Computational Electromagnetics and Applications
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Lecture No 18
Variational Methods

So this module is going to be a little bit mathematically heavy. And for people who are not interested in the deep mathematics of this variational method you can very well skip it. But people who wanted to know a little bit more about how to apply this variational method for various applications and see the mathematics behind it. We are going to look into the method of weighted residuals which will also be very fundamental for us to understand the method of moments which will look into at a later stages. So without further a due let us look into the method of Weighted Residuals.

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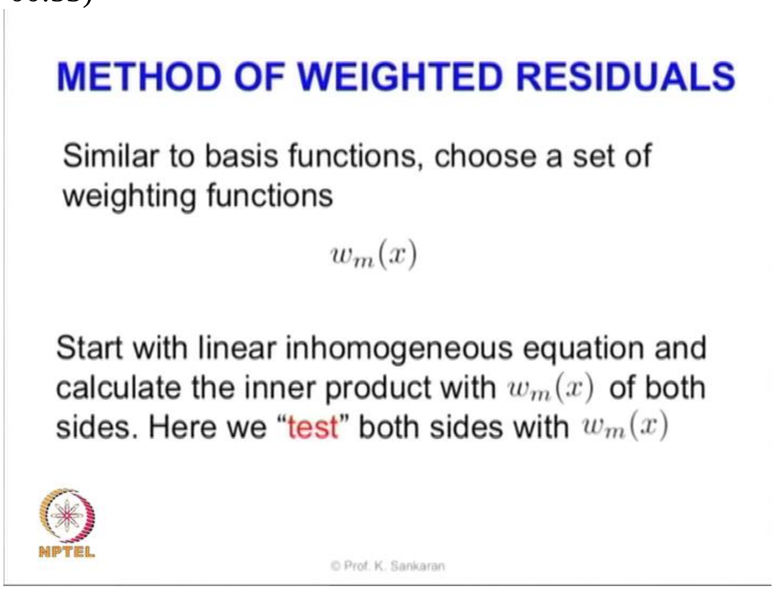

Similar to the basis functions which we have used, earlier case we used the v n as the basis function. We are going to choose a new set of function which is called as a waiting function. The waiting function is written with w and the basis function are written with v n. So we start with the linear inhomogeneous equation and calculate the inner product with this waiting function on the both sides. Here we have the test both sides with we are testing both sides with v n.

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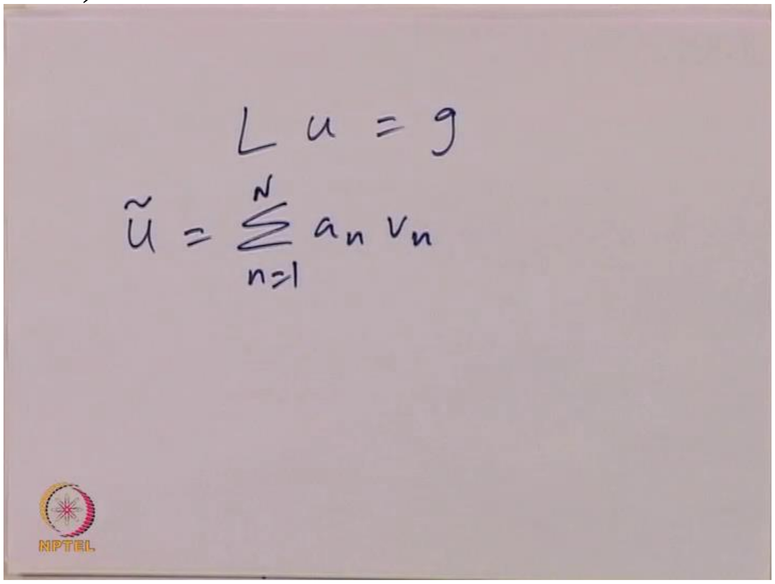

So what we are doing here is remember our first equation is L u is equal to g and u is written as sum of certain a n v n and certain number going from 1 to N. And we have g which is the forcing function it is known.

