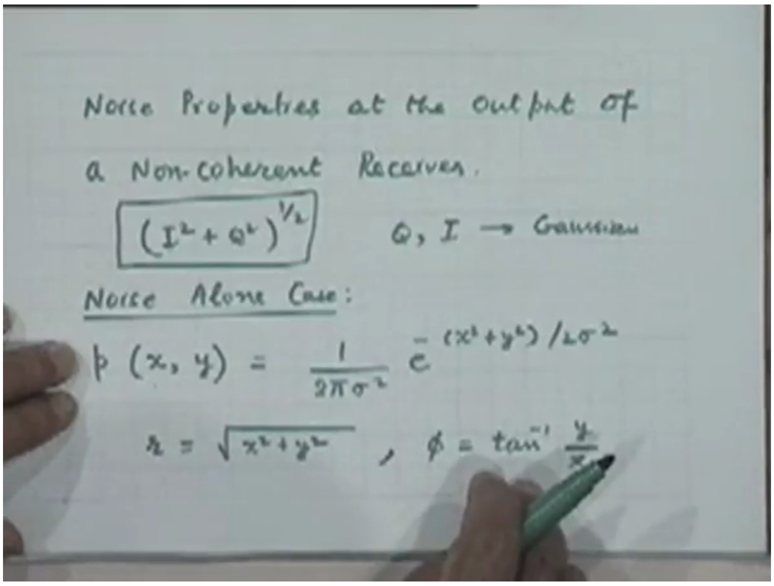

So let's start with a discussion of the noise properties at the output of non-coherent receivers ok.

What are we doing in the case of a non-coherent receivers? Basically if you recollect what we discussed last time, the input signal is being brought down to at the baseband form it you take that form that is a most convenient form to analyze and then the each of the I and Q outputs are being squared and summed and then we take the square root of this right. As far as the two outputs I and Q are concerned they are matched filter outputs whose properties we understand in the same way that we just discussed right each of them behave exactly in the same way.

It will have some mean, some variance right you will have to identify what is a mean and what is a variance, lets infact to start with will assume that only noise is present so the mean at the output of the matched filters in I and Q both channels will be zero and the variance will also be identically equal to sigma square and we also know that because their quadrature channels the corresponding noises will be uncorrelated right. So will start with this understanding of the I and Q outputs and finally our decision statistic is really going to be in terms of this I square plus Q square, square root of that.

So what we like to understand is how does this quantity behave statistically? Ok that is the basic, because this is what we are finally going to sample and compare with a threshold for making a decision, or compare with another similar quantity for making a decision right, in the case of orthogonal signaling schemes will be doing this computation for both the signals separately and then comparing the two together. For example in FSK right, for each other two signals that you might be transmitting you will do this computation and then mutually they will be compared whichever is larger will take the decision in favor of that fine.

So basically this is our problem at N and let us to start with consider the noise alone case and then will add signal to it later. So as I said I is Gaussian individually so is Q right that we already discussed a number of times now and they are uncorrelated which in this case will also imply that they are (())(12:34) independent because we are assuming they are Gaussian right. Look at the input is a Gaussian process with they are Gaussian. Therefore really speaking if I (instead) I will not use a notation I and Q for the timing I will just go back to X and Y. The outputs of the two quadrature channels I am denoting now these outputs by the random variables X and Y right.

We can think of these two random variables having a join probability density function which is essentially very simple kind of two dimensional Gaussian density function and since they are independent this two dimensional Gaussian density function is obtained by simply multiplying the individual Gaussian density function right. So it will be 1 by 2 Pie sigma each of them has a same variance what I am say there is E to the power minus X square plus Y square upon 2 sigma square ok and what is our interest? Our interest is in let me define a random variable r which is obtained from these two random variables X and Y in this manner.

Will like to understand what are the properties of r, do you know what the answer is? If you know the answer then we can skip this analysis, is that ok with everybody? We can skip it or, does everybody know that r will be a rather random variable? So I can skip it? We know it and still (())(14:21)

Student: sir we had known it but we have never derived it

Professor: You don’t know how to do it? Alright lets go through it, it is very simple let me define r in this manner let me also define another random variable Phi which is tan inverse r tangent of Y upon X ok, so essentially what I am saying is we are going to do a coordinate transformation to represent the same X Y values by corresponding set of r Phi values right, we just done a rectangular to polar coordinate transformation of these two random variable. The same point in the two dimensional space is now being represented in term of its polar coordinates r and Phi right and the basic idea of doing this kind of studying this kinds of transformations for their (())(15:20) properties is to consider the probabilities of occurrence of an event which lies in an elemental area in the same space.

Both in terms of this density function as well as in terms of a density function of the transformed (variables) r and Phi ok.

