Computational Electromagnetics and Applications
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Exercise 10
Variational Methods

Today we are going to solve a very simple problem that we have been looking in various methods the Poisson Equation where we are going to usse Rayleigh Ritz method a variational method to solve such a problem. So let us look into the problem geometry itself.

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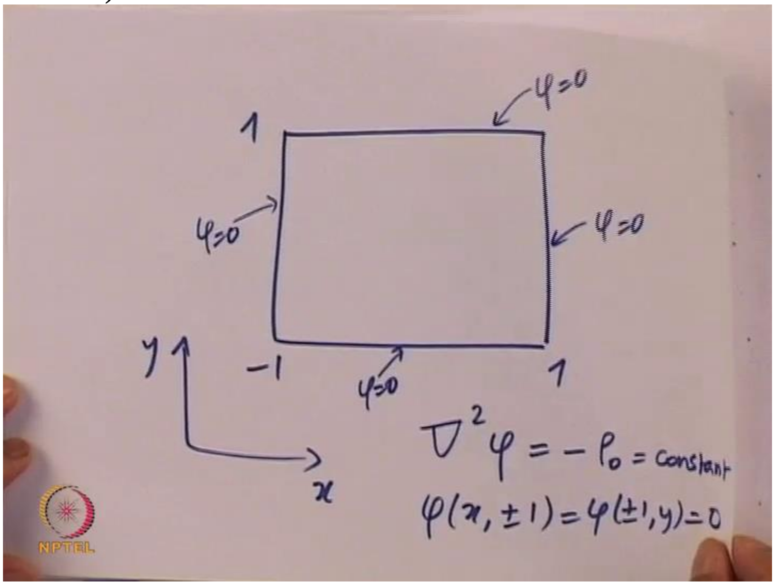

So we are going to have a square, so the square is going to be minus 1 so minus 1to plus 1 and in both direction in x and y direction. So if this is the x direction this is the y direction. So if this is going to be the x axis this is going to be the y axis so this is going to be our problem geometry. And we are going to solve the Poisson equation. So this is going to be Del square Phi is equal to minus Rho 0 and for the problem we are going to consider Rho 0 is equal to constant So this is going to be constant across the entire domain and we are going to solve this problem subjected to some boundary conditions The boundary condition is Phi (x, plus or minus 1) is equal to Phi (plus or minus 1, y) is equal to 0. So that means for all x but plus or minus 1 y we are going to have 0. So the potential is going to be 0 here, potential is going to be 0 here.

Similarly for all y x equal to minus 1 and x equal to plus 1 it is also going to be 0. So along the boundary the potential boundary is going to be 0. So Phi equal to 0 Phi equal to 0 Phi equal to 0 and Phi equal to 0. So this is the problem definition and let us go and look at how one can solve this problem using a simple technique of Rayleigh Ritz.

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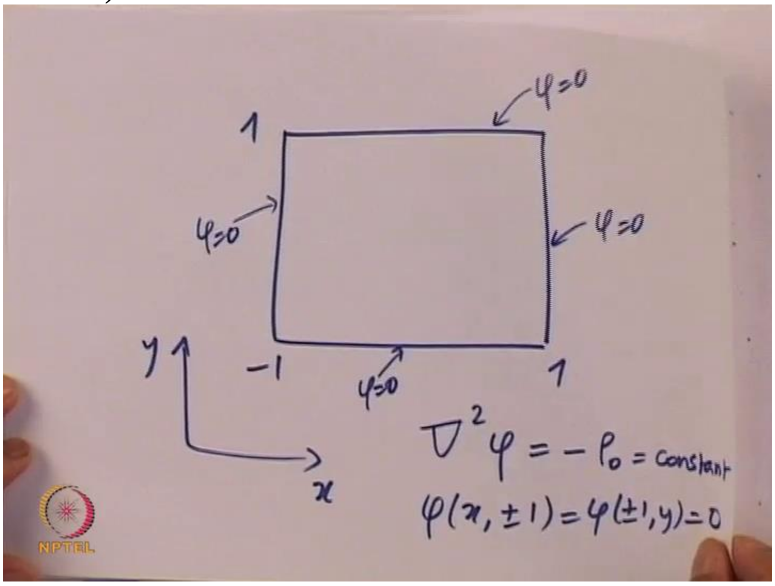

So we are going to take the problem from its symmetry. So if we look at this problem there is a kind of a symmetry so it is having x and y symmetry so we are going to use a basis function based on symmetry.

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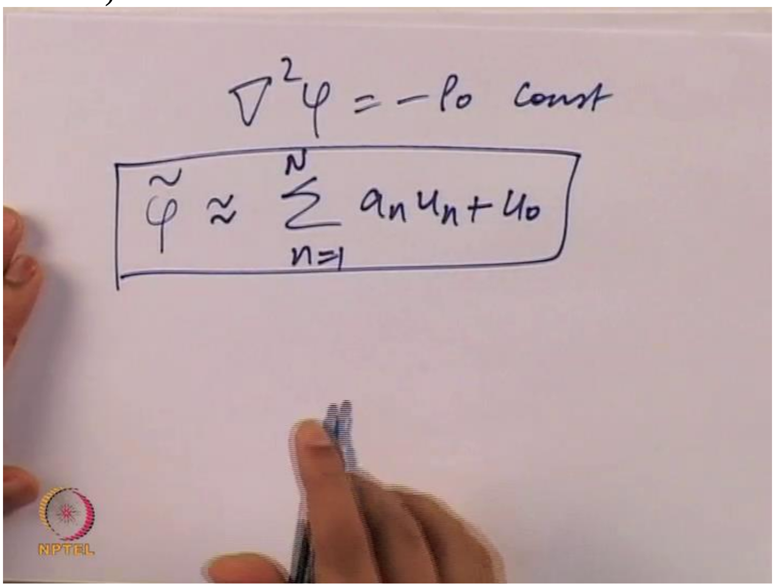

So the solution to this problem will be of the form. So we have the laplatian equation and we have set this is equal to constant the general solution to this thing will be of the form where we are searching for an approximate solution which is Phi tilde. It is going to be of the form sigma n equal to 1 to N a n u n plus u 0. So this is going to be the general form. And of course depending on the number of basis functions this is going to increase in terms.

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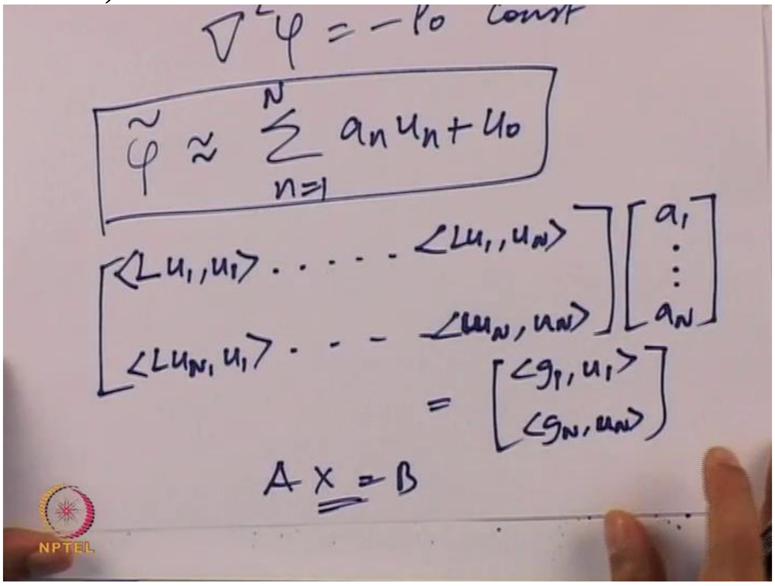

So what we will have finally is of the form matrix equation which will have the inner product of <L u 1, u 1> .... and the last term on the first row will be < L u 1, u n >. Similarly here first term on the last row will be <L u n, u 1>........ The last term here would be <L u n, u n>. So this will be the first on the left hand side multiplied by the basis functions that multiplied by the coefficients which are going to be [a1.... a n] and on the right hand side it is going to be a vector of <g1, u 1> in a product of g1, u1. The last term is going to be <g n, u n>

So this is nothing but in the matrix form AX equal to B. And we are interested in finding the value of X by taking the inverse. So this is going to be the way we are going to do the problem what we are interested is in finding these coefficients. So that is the way we are going to solve it and we will see what the coefficients are going to be in the following steps.

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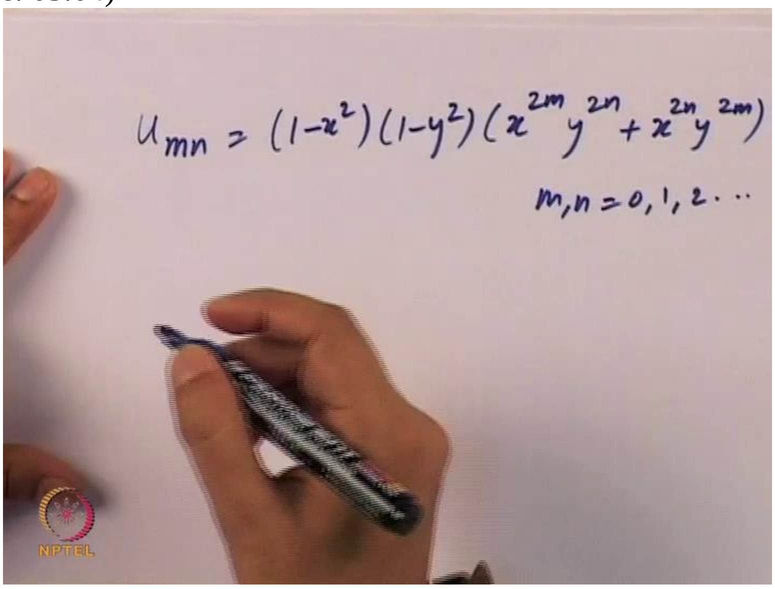

So since the problem is symmetrical we can set the basis function we are going to use, it is going to be equal to u mn is equal to (1 minus x square) multiplied by (1 minus y square) multiplied by x power 2m y power 2n plus x power 2n y power 2m) So for m,n equal to 0,1,2.... So on and so forth. So this is going to be the basis function we are going to use.