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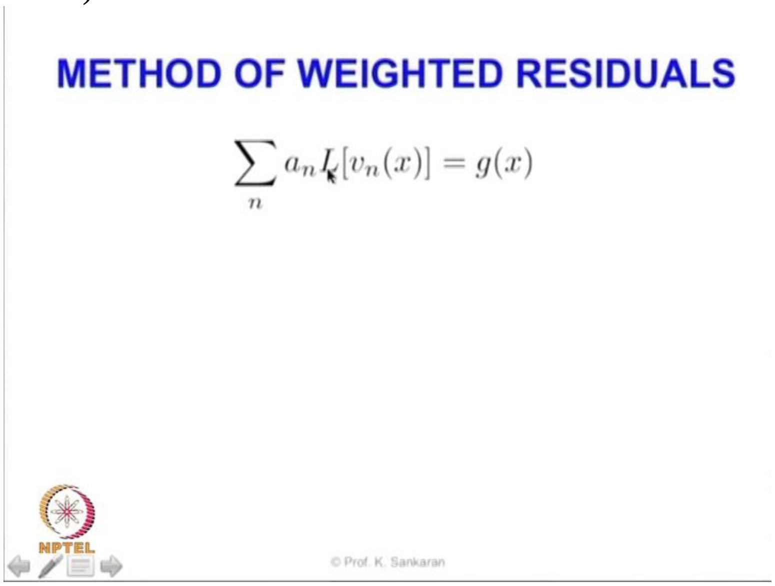

So what we are doing is we are taking the value a n v n and we are bringing the operator L inside the summation.

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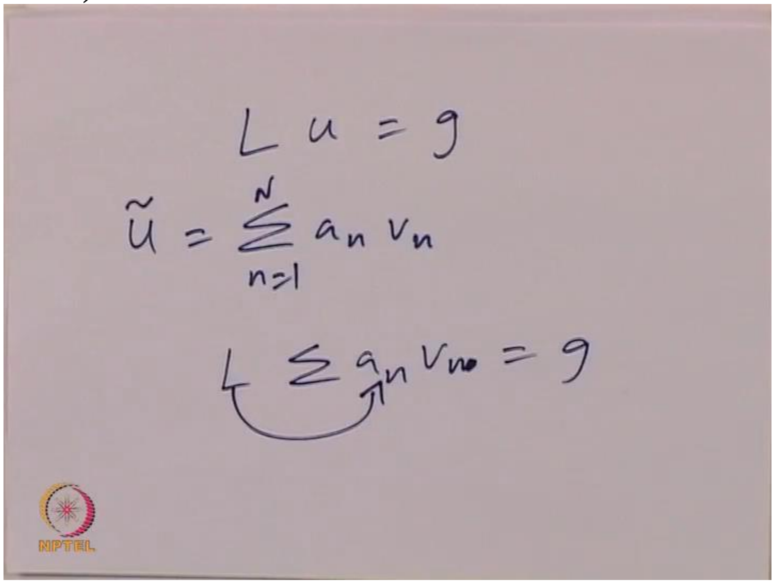

So we are starting with L u a n v n is equal to g and we are bringing this L inside the summation. And that is what we see here in this particular slide.

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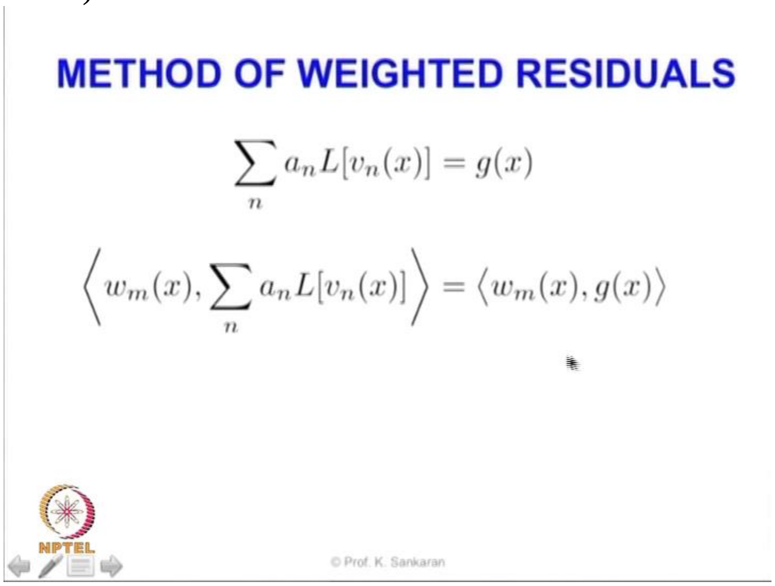

So where we have the L inside the summation and they are testing it in the sense we are doing the inner product of w m on both sides. So remember the inner product is in a way testing the similarity between both of these things in one way or the other. When the inner product is equal to 0 I said that they are orthogonal, when they are similar the number will be a very large number and when they are in between that means they are in between orthogonal and similar. Similarly this is also done on the right hand side.