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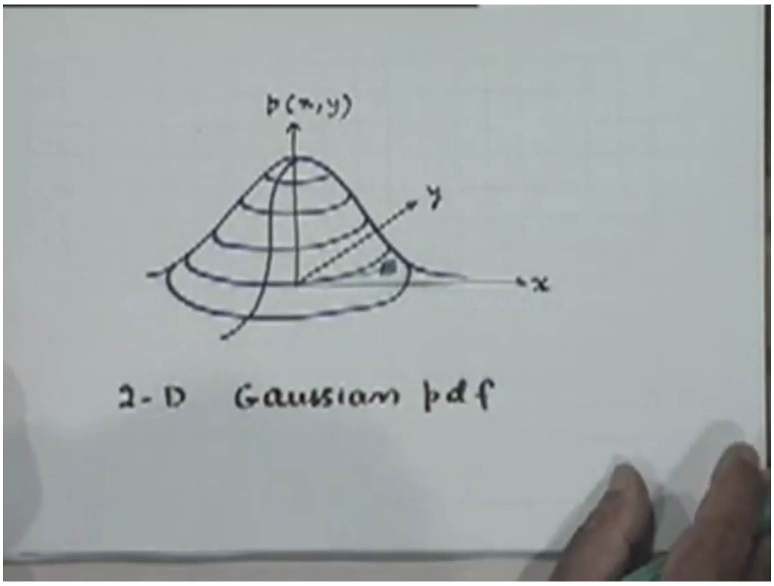

Incidentally this density function is something like this I mean you have a Gaussian density function in two dimension, this is your X this is your Y dimension and you have some kind of a inverted pyramid oh it is a kind of a pyramid actually except that is more like a tent and so if you are somewhere on this plane X Y plane you can as well represent it by r and Phi and the relationship between the two density functions the two joint density functions in terms of X Y as well as r Phi is obtain by equating the probabilities of occurrence of corresponding to an elemental area in this X Y plane.

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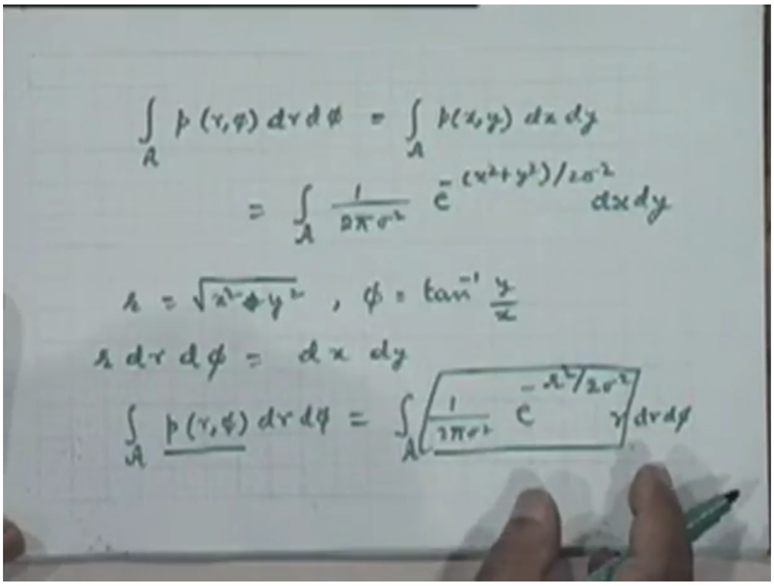

So if we do that if we consider an elementary area lets say A and in consider the probability that r and Phi will lie in this elemental area right, tell me this ok, where k is a small elemental area with r and Phi as a center. Similarly we can also compute the same probability by considering the corresponding P X Y right, basically this is a starting point of most of this kinds of analysis from fundamentals. Let’s substitute for P X Y that is 1 by 2 Pie sigma square E to the power minus X square plus Y square upon 2 sigma square dx dy right and now let me make this substitutions that we just discussed, that is r equal to this and Phi equal to this.

If we do that we can write this as before so let me just rewrite it for you for completeness sake here r is this phi is this and will also have to do something about d r d Phi, d r d Phi is infact r d r d Phi is equal to dx dy right. So making this substitutions will get 1 by 2 Pie sigma square this will become r square by 2 sigma square right and this dx dy will be replace with r d r d Phi ok. Now if you compare the two integrals it is obvious that Phi r phi the joint density function of r and phi is nothing but this quantity alright, because you got it in the same form right. Both the integrals now with respect to r and phi.

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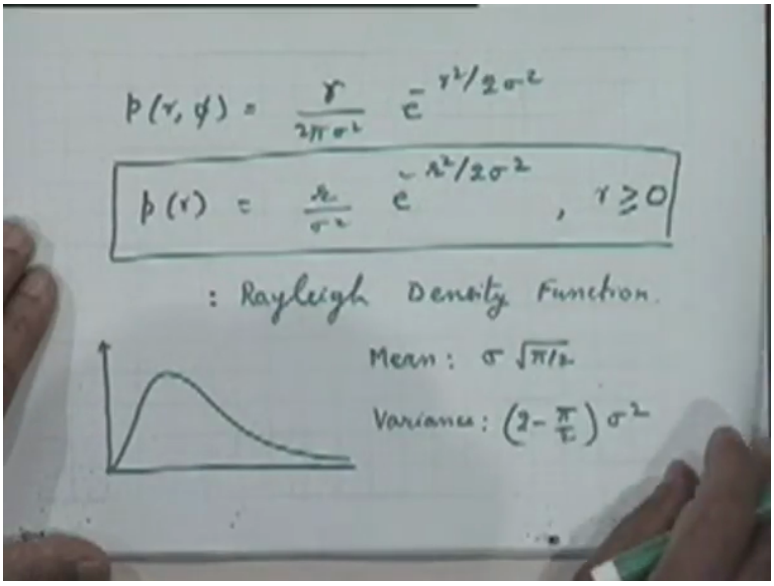