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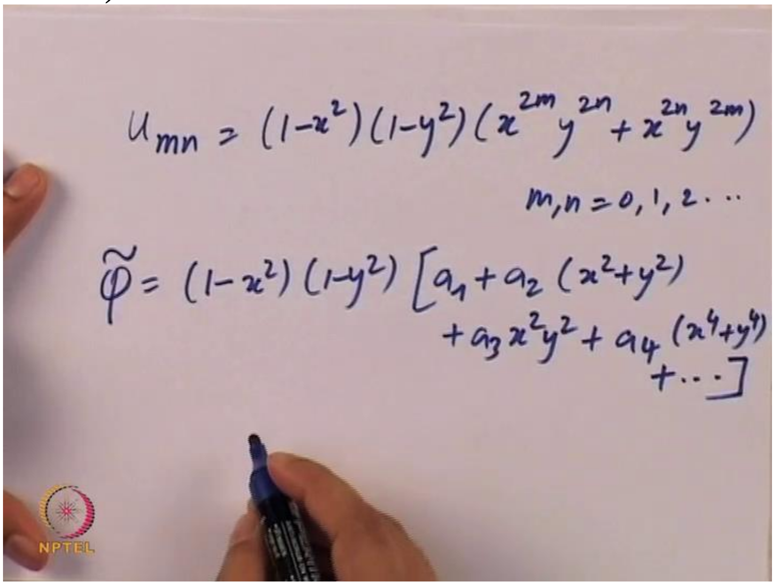

So let us say we have the problem using this basis function, so what we will get is we will get the solution itself as an approximation. So the approximate solution is going to be Phi tilde is equal to of the form 1 minus x square multiplied by 1 minus y square) multiplied by [a1 plus a2 multiplied by(x square plus y square plus a3x square y square plus a4(x power 4 plus y power 4) plus so on and so forth]

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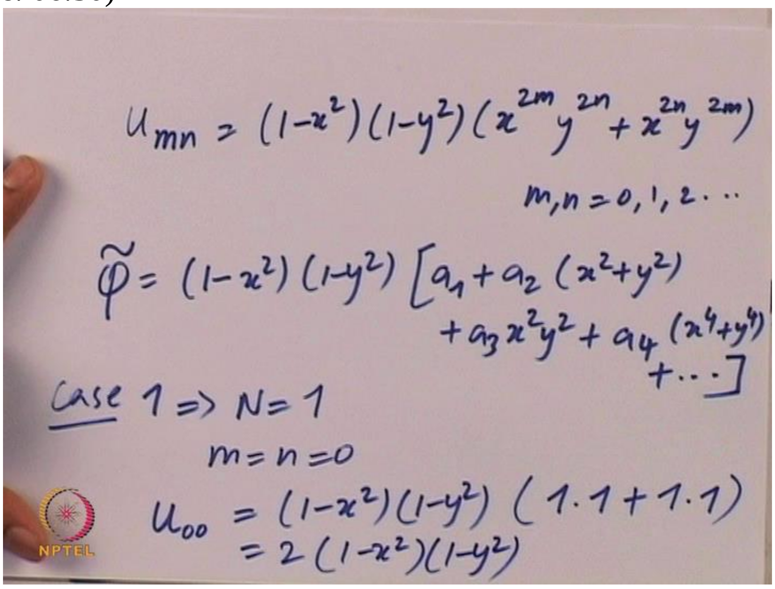

So let us take a very very simple case where m is equal to n is equal to 0 the first case. So the case 1 is equal to the number of basis function is going to be 1 so that means m is equal to n is equal to 0. So we have that let us club this into this equation what we get is u (0,0)is equal to (1 minus x square) multiplied by (1 minus y square) and putting m equal to n equal to 0 what you will get is this value will become 1 and this value will become 1 plus this value will become 1 multiplied by this value will become 1. So the entire thing will become 2 times 1 minus x square multiplied by 1 minus y square. So this will be our basis function. Once we do that we can multiply this into our equation in a clever way by taking the inner product.

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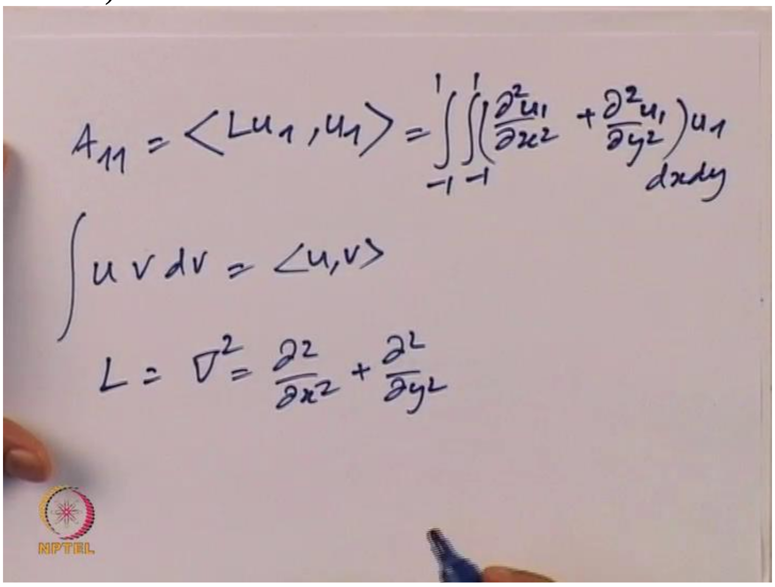

So we will have the first coefficient which is given by <L u 1, u 1 >. Remember the inner product is defined as u v dv, if we say this is the inner product of u,v. So in this case it is going to be a double integral for minus 1 to 1, minus 1 to 1 for x and y. And the L operator in our case is equal to the laplatian operator. So we can say the laplatian operator is going to be Doe square by Doe x square plus Doe square by Doe y square. So we are going to write the Laplatian operator here and we are going to multiply it with u 1 put this into a bracket and you have u 1 which is going to come. Remember this is the way we define inner product and multiply it by dxdy. So this is going to be the first step.

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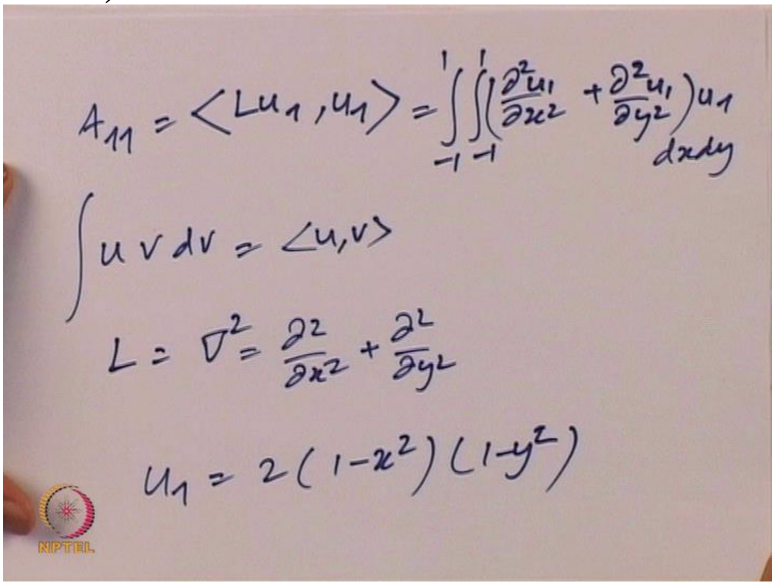

And the next step is to introduce u1 and we have found the value of u 1 in our previous step which is going to be 2 multiplied by (1 minus x square) multiplied by (1 minus y square). And I am going to differentiate this twice with respect to x and add it with the result I get after differentiating twice with respect to y square and multiply it with u 1. And this is going to be again multiplied by dxdy and that is what I am going to do exactly now.

There is some level of symmetry in the problem and the symmetry is here very clear that it goes from minus 1 to 1, minus 1 to 1 on both sides. So I can say 0 to 1 0 to 1 and multiply it by 2 twice. So that is what I am going to do here.

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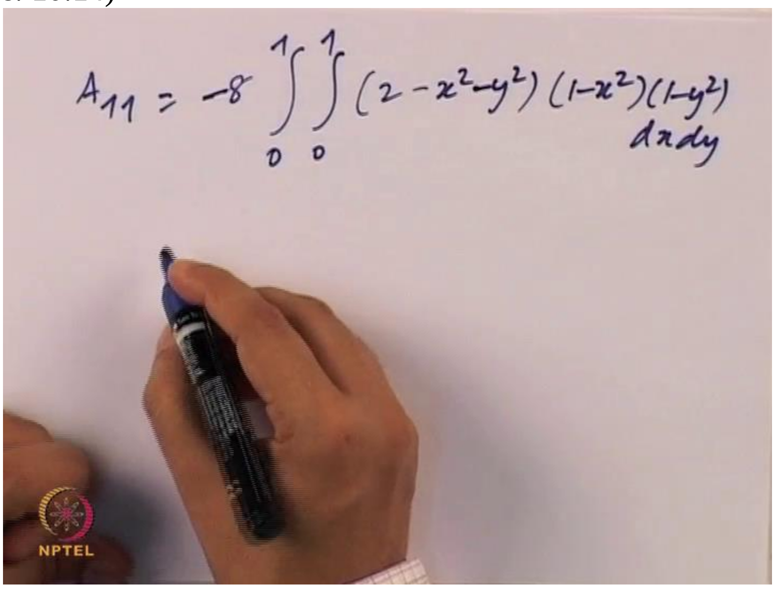

So I am going to write it in a simplified form A 11 is equal to minus 8 integral 0 to 1 integral 0 to 1 I have taken half of the integral and I am going to multiply it with 2. And I am going to do that also for x and y. So the outside multiplier will be minus 8 and the result of differentiating with respect to x twice and adding it with the differentiation with respect to y twice and multiplying with u1 will lead to this form. So (2 minus x square minus y square) multiplied by u1 which is (1 minus x square)(1 minus y square) dxdy. So this is going to be the first thing and we will see in the Matlab code how we can reduce it and get the value for A 11. So this is going to be the first one.