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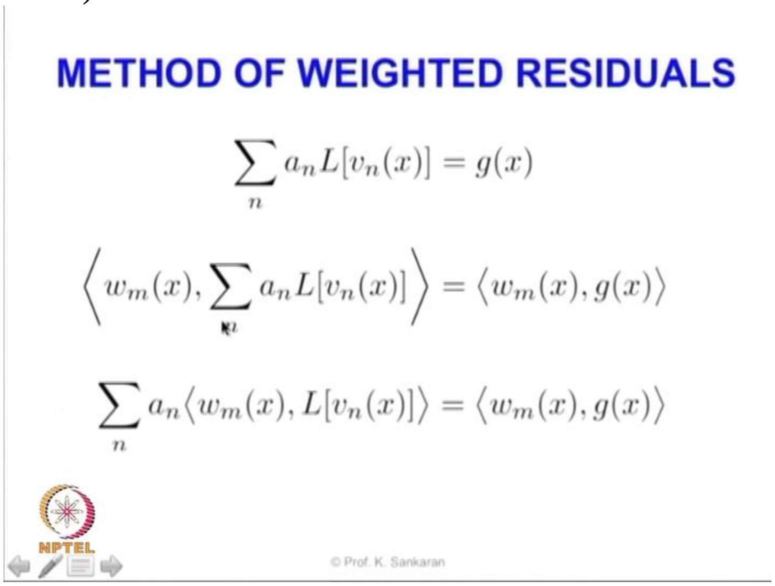

So we can take out the summation out. And what we will get is we will take out the summation and expansion functions out so it will be inner product between w m (x), L [v n (x)] is equal to the right hand side.

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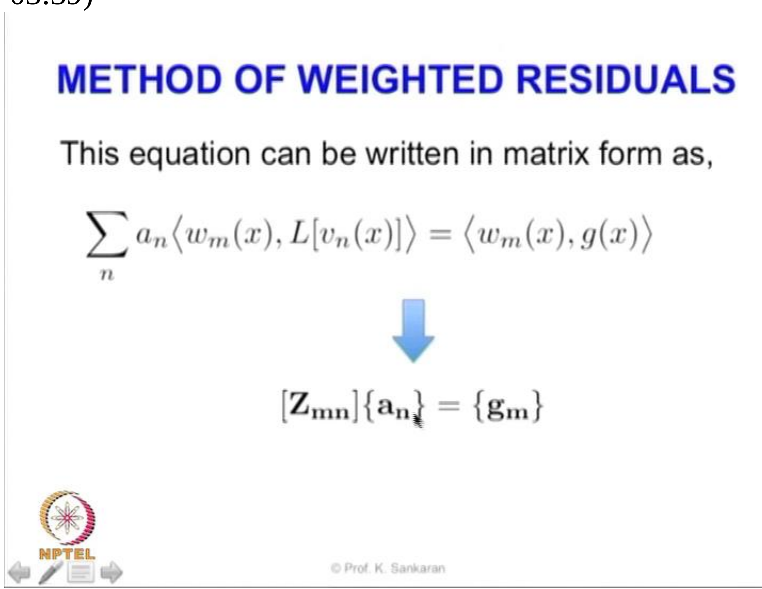

And this equation can be written in the matrix form as follows: So what we have got here is this a n which is a vector which is coming from here and z mn which is nothing but this particular inner product term which is a matrix of size m, n because it has m and n here and then the right hand side which is again a vector which is purely depended on m.