So we can write the expression for p r phi as 1 by 2 Pie sigma square E to the power minus r square by 2 sigma square sorry into r also this r also is to come here fine, just look at this again, this r into this alright, But what is our interest? We are not really interested in Phi at all because finally we are only looking at r so we want to obtain the density function of r from this joint density function of r and phi, how do we do that?

Integrate out the other random variable in which we are not interested, if you do that integrate simply carry out an integration of this over phi infact it is not a function phi at all right it is a constant with respect to phi essentially it will be from 0 to 2 Pie into d Phi integral of this into d phi and that will essentially become this 2 will disappear right. So you be left with p r as r by sigma square E to the power minus r square by 2 sigma square. Offcourse it is understood that r is going to be greater than or equal to 0 because of the way it being defined.

This is precisely what is known as the Rayleigh Density Function, Rayleigh (())(21:01) ok. So now you know how it is derived it is really very simple. So what we have learned is that when we pass or when we take the envelope of a Gaussian random process that envelope will have a density function which is Rayleigh that is essentially what it corresponds to. If you were interested in Phi you could also obtain the density function of phi by integrating over r it will turn out that Phi will be a uniform distributed random variable between 0 to 2 pie right, that their density function of 1 by 2 Pie between that.

But we are not interested in that so I will leave at that. Just a few very important properties very basic properties of this Rayleigh Density Function this no longer a zero mean random variable it is obvious isn't it? Because the density function is defined only for positive values of r right infact this is how it looks like it is not symmetric Gaussian was symmetric this is no longer symmetric and obviously it is not zero mean infact it can be easily checked then I like it and do that yourself that the mean of this is sigma times root of Pie by 2, the input process was zero mean Gaussian, input noise was zero mean Gaussian the output is really random variable whose mean is not zero right.

It is this value difference in the variance, similarly it is variance is smaller than the input variance is given by 2 minus Pie by 2 sigma square, this is something you can verify yourself easily. The just like we have a stabilized form for the Gaussian density function you are familiar with that, standard form for the Gaussian density function in which the mean is taken to be zero and the variance is taken to be one. We similarly have a standardized form or a standard form for the Rayleigh Density Function in which sigma square is taken as unity and I will just pay for the sake of completeness write it for you.

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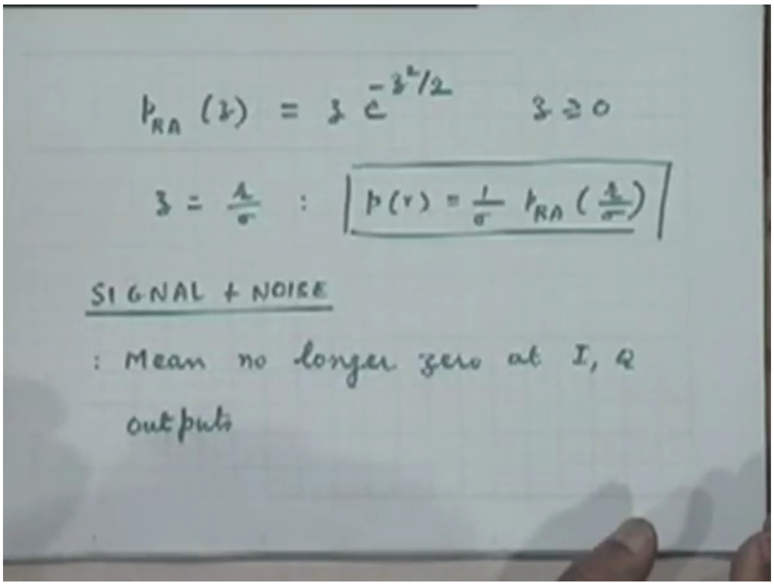

The standard Rayleigh Density Function P R A lets say in terms of some random variable Z is defined as Z E to the power minus z square by 2 right just same expression except that you have put sigma square equal to 1 or z greater than or equal to 0.

So in terms of this standard density function standard Rayleigh Density Function P RA as we just designed are can be obtained by putting this Z equal to r by sigma right, so essentially we can say that your P R is if you make this substitution 1 by sigma P R A, ok all I am saying is this P r can be obtained from z by replacing r by sigma and then offcourse you will have to scale it up with 1 by sigma to get the same expression ok. So in term of the standardized Rayleigh Distribution Function the non-standardized distribution with a non-unity signal is written like this that is all.