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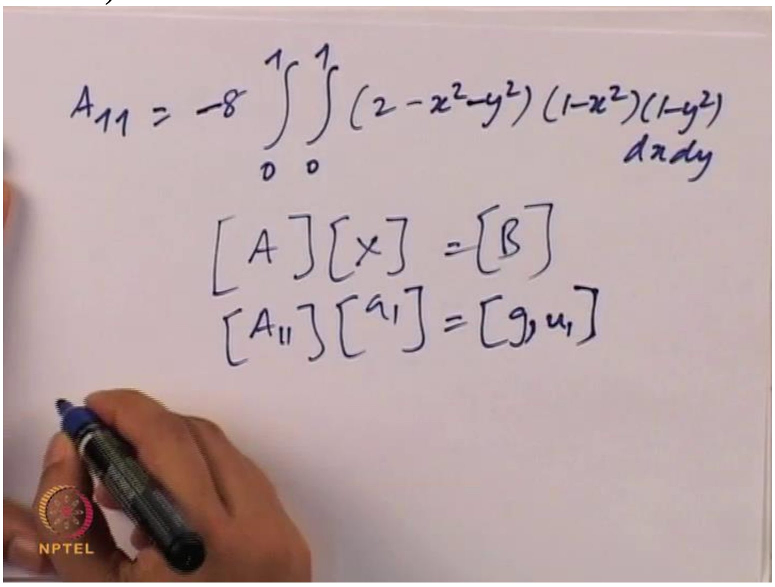

The second one will be the right hand side. So what we will get on the right hand side will be. So this is the first coefficient that we are computing A 11 and we will also have the first term on the right hand side. So remember this is if you are using 1 basis function the matrix of [A][X] equal to B will have only A11 term. And the x term is going to be [a1] is equal to [g1, u1]. Since g is constant here g is nothing but the value of Rho itself Rho 0. So its a constant we are putting as g 0, g 1, g 2, g 3.

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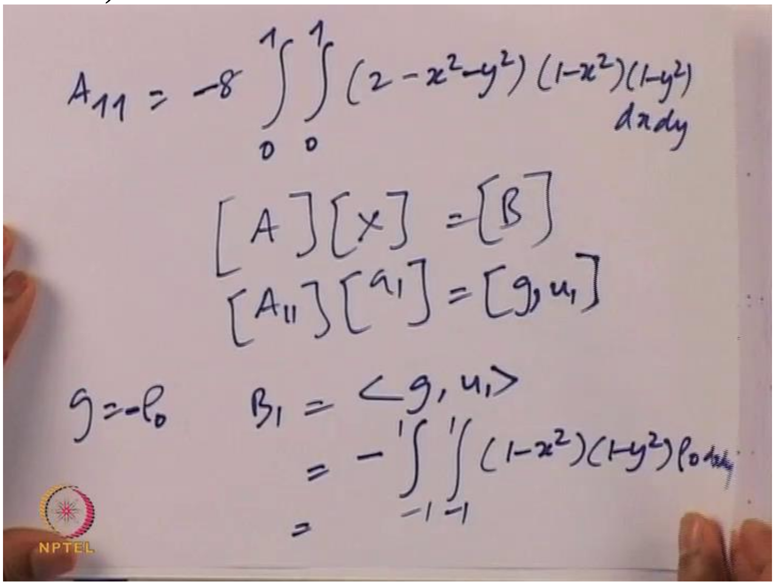

So here A 11 is the first term and the only term that we have and we are interested in computing the right hand side term which is B term for B1we write it B1 because we are in the first case with only one basis function N equal to 1 is equal to <g,u 1>. So this is going to be written as minus because g is minus Rho 0 in fact I forgot the minus sign we have to put a minus sign here. That is how we have defined the problem initially.

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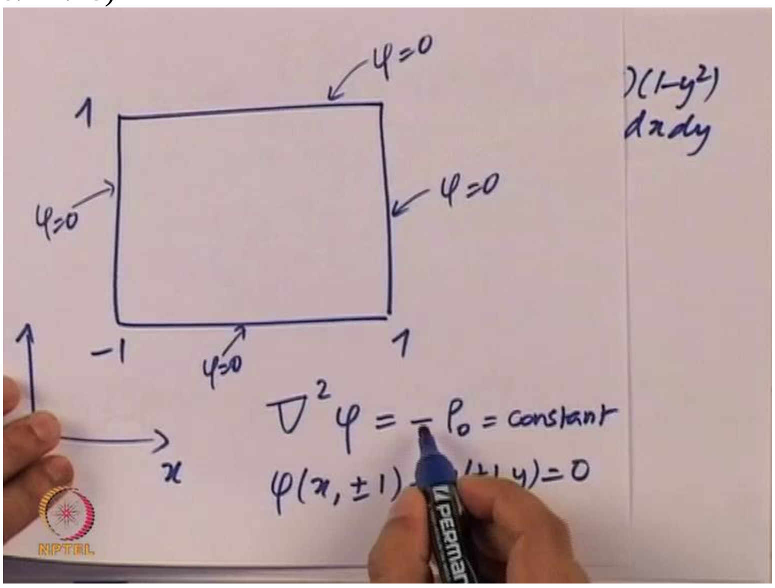

So the problem definition is the Poisson equation and the Poisson equation has a minus sign here.

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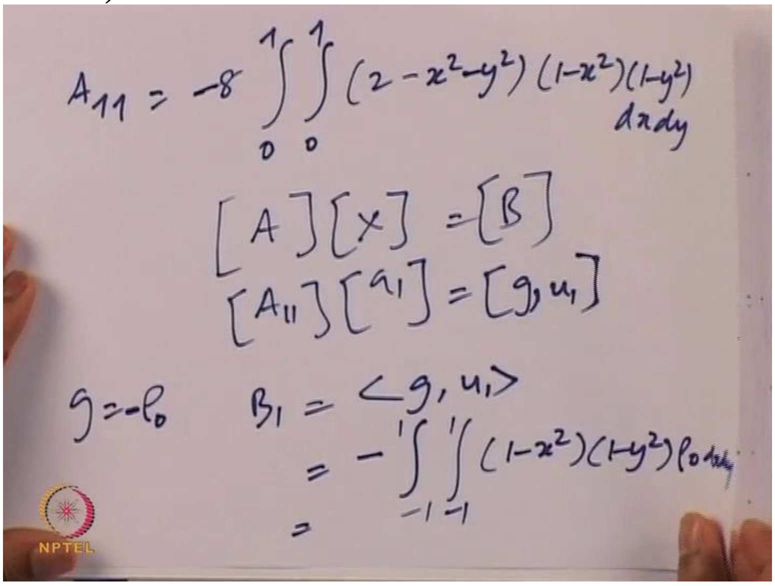

So that minus sign should be the value of g here so minus sign comes out. Minus 1 to 1 minus 1 to 1 (1 minus x square) (1 minus y square) Rho 0 dxdy. And when we simplify this what we get and we when we simplify this and apply the value of x and y what we get is B1 is equal to a value that we will compute in the program and we can also do that numerically by hand. But we are going to do it in a Matlab code and I will show you how to do this.

So this is going to be for the first approximation. So now let us look at if I increase the value of n equal to 2 and n equal to 3 n equal to 4 so on and so forth.

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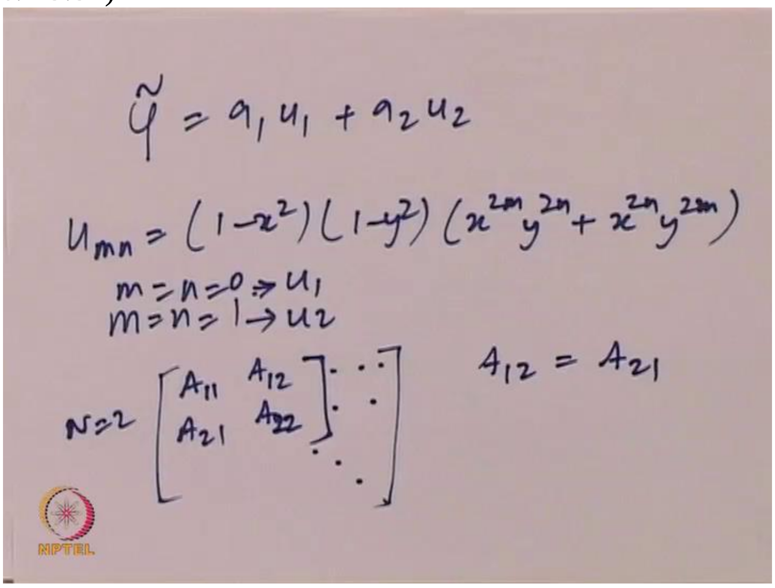

So if n is equal to 2 what you will have is 2 basis functions. So the Phi tilde is equal to a1u1 plus a2u2. And you will compute the value of u2 from the general equation of u m,n which we described it as (1 minus x square) (1 minus y square) multiplied by x square m y square m plus x square m y square m). So you are substituting the value for m equal to n equal to 1 and for the second case m equal to n equal to 0 for u1 and this is for u2.

So this is what we are going to do and we will get accordingly the different values for un and u2 and based on that we are going to compute not just 1 term but a series of term. The series of term will be [A 11, A 22, and here A 12 and A 21]. So this is the size of the matrix n equal to 2. The size of the matrix will increase for n equal to higher numbers having more and more terms on the matrix. And there is going to be some symmetry between the elements. So A 12 and A 21 should be same. So A 12 should be equal to A 21. Whereas the diagonal elements are going to be unique. And that is what we are going to see in the Matlab code.