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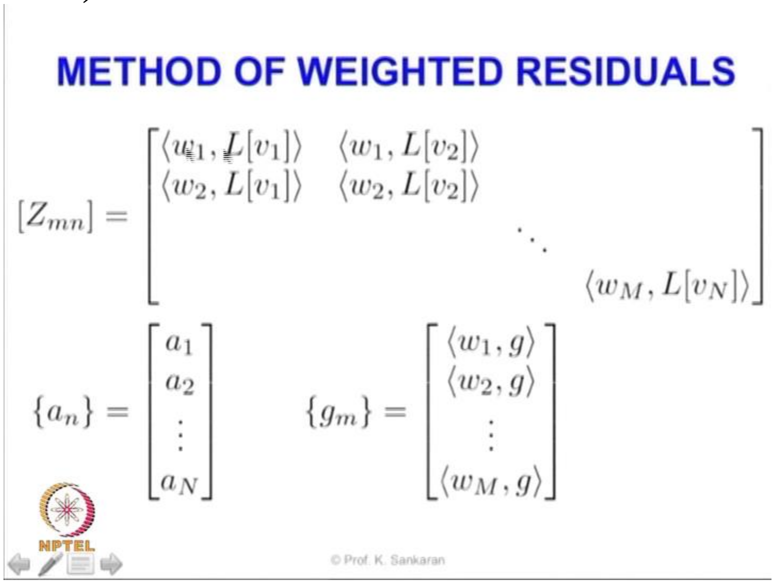

So when we are expanding this we will see what we will get is the terms as follows: this is a starting point where we say the method of weighted residual is we are weighing it with respect to a test function and we are doing the testing on both sides. How are we going to go from this point to the Galerkin method is a very simple assumption. The assumption is when I say when the w terms are same as the basis function.

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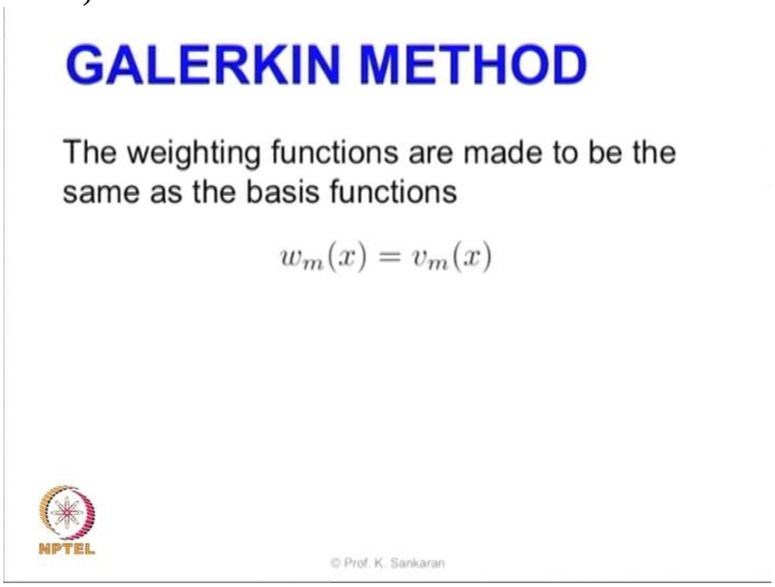

So remember these are the basis functions which are used to expand my unknown and these are the weighing functions I am going to use as a testing functions on the left hand right hand side. But when I say they are both same I fall into a method called Galerkin method. It is due to one of the greatest mathematician from the Russian republic who has contributed enormous amount of techniques and tools to applied mathematics. And this is one of his classical examples where he has just said by using the weighing function and the testing function or the weighing function and the basis function to be same we end up into a very very simple form.