Now that was a case when the input to this system is envelope (detect) matched filter envelope detector combination was only noise. Now that is all right for situations like we are only interested to know what is the output noise properties, but our interest is to know both how to behave with noise as well as when it is coming with signal as well as noise right. So now lets consider the case of signal plus noise. Same system but the input signal as well as noise. Now what will be the essential difference at the IQ level at the XY level.

Student: mean will not be zero.

Professor: Not zero right when the signal is their at the IQ level at the two quadrature matched filters levels, again will have Gaussian random variables but will be non-zero mean Gaussian random variables. Let's say so the let me write it down, mean is no longer zero at the IQ outputs, yes please

Student: sir how can we say that the both I and Q will be Gaussian variable because the signal as well as the noise, (()))(26:36) matched to be Gaussian process, signal (())(26:39)

Professor: Signal we know what effect the signal has, we already discussed at the beginning of to this lecture also. The signal the presence of signal with noise will essentially change the mean of the because the signal is a deterministic signal, it is not random at all and after matched filtering suppose signal alone was present at the sampling instant we get a value A with noise we get that value A with noise fluctuations right. So the mean value is A with some variance depending on the noise input. So it remains Gaussian on its mean is different in the presence of signal because is a linear processor right, matched filter is a linearly filter.

So since the input is essentially non-zero mean Gaussian noise in the presence of signal it is a input mind you is a time has a mean which is time variant it depends on signal S t, so does output its mean is time variant, but we are only interested in the output random variable that results from sampling the output at t equal to capital T, that has a very definite value equal to A. No matter what is the shape of the signal right, so the output process at the output of the matched filter if the Gaussian process in general with a time varying mean right because output has a definite waveform right, so the mean is really time varying but at the sampling instant at which we are interested that values known to us and you are really looking at that value right.

So the essential difference in this case to start with is that at the IQ level at IQ outputs we have Gaussian random variables with non-zero mean ok this is our starting point.

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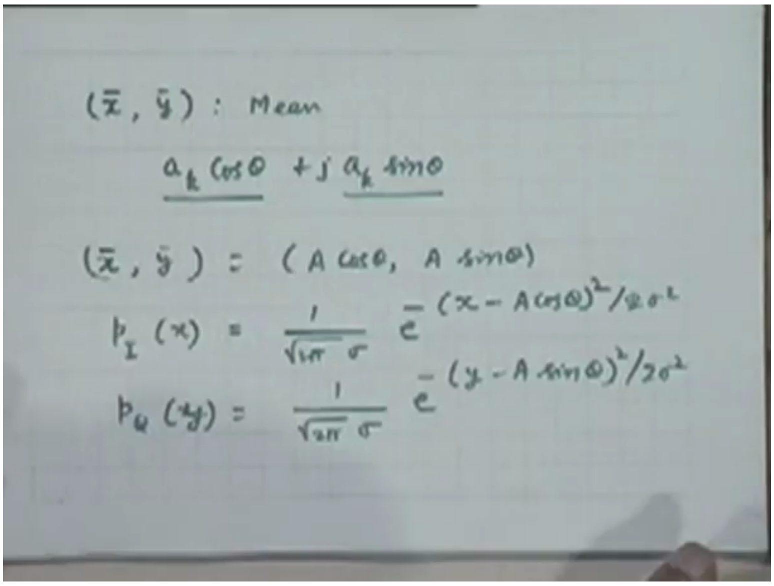

So therefore lets consider the general problem of combining two Gaussian random variables X and Y such that X has a mean X bar and Y has a mean Y bar so this is the mean values lets say assume that the means with this and we are now interested in finding out what are the properties of square root of I square plus Q square when I and Q are non-zero mean Gaussian random variables, that is the question, I am considering a general case infact in (gen) they will be in general they will be different in the quadrature channel, can you justify why?

Yes question was, would these mean be same or different? That was the question Vivek

Student: (())(29:39)

Professor: Ok lets recollect the case of On Off Keying the simplest case right, if you remember the output was represented by A K cosine theta K plus j a k sine theta, where theta is your unknown phase. So the mean in one case one channel is A k cosine theta in other channel which is A k sine theta right. So in general they (could) they can be different depending on what is the value of the unknown phase right. Offcourse if theta is eliminated beforehand by doing a phase recovery and phase compensation then they can be identical also right. So in general they can be different or same.

So let us represent this means also in terms of polar coordinates you have some arbitrary means X bar Y bar you can also represent them in terms of let us say some values A cosine theta and A sine theta with an appropriate pair of A n theta values fine. Because we are going to go from rectangular coordinates to polar coordinates, lets represent these mean values are also in terms of polar coordinates. The rest is everything rest is absolutely same. So theta here is some unknown phase angle and we start with let say p I x that will be 1 by root 2 pie sigma E to the power, what will be the difference? Will now have x minus A cosine theta whole square by 2 sigma square.

Similarly p q x or p q y let me call it will be 1 by root 2 pie sigma ok and then also be uncorrelated so we can write down the density function joint density function of these two as a product of these two and proceed in absolutely the same way as we did for the noise alone case right and do some simplification.