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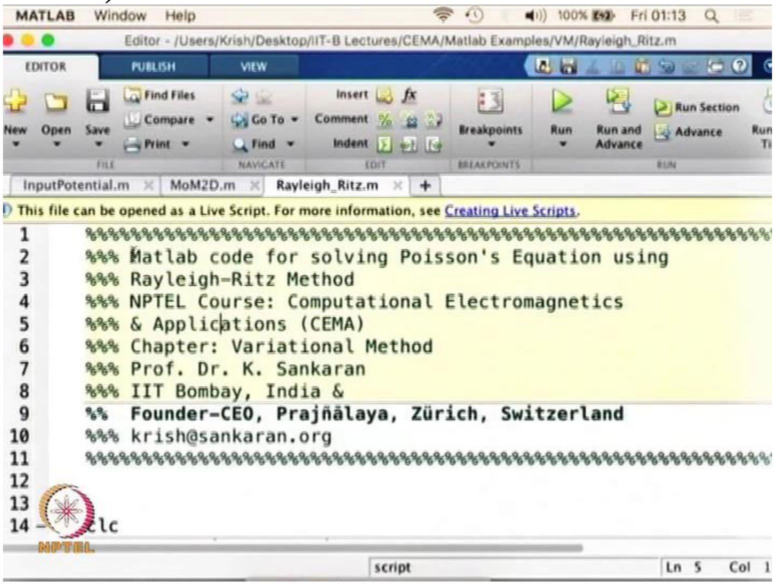

So now let us go and see the Matlab code itself. So this is going to be the Matlab code that we are going to use we are going to use Matlab code for solving Poisson equation using Rayleigh Ritz method and we are going to use a specific tool box which is called as symbolic tool box.

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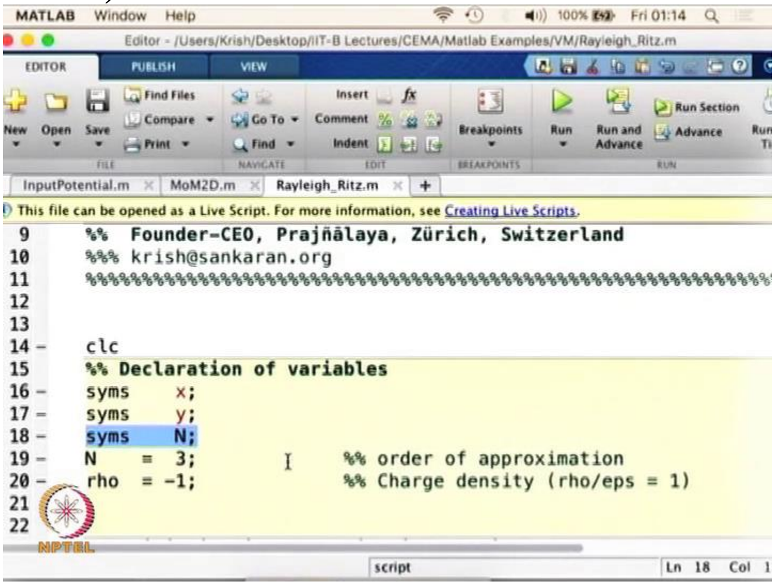

Some Matlab versions do not have the symbolic toolbox. If you do not have it you have to install it. It is very easy and elegant to do with symbolic toolbox that is why we are showing this. And for simple problems you can also do it differently but symbolic toolbox will be a very elegant thing because we wanted to see the expressions as well.

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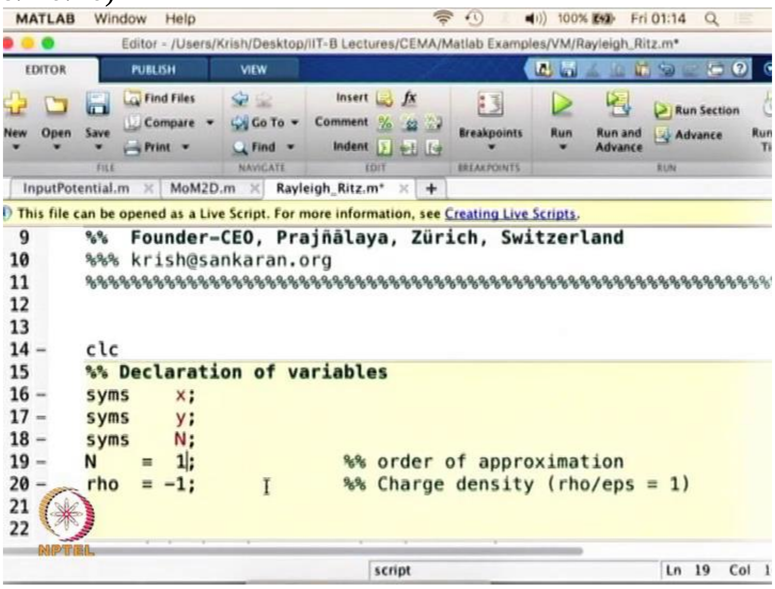

So for the first case we are going to put n equal to 1. IF we put n equal to 1.

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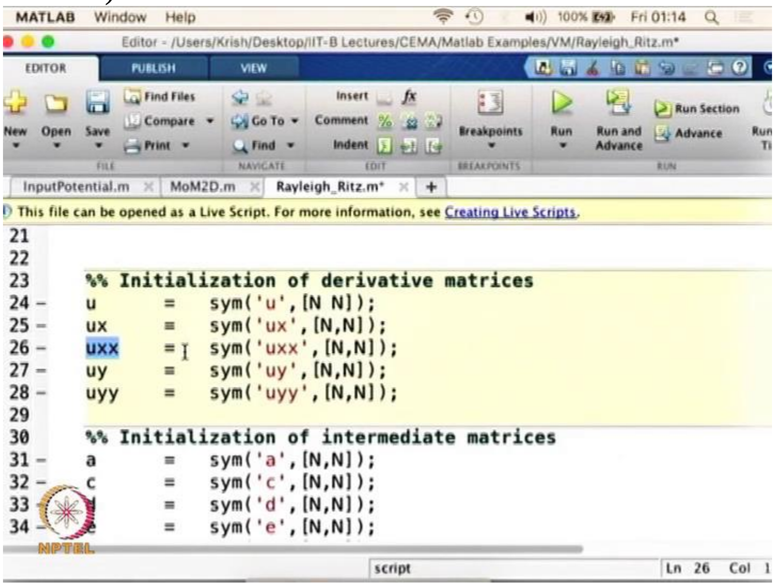

And we are going to initialize the derivative matrix so the first derivative matrix the second derivative matrix because we need them in our definitions in our expressions. So if you see this part we are going to have both the derivatives. So we have to compute the second derivative. In order to compute the second derivative we have to compute the first derivative. And that is what we are doing here. So our goal is uxx and uyy. But to get uxx you have to derive ux and in order to derive uyy you have to get uy.

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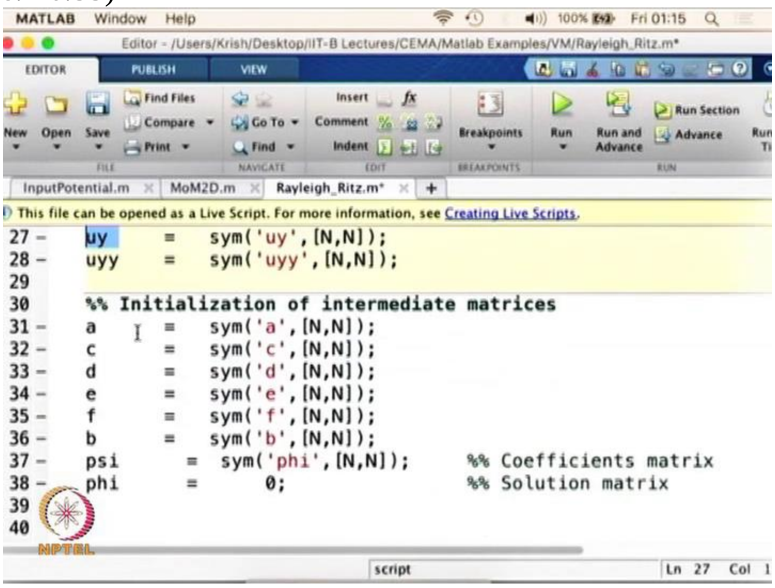

And you are initializing some of the intermediate matrices and the coefficient matrix is going to be c and the solution matrix is going to be Phi. Initially you set solution to be 0. So this is the way we start doing the problem and while we simulate the problem and get the solution. The solution will be different so initially we set it to 0.

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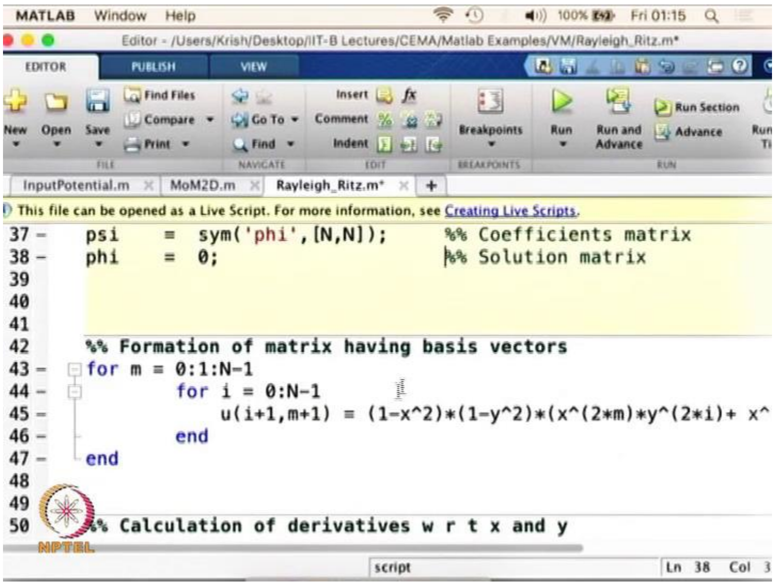

So we are forming the matrices and the matrices are formed based on the equation we have (1 minus x square) (1 minus y square)(x power 2m) (y power 2i) and we are using i instead of n here because we are using n for a different variable so we are using i instead of n, so wherever there is i we have to substitute it in the derivation which we showed before as n.