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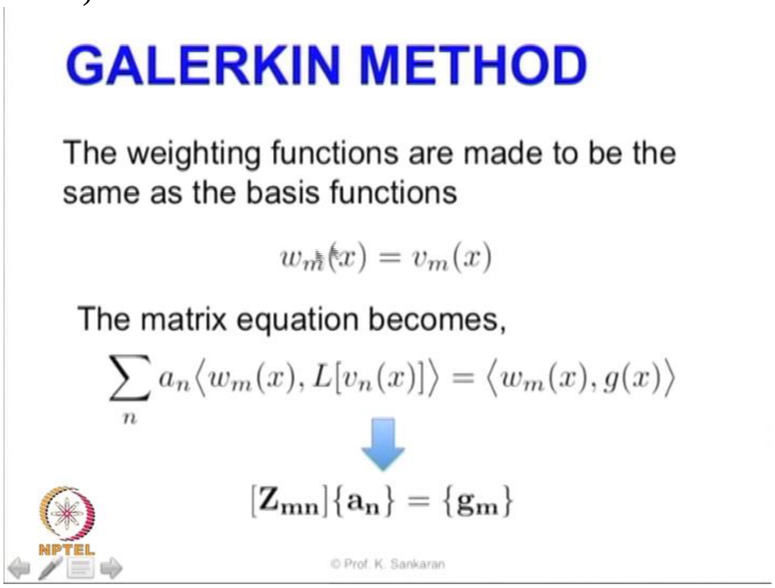

So the matrix becomes very simple as in the slide here where we got the left hand side as only in terms of w m. So we can say w m equal to v n. So the terms will become very simply only as a function of v m.

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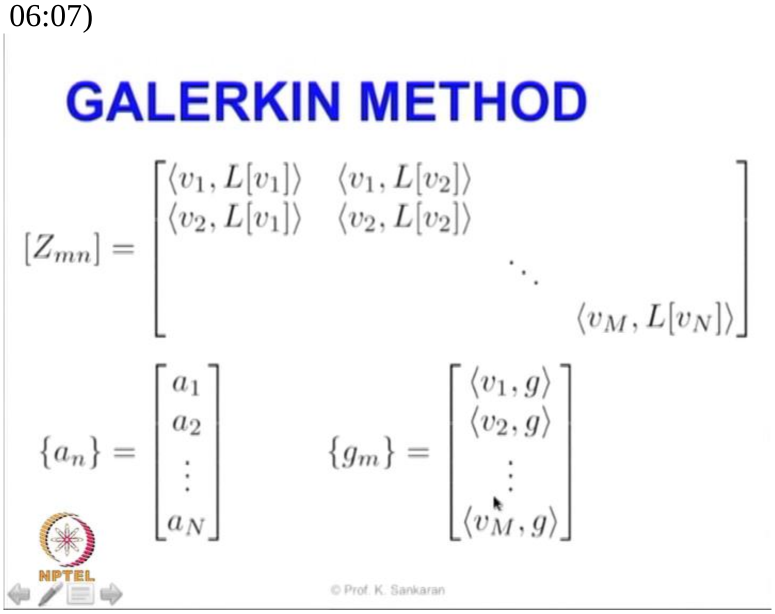

So remember in the old case our left hand side was the function of w and v but here we say we are having them both same so you will have only v on both in the first and the second term. And also on the right hand side instead of w we will have the basis functions v m.

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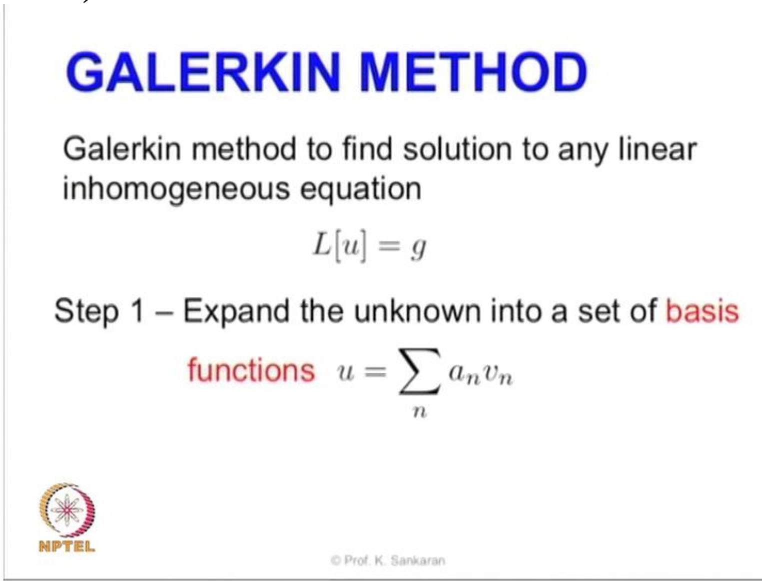

So Galerkin method to find a solution to a linear in homogeneous problem is simply starting with the initial PDE and we are expanding it using the set of basis function like in the case before. Where we say our initial set of values are given by the basis function u is equal to sigma n a n v n and we are just bringing the L inside the summation and we will have the simple equation given as follows.