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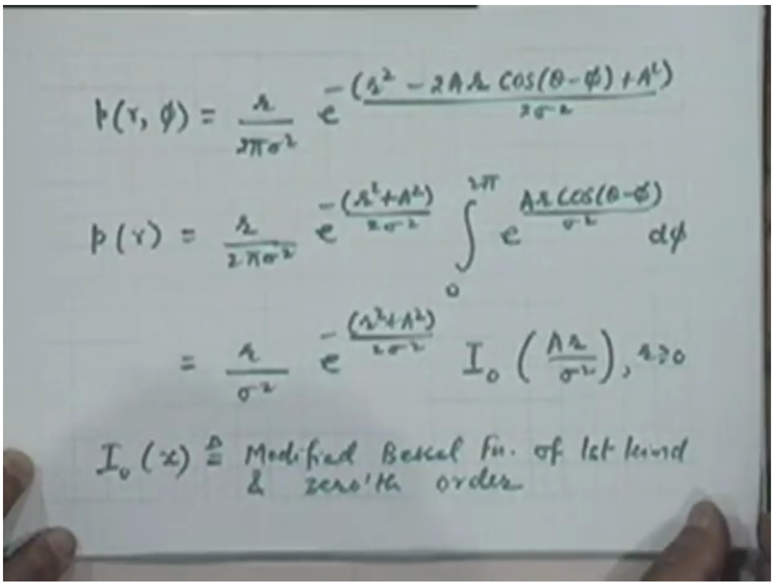

If we do that, I will skip those identical steps the absolutely identical that is no difference just a little bit of trigonometric manipulation which you can do yourself, you will find that you can write down the expression for a joint density function of r and phi where r and phi are defined in the same way as before that is r is square root of X square plus Y square and phi is tan inverse Y by X will be this density function will be r upon 2 pie sigma square E to the power minus r square minus 2 A r into cosine theta minus phi plus A square this whole thing this whole exponent divided by 2 sigma square ok.

I am sure you will be able to verify that very easily but you understand the result, offcourse one can obtain density function of r and phi individually by integrating the output the other variable which is what will do now. But upto this step is straight forward substitution and a little bit of trigonometric manipulation not much. So ultimately since phi is not of much interest to us in our application will integrate out phi and see what happens to p r, you can write this as r by 2 Pie sigma square since I am integrating over phi all this are constants I can take it to outside integral minus r square plus A square upon 2 sigma square into integral from 0 to 2 Pie of e to the power this minus and minus will become plus, this 2 and 2 will cancel right.

So you can write A r cosine theta minus phi upon sigma square, we are integrating over phi. Actually I had written phi minus theta instead of theta minus phi because cosine is even function. Now this we can write as r by sigma square into E to the power minus r square plus A square upon 2 sigma square into now this particular integral is a standard integral which is nothing but the so called modified Bessel function of the first kind and zero order, it is denoted by I sub 0, with this as the parameter. Let me define what is I 0 formally, a function I sub 0 x the so called Modified Bessel Function of first kind and 0^{th} order is formally defined as follows.

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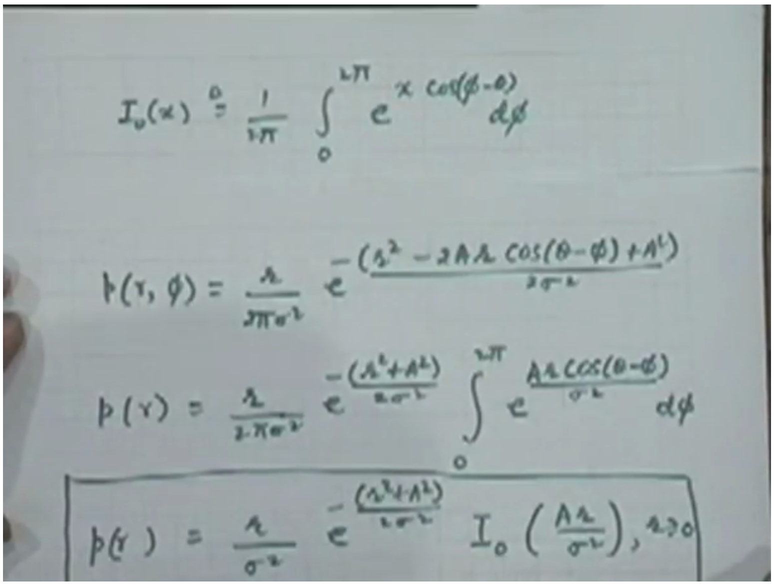

I sub 0 x is 1 by 2 Pie 0 to 2 pie E to the power x cosine phi d phi, so this is the formal definition you can apply the definition and see that this is what you will get here. What is the difference between this integral and what we have defined here? There is a theta there, this phi here can be brought to this side, it will made phi mean minus theta rather than theta minus A so that doesn't make any difference but actually it is also independent of theta. This same thing I can put a phi minus theta and go from 0 to pie because it is actually a periodic function it doesn't really matter.