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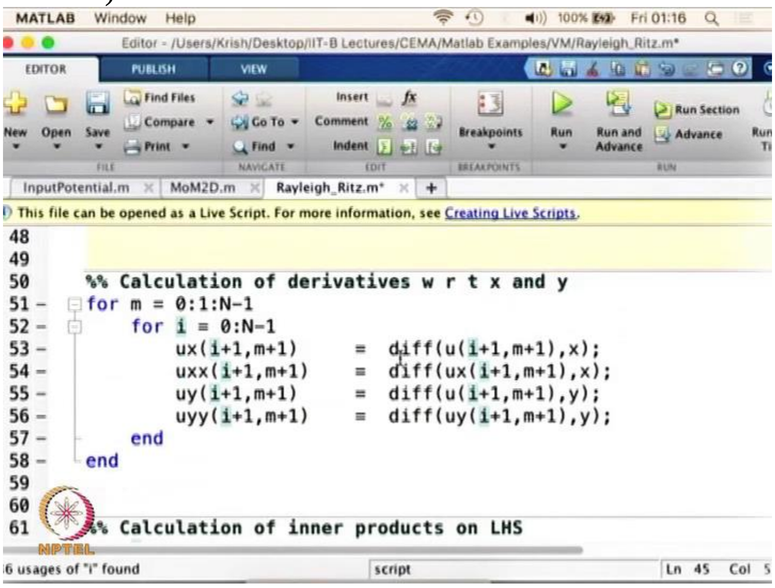

And once you do that you do the derivation of it using inbuilt function called diff.

So this is a Matlab function which allows you to do that.

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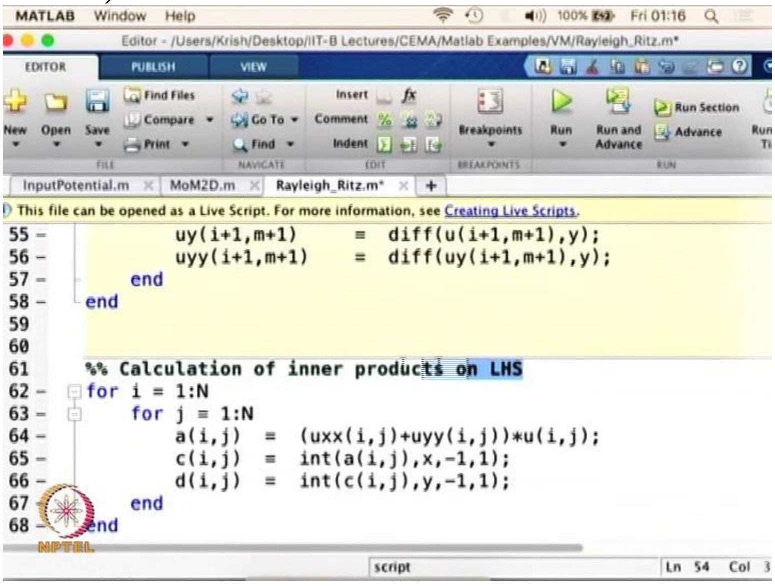

And you are computing the inner products on the left hand side;

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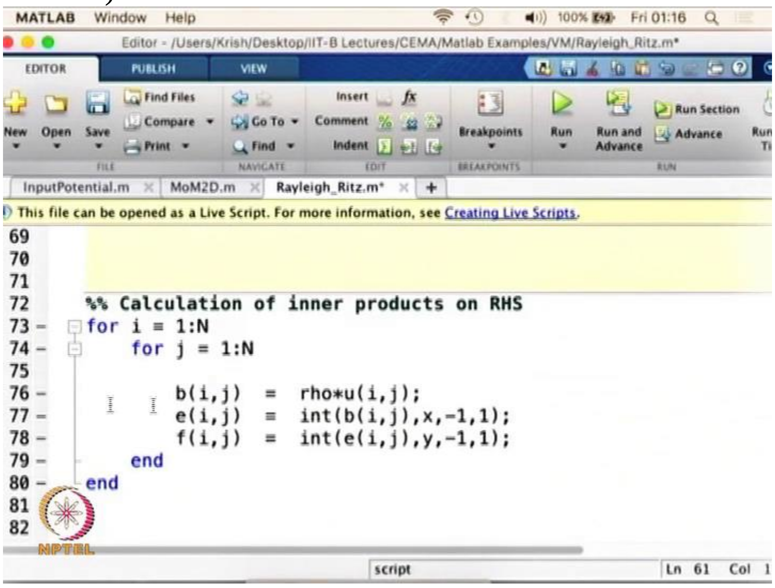

And you are computing the inner products on the right hand side as follows.

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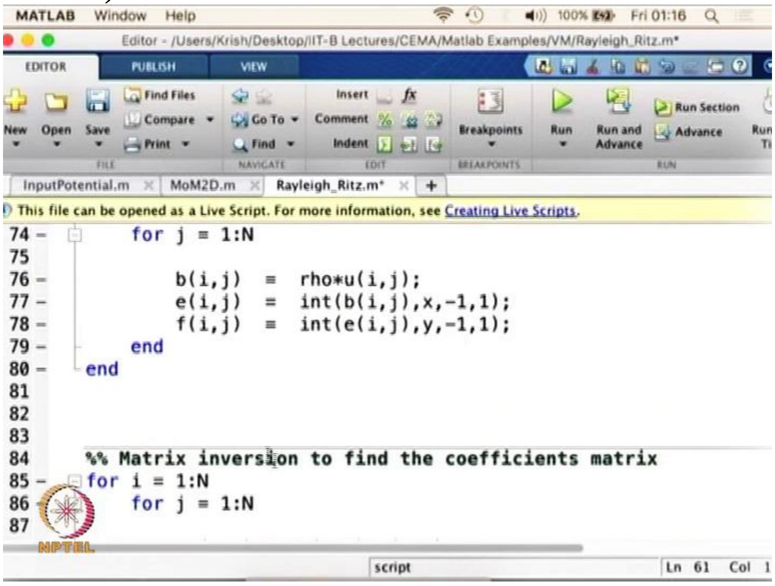

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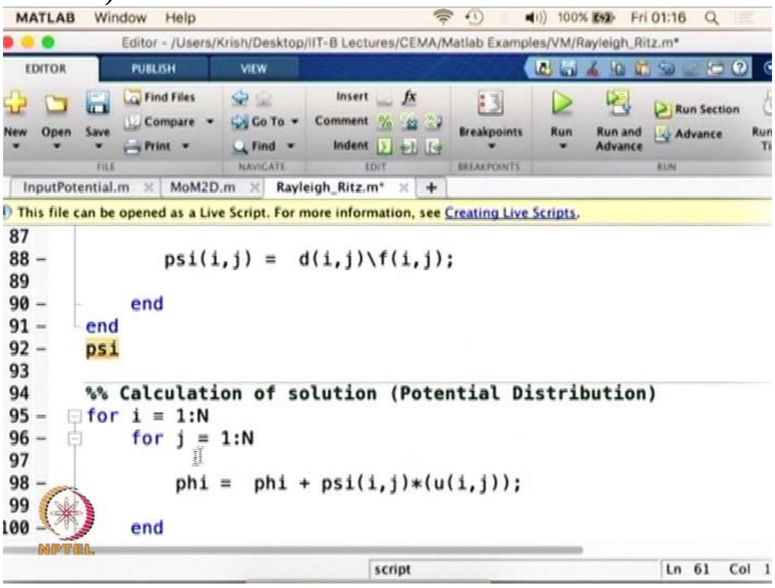

And once you do that you do the matrix inversion to compute the value of psi and you compute the solution for the domain.

So this is a very straight forward and we are going to print the solution in the final step. So now we are going to do this for n is equal to 1.

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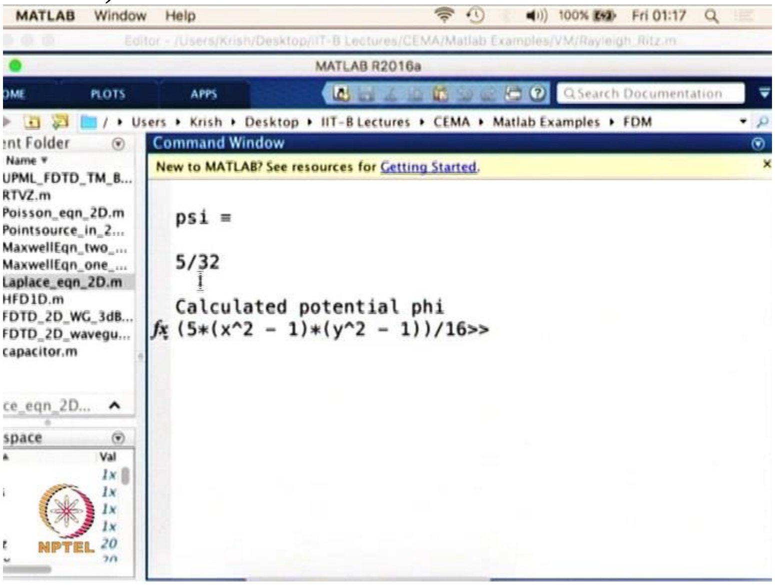

So let us run it and let us go back to the code itself here. What you see is you get 5 by 32 as a value and the potential itself is going to have this function.

So what it means is the Phi value is going to depend on the value that we are going to compute here 5 by 32 is a coefficient that we have which is coming here as 5 by 16 because do not forget the value which we have chosen for u1 is going to be.