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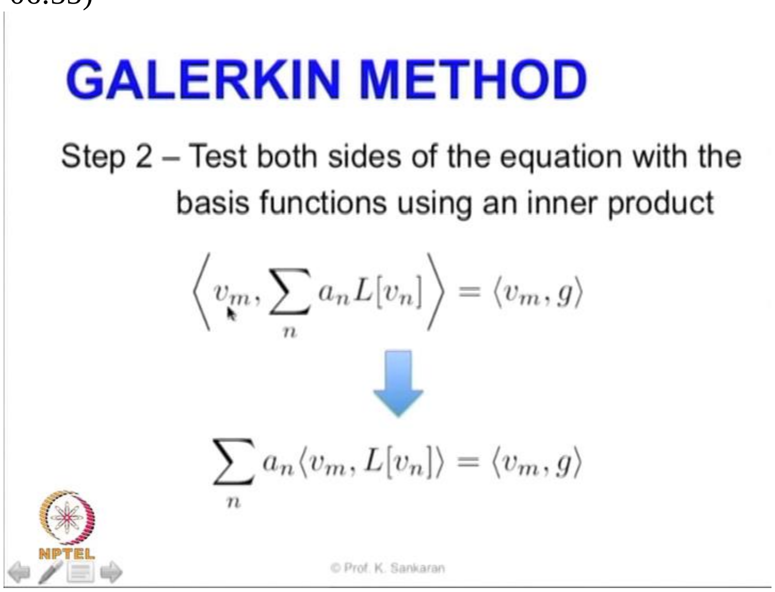

And from here we can test both the sides using the basis function itself and we end up in a very simple form. So remember we are saying the weighing function and basis function are the same. So we have only the basis function on both sides and we end up in a very simple matrix form as follows:

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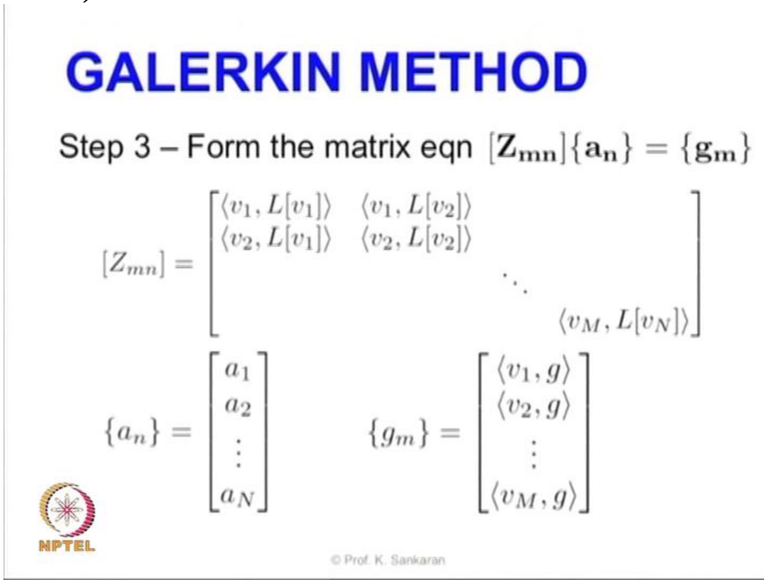

So what we have done so far we have given you the logic behind the method of Weighted Residual and how we can go from method of weighted residual to the method of Galerkin. And with that we are coming to a very nice stop. But before going to a stop what I would also like to do is a very simple example. Because we started with the action principle and from the action principle we are going into the PDE or the Euler equation. You remember how I derive it in the earlier modules. But I would also like to look into how to derive the functional starting with the PDE itself. So this will be a kind of an exercise which I would like to do in this particular module before we break. So that will kind of get you an understanding how we can go from a PDE to the functional and also starting from a functional to a PDE. So both ways we are covering the main points.