In phase offset because ultimately you are going to consider the whole period over 2 Pie, so it turns out to be independent of theta this integral right. So this is our final result, this one ok. So as you can see this density function depends on, yes please.

Student: (())(38:11)

Professor: This definition if I put a phi minus theta here, it doesn't affect the value of theta the integral. It is a periodic function right isn't it? Because phi minus theta no matter what the value of theta is when you consider cosine phi minus theta it will go over one full period right because of this cosine factor it doesn't depend on theta. Yes as you can see this density function depends on A which was remember how A was defined? A was the output amplitude of the mean value, this is the mean value in terms of amplitude polar coordinates in polar coordinates depends on noise variance.

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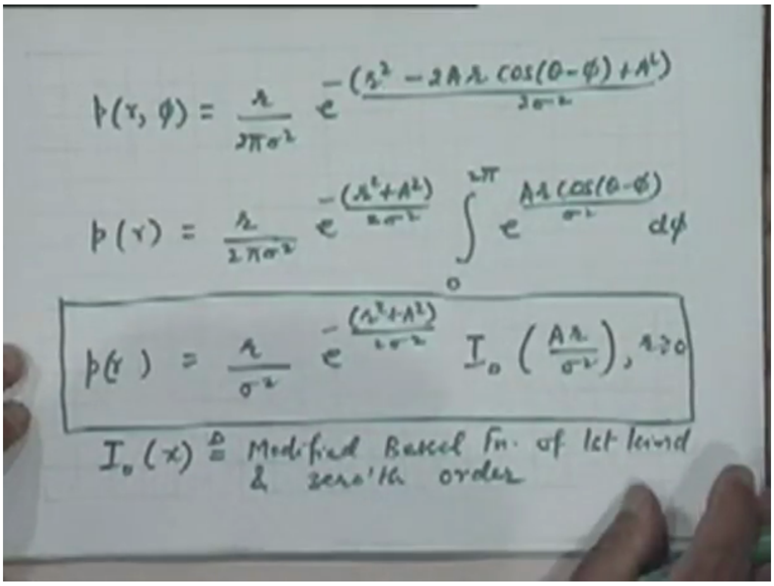

Offcourse you can see that well A is zero, what happens? It turns out that ok, this will be one, I 0 at X equal to 0 will be 1 which is very easy to check because this will become 1 and this 2 Pie and 2 pie will cancel and that is 1 fine. So this will reduce to the Rayleigh Density Function when A is equal to 0 right. So actually speaking this is so called Rician Density Function derived from the name of communication theories under name of Rice R I C E it is called Rician Density Function and it is not really a single density function it is a whole family of distribution functions the family being defined by the parameters A and sigma right.

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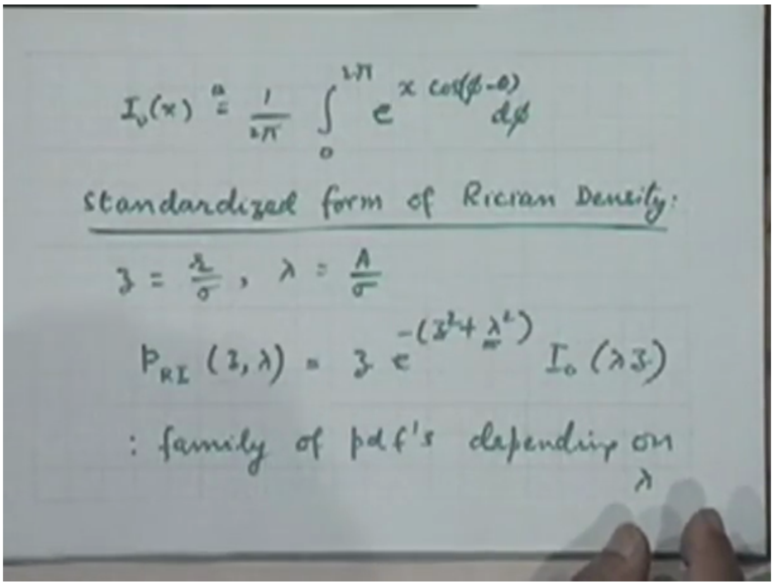

So the standardized form of the Rician Density Function is as follows, are you familiar with the Rician Density ok, now you are, so again if we define Z equal to r by sigma and lambda as A by sigma then we define P Rician in terms of the standard variables Z and lambda as Z e to the power minus C square plus lambda square I0 lambda Z ok. So if we normalize r by sigma as well as A by sigma right that same density function will reduce to this form, this will reduce to this form which is known as a standard form the Rician Density Function.

So as you can see it is a now it is one parameter family right, it is parameterized by the parameter lambda. It is a family of PDF's depending on lambda. I am sure you like to look at what this family looks like. I think before that I should also show you what this function looks like I 0 X, do you have any idea what this function is like?