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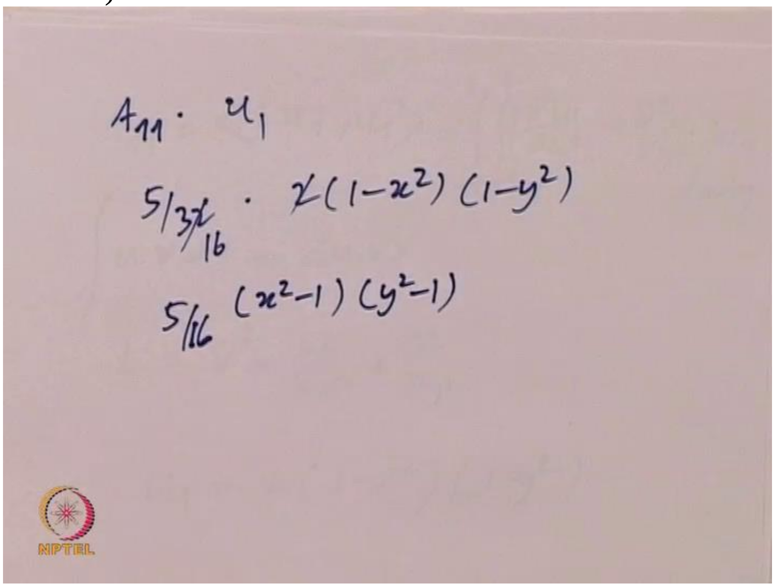

So let me explain this. So the coefficient is A 11 multiplied by the value of u that we are having. So here the value of A 11 is going to be 5 by 32 and u 1 is going to be the value that we have which is 2 multiplied by (1 minus x square) (1 minus y square) 2 and 32 it goes away it becomes 5 by 16 . And if we swap it 2 x square minus 1 there will be a minus sign and this minus sign goes here and then we swap it one more time and gets cancelled. So we will get 5/16 (x square minus 1) (y square minus 1).

So that is what a solution what you get on the Matlab program here. 5 by 16 (x square minus 1) (y square minus 1).

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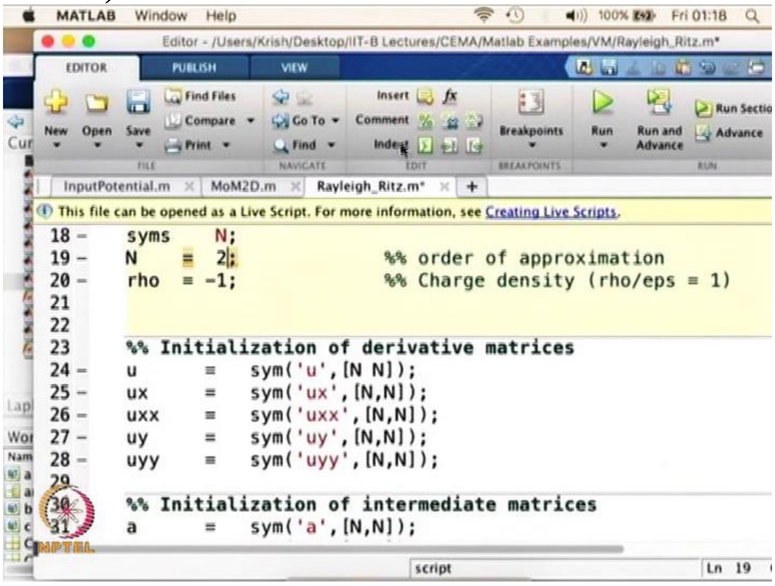

Now we will do it for n equal to 2. So for n equal to 2 you will have 4 values in the Matrix of A so when we run it.

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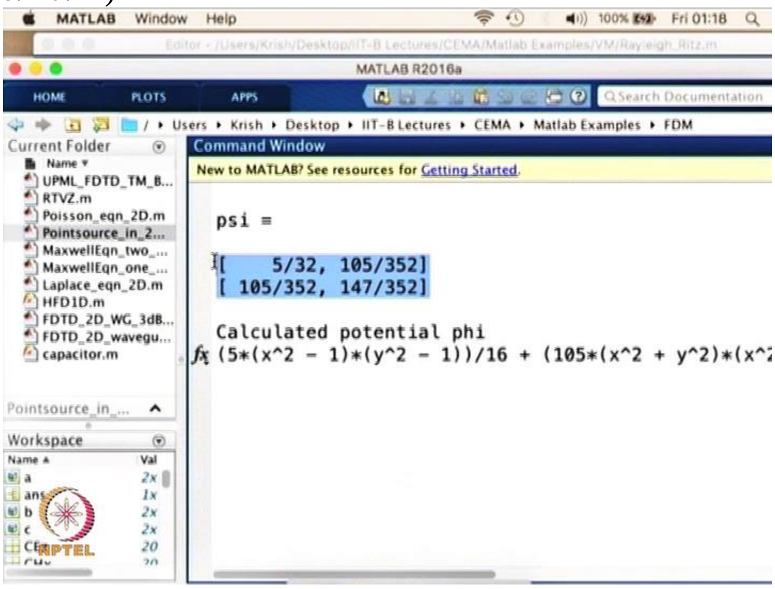

So we will see that there are four values the first value is the same value that we had before 5 by 32 which is good and we are getting another value which is 147 divided by 352. And as I explained before A 12 which is 105 divided by 352 is the same as A 21 which is also 105 divided by 352.

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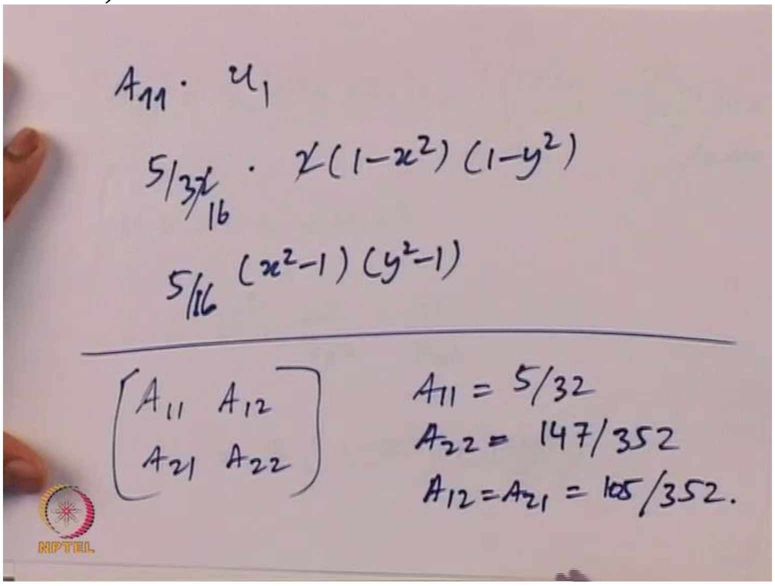

In other words I write it again A 11 A 22 A 12 and A 21. This is going to be the matrix and we are getting A 11 in the first case and also in the second case as 5 by 32. We are getting A 22 as 147 divided by 352 and we are getting A 12 is equal to A 21 is equal to 105 divided by 352. And this is what we are seeing in the matrix here.

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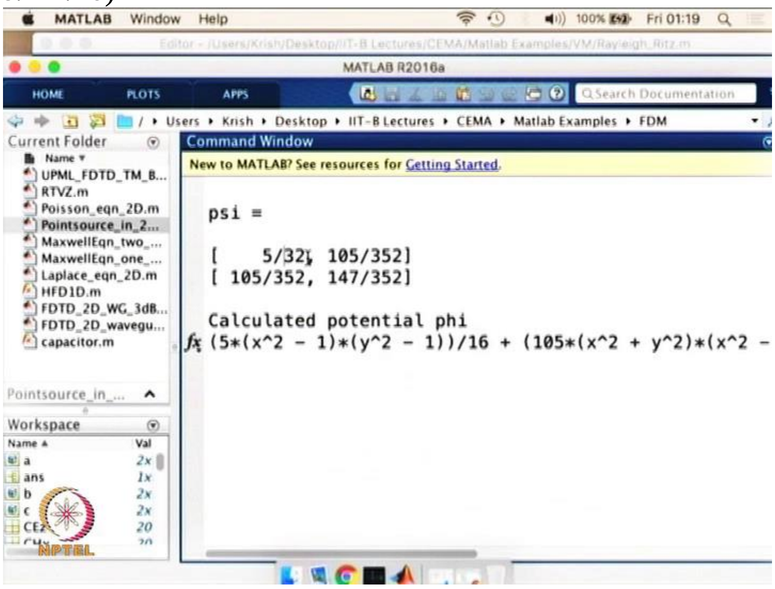

In the C matrix in the equation and as we had before we are going to have one term because of this element

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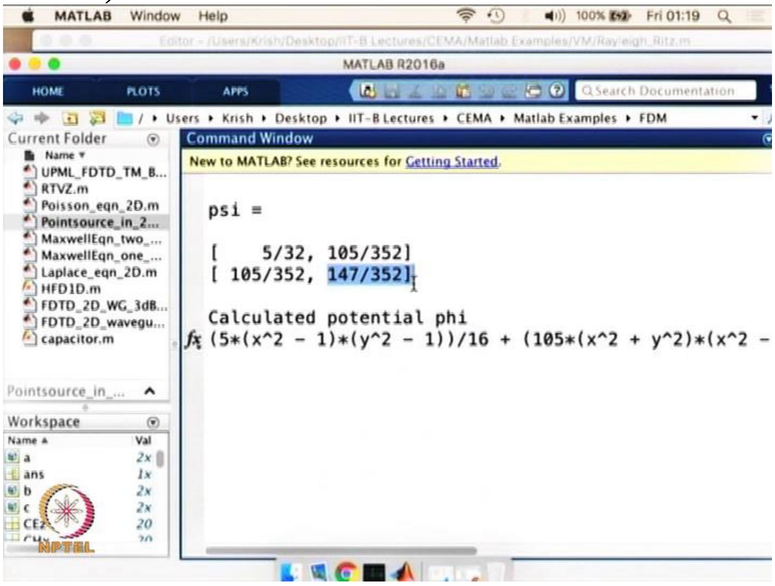

One term because of this element

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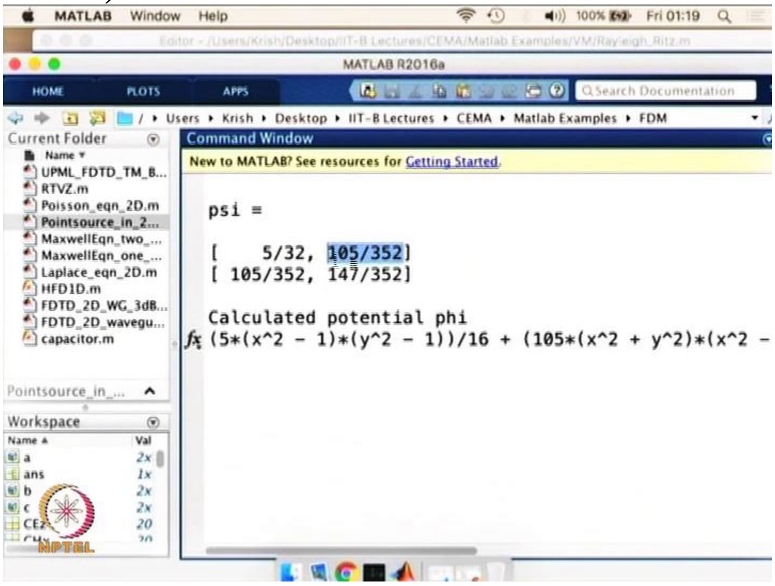

And one term because of this element because it is getting repeated so it will get added up. So we will have totally three.