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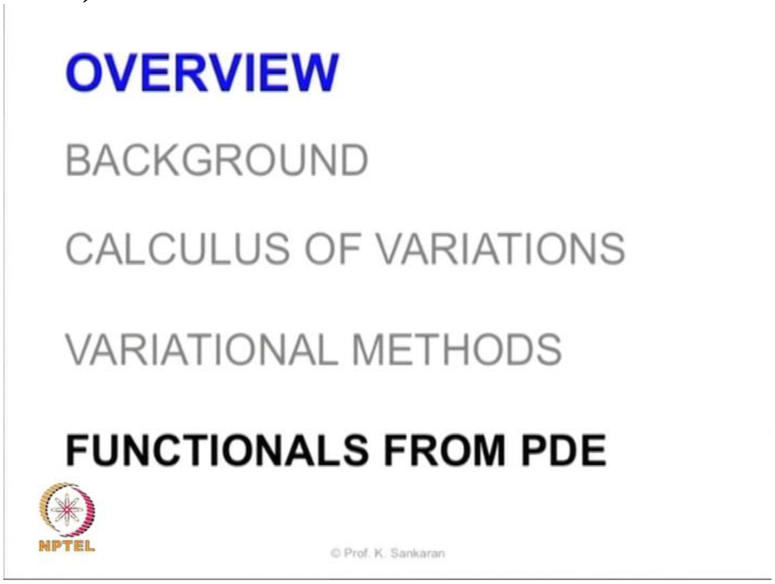

So with that being said let us look into a simple example where we start with the functional from a PDE.

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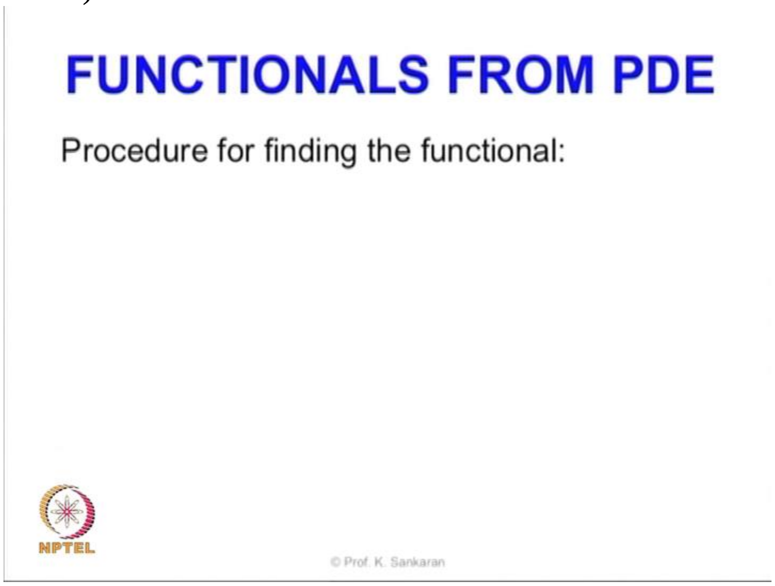

So let us assume that we have been given a particular PDE which is written of the form.

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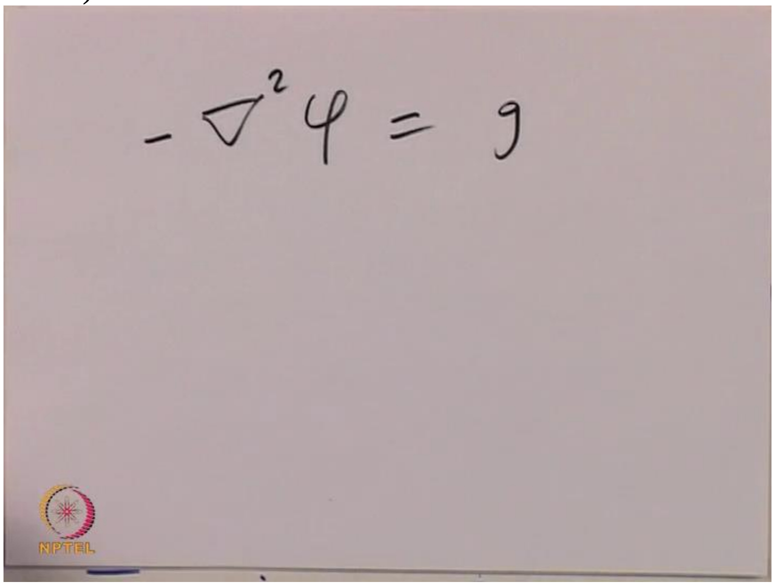

Let us say Poisson equation where we have the laplatian on the potential is equal to. So I will take the minus of the laplatian is equal to some forcing function g.