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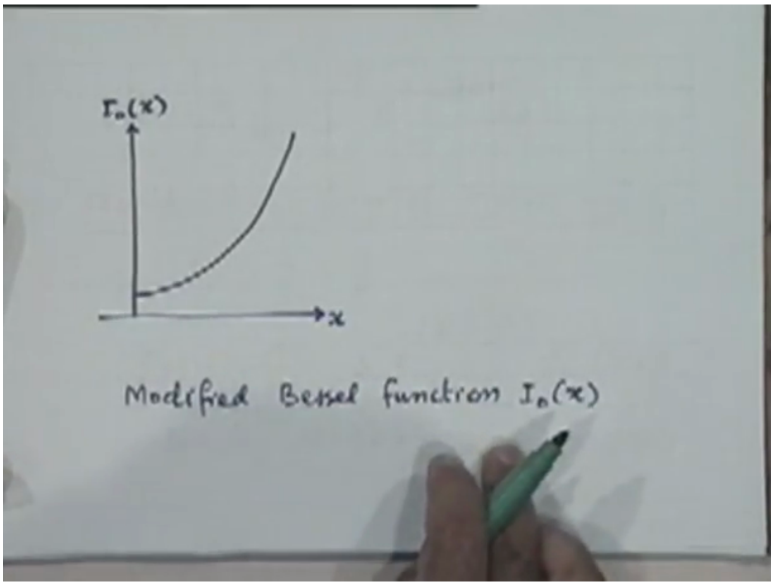

It is a monotonically increasing function looks like this ok I 0 X if you plot it as a function of X this is how the modified, yes

Student: (())(42:29)

Professor: Ok let me just check that is any mistake anywhere, yeah I think two has eliminated from me, then I should show you what this family of density functions look like.

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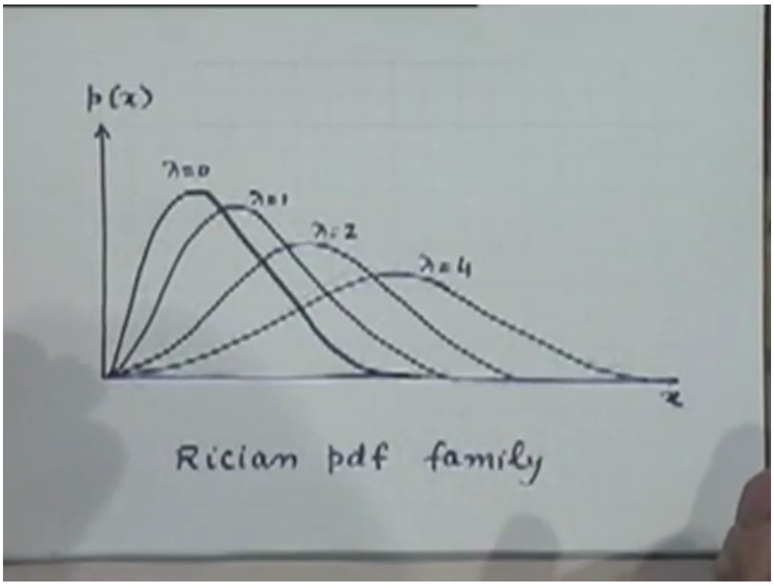

This is a typical set of plots for the Rician period family. As you can see for lambda equal to zero you have the same shape I have discussed before namely the Rayleigh and essentially for different values of lambda, basically what is lambda if you remember? A by sigma and A will depend on the input signal amplitude, A by sigma is some kind of a signal to noise ratio measure right.

So as a signal to noise ratio increases at the output you will find density function which shifts to the right and gets flattened out right, its tail become longer and longer right it extends more and more towards infinity and infcat you might see a resemblance of this density function with something that is known to you looks like a Gaussian density function right.

So asymptotically as the signal to noise ratio becomes large the Rician Density Function essentially is like a Gaussian density function, it not exactly a Gaussian density function because the Gaussian density function will never become zero at any point whereas it always starts from zero but it resembles the Gaussian density function more and more ok. So with this as a background we are now is to our error rate analysis, do you have any questions first before we proceed further?

So what we have learned so far is, how to describe statistically the random variables which will be working with at the in the case of coherent as well as non-coherent receivers, in the case of coherent receivers will always be working with Gaussian and the variables either zero mean or non-zero mean right. In the case of non-coherent receivers when you are considering the case of noise alone the random variable will be considering the output is Rayleigh, if you are considering along with signal it will be one of the density functions belonging to this Rician family that is essential point to remember.

So obviously as you can see coherent receivers are easier to analyze as compare with non-coherent receivers right, because non-coherent receivers bring in this additional complication.

Student: (())(46:00)

Professor: No but ultimately what is this Rician Density Function? It is R it is a density function in the random variable R equal to square root of X square plus Y square where do we do that computation? In non-coherent receivers right. We don’t do that computation for coherent receivers, we don’t look at envelope in coherent receivers, we look at envelope in non-coherent receivers right and this is a density function of the envelope right and an envelope we looked at only in non-coherent receivers were phase is not known or cannot be known, phase cannot be measure and therefore we want to eliminate phase through this envelope operation right. Square root of X square plus Y square operation ok.