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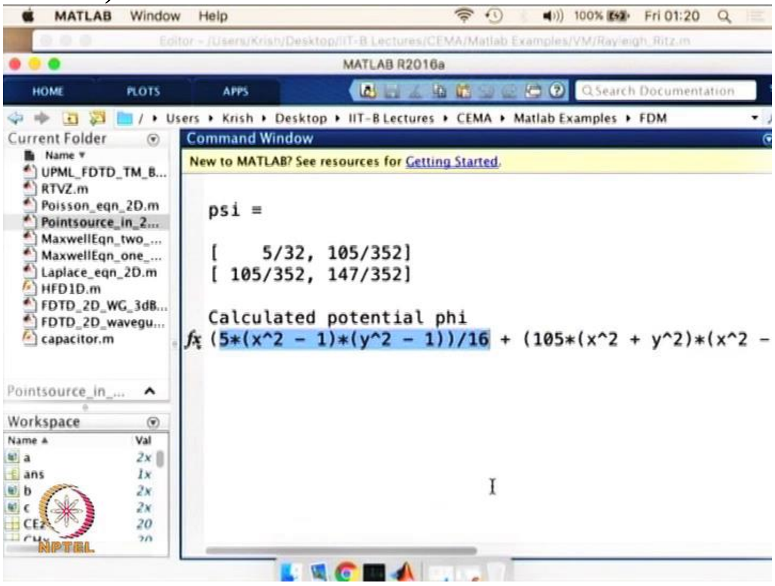

So this is the first term and this is the second term so until here is a second term

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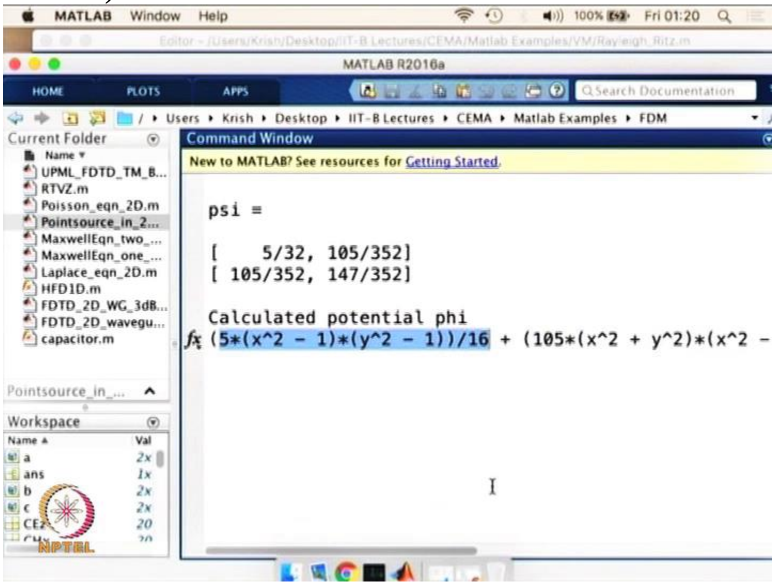

This is the first term

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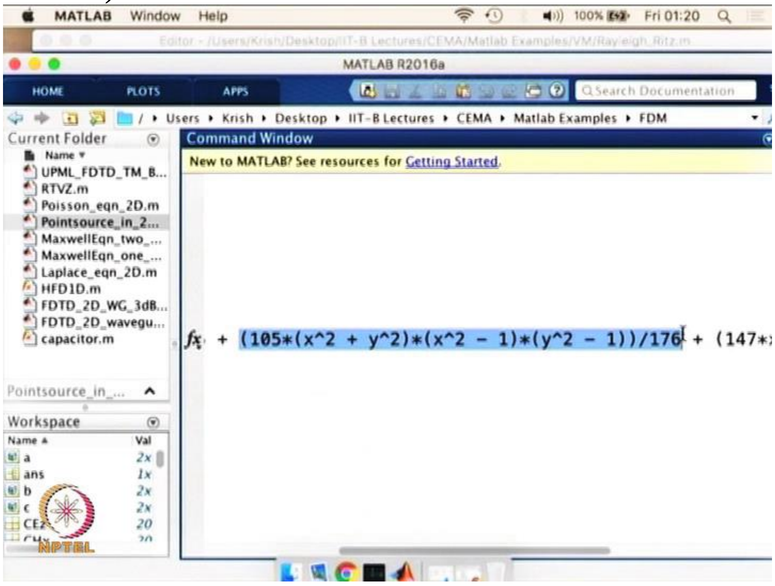

This is the second term

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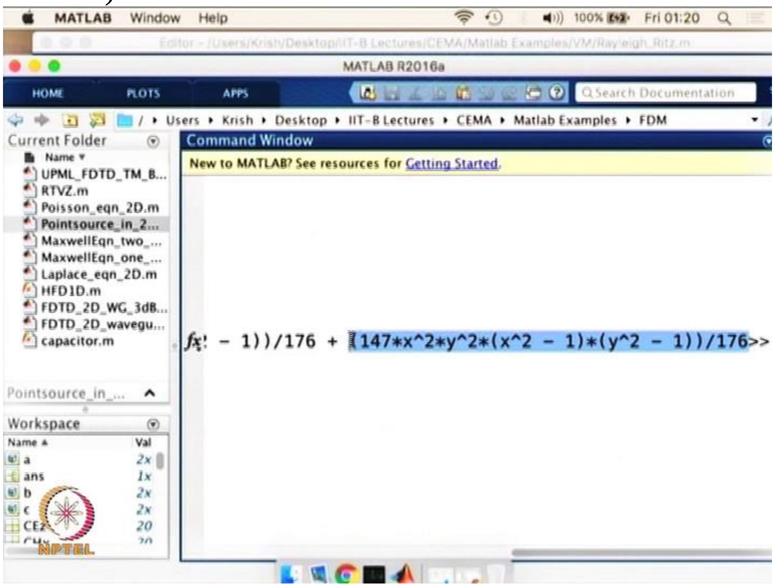

And you have the third term which is going to be from here until the end of the equation.

So this is the third term

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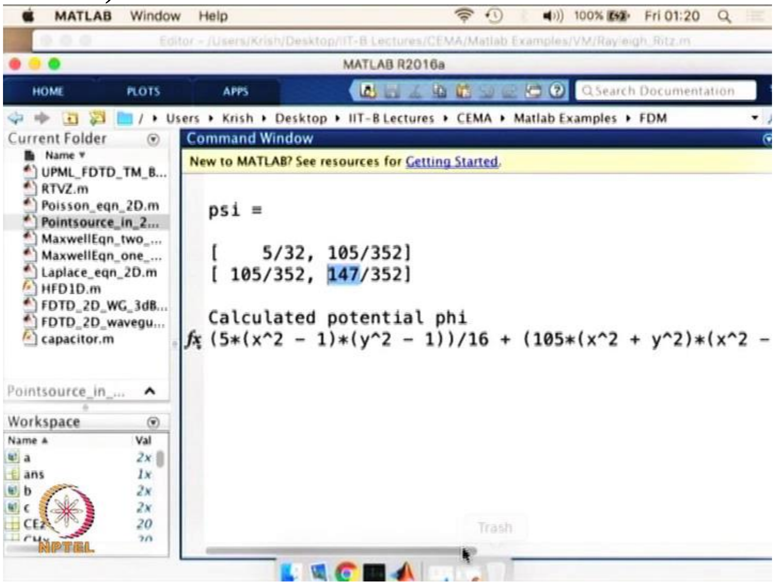

And the numerators have to be same as here and the denominators will be divided by 2 because there is a numerator 2 from u n definition itself. So anyway so this is the way we have computed for n equal to 2. And you can go in higher orders. The beauty of doing it with symbolic tool box is it allows you to have different orders.

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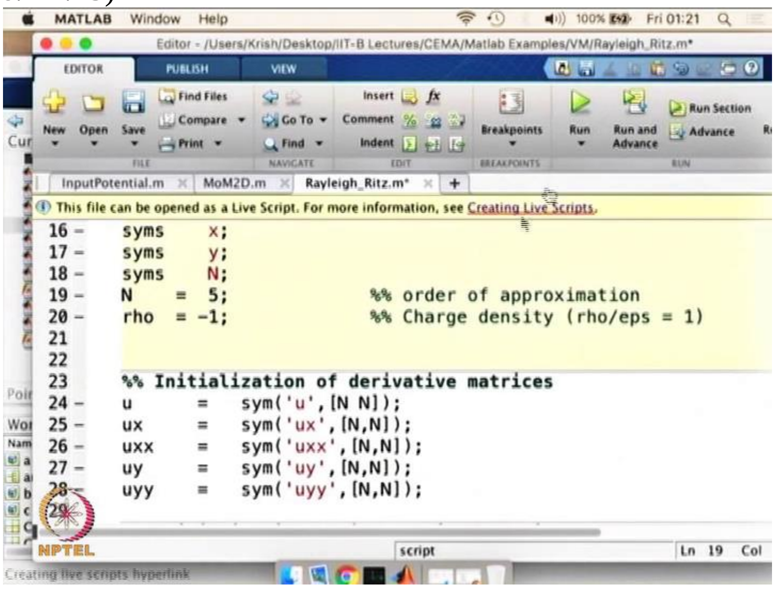

So for example if you have n equal to 5 you simulate the program what you will get is 5 terms on the leading axis for c.

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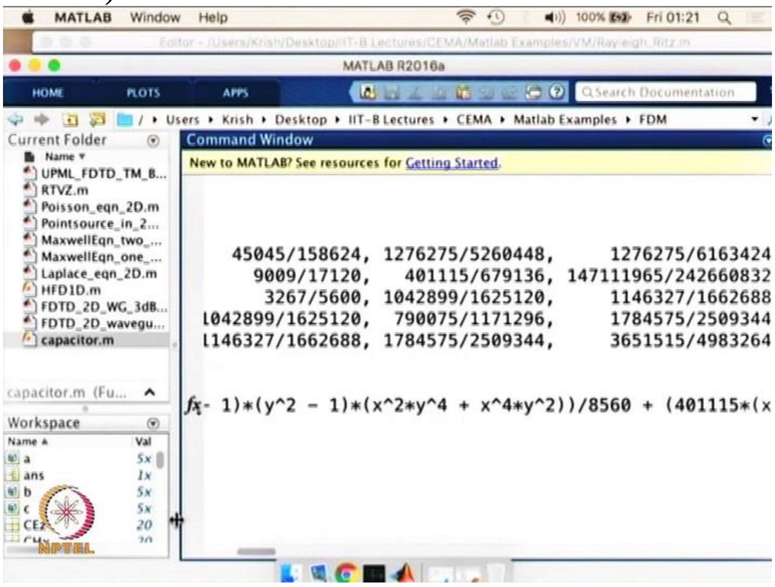

So one term here one term here one term here one term here and one term here. So these are the five terms.

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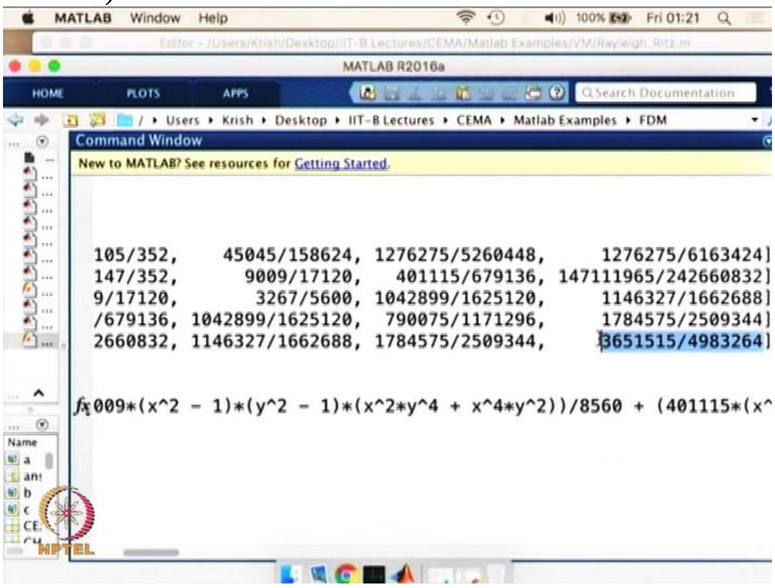

First this is the last term

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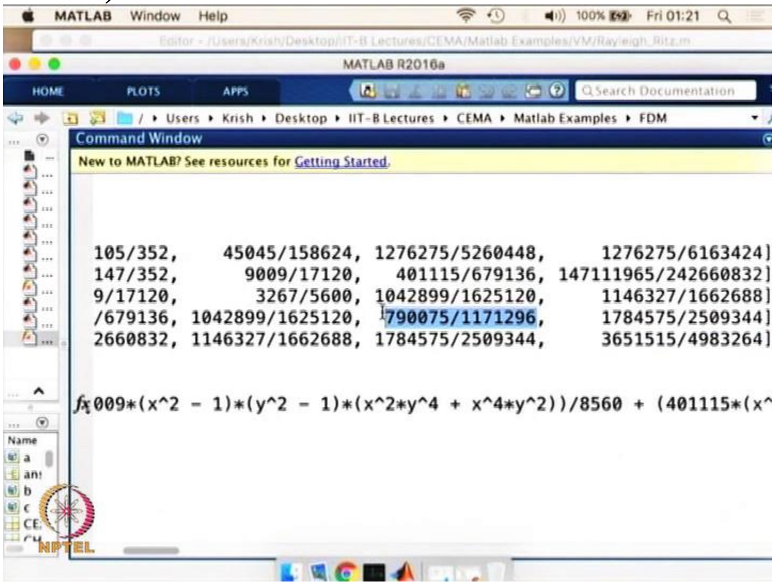

And this one term above,

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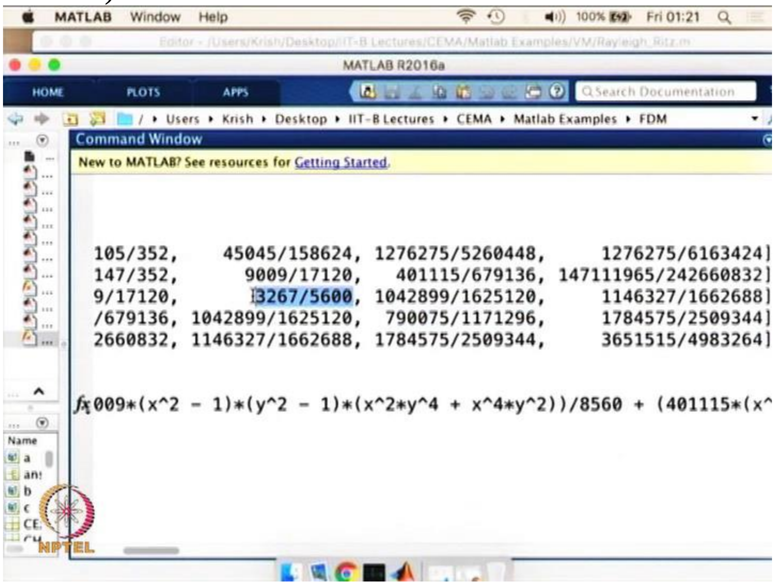

This is the third term

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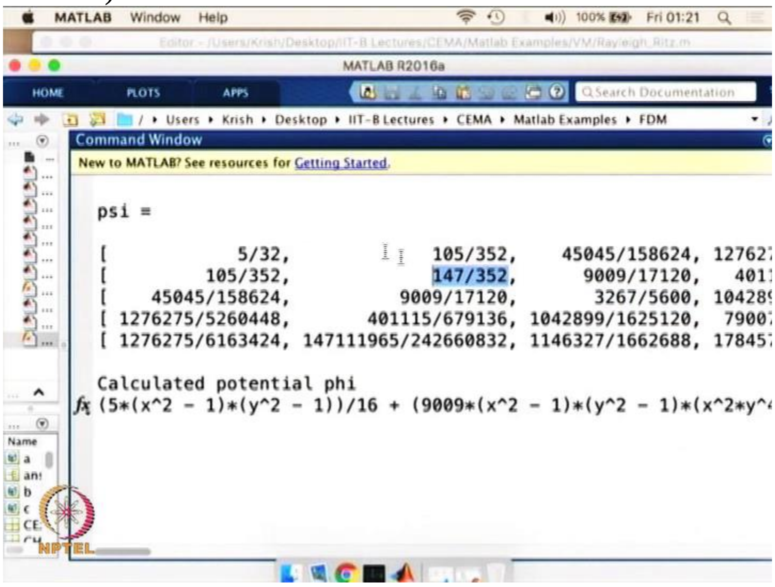

And this is the fourth term

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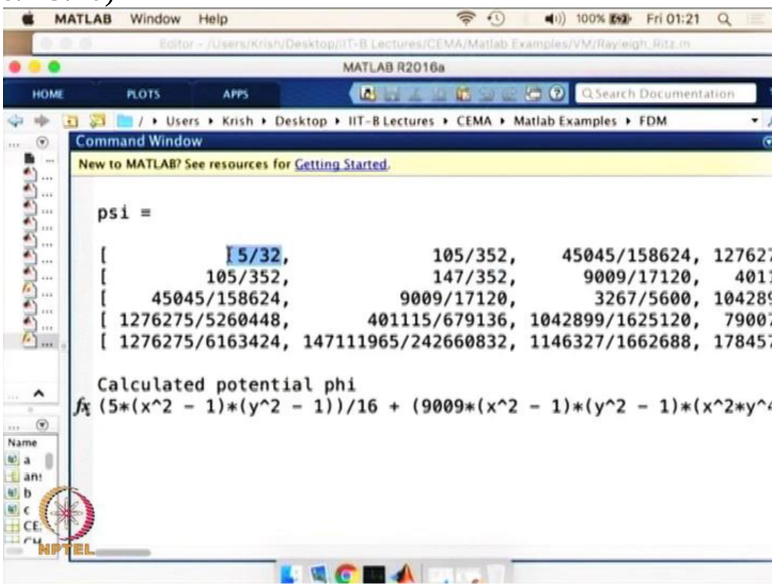

And this is the fifth term.

So these are the leading diagonal terms and you can see 5 by 32, 105 by 32 , 105 by 352, and those terms are repeating and that is how it should repeat because we are going higher in number of basis function whereas the lower basis function values will get repeated.

So what we have done in this problem is we have simulated standard Poisson Equation using a very useful toolbox called as symbolic tool box from Matlab and we have simulated it for very many basis functions. The reason for going for symbolic toolbox is to allow you to create a general program and you can do it for different number of basis function.

And we have showed for this case please take the code and try it for yourself and as I said before you need a specific toolbox which is a symbolic toolbox which is something you can download for Matlab. And try it out for yourself how one can use the in built functionalities cleverly to solve such simple problems. Thank you!!