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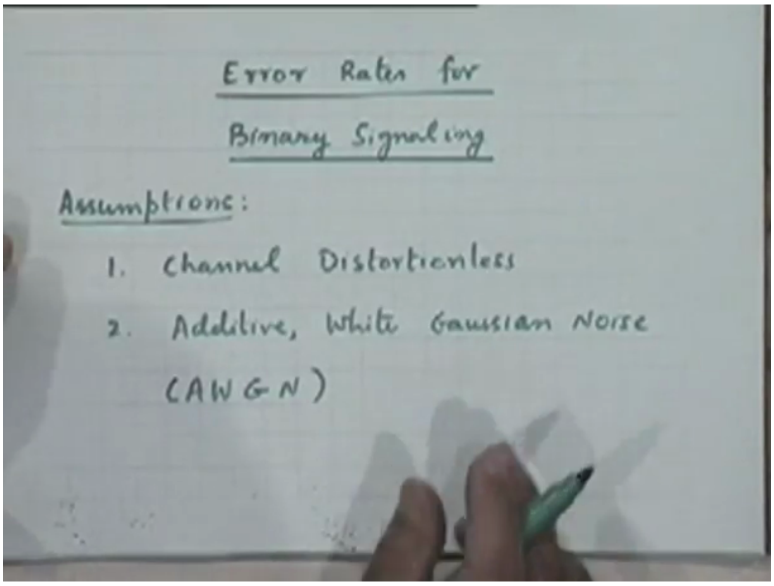

Now we ready to discuss error rates for binary signal yes Varun, ok, let me proceed with analysis now will make a very simple assumption which you have been making even at the time of developing the receivers discussion of receivers itself, one is will assume that the channel is distortion-less so that doesn't come to the picture. So we are not looking at the effects of any known identities of the channel.

Second is that the noise is of the so called additive white Gaussian type right, we could also have what is called multiplicative noise, noise which multiplies the input signal their models their situations where that kind of model appropriate, however most of the commonly used channels will exhibit this kind of a noise addition and then this is sometimes also known by the name of A W G N channel Additive White Gaussian Noise channel. So will be making these standard assumptions in this analysis then proceed on to calculate the error rates.

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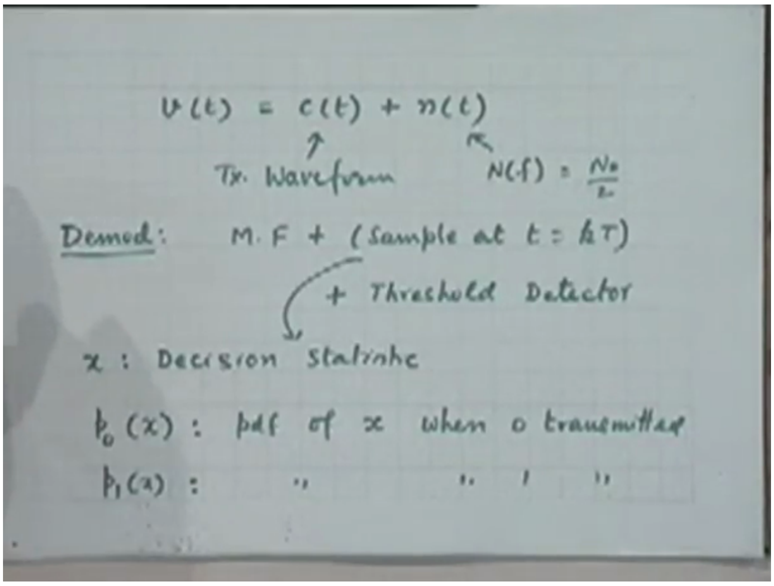

So let us write our received signal V t as the sum of the transmitted signal lets say C t plus this noise N t and there is no other distortion, C t comes along as a as such except for the addition of White Gaussian Noise at the receiver. So this we have transmitted waveform this is noise with a power spectral density function N sub 0 by 2. The model for our de-modulator is, in a case of coherent receivers to start with I will consider coherent receivers because they are easier to deal with. We have a matched filter followed by a sampling device so it will, will do an operation of sampling at time instance t equal to K T right plus this is followed by a comparator or a comparison with a threshold, threshold detector and it is this value the sampled value which is the decision statistic which I will denote by X.

So X is a decision statistic which is the sampled value right. Now as we have just try to appreciate this decision statistic X will in general have different distributions when a zero is transmitted and when a one is transmitted right. For example suppose it was On Off Keying right the matched filter output in both cases this decision statistic in both cases will be Gaussian or in one case it will be zero mean in the other case it will be with the mean of A. So they are both Gaussian but the detail distribution is different in each case right. So in general the test statistic or the decision statistic will have different distributions for a 0 transmission and for a 1 transmission.

Let me denote these two possible distribution as P sub 0 x and P sub 1 x alright let me just complete this, so this is the PDF of x when 0 is transmitted and that is a PDF of x when 1 is transmitted, so start from here next time, abhi to time tha pach minute jaldi rokh liya.