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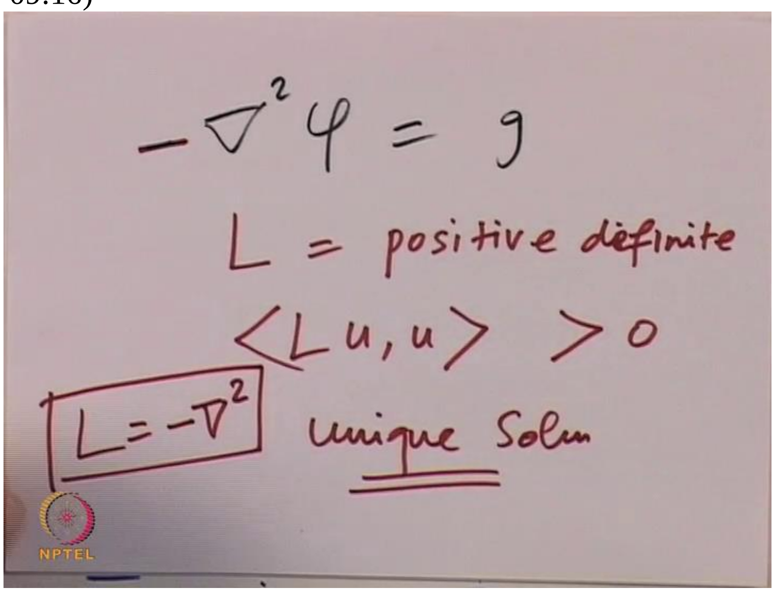

The reason I am taking a minus because of making the operator L as positive definite. So positive definite is a fundamental requirement on the operator itself and if the operator L should be positive definite. So when we operate the operator the inner product so we will get the value that is more than 0. So that is the definition of the operator being positive definite. And if the operator is positive definite the solution to this particular problem will be unique. So the reason why I have put the minus here will make the operator L is equal to minus Del square as a positive definite operator. So that is the reason for this minus it is not that important but it is important for you to know the reason behind it.

The reason behind it is to get a nice function and the uniqueness of the solution.

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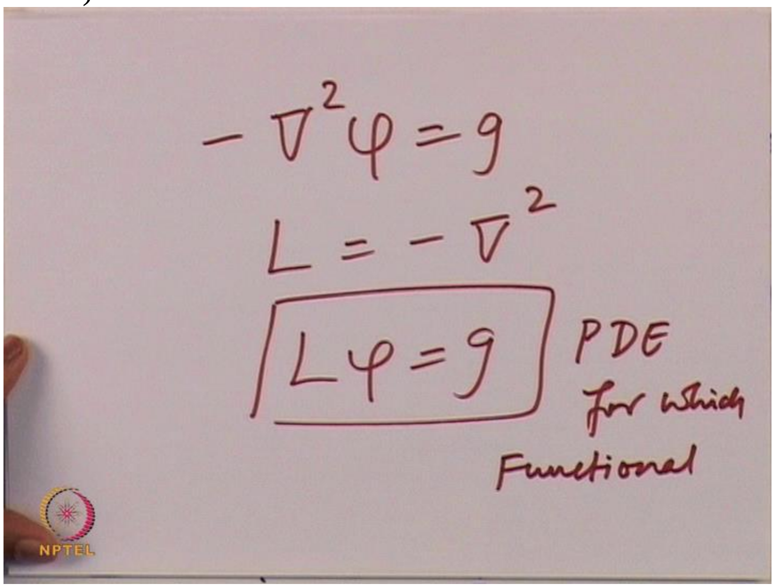

So let us start with this Laplace equation where we have got minus Del Phi equal to g. I say L is equal to minus Del square. So my expression will be L of Phi is equal to g. So this will be my starting point, this is the PDE for which the I want the functional. The functional for this particular PDE.

So the first step of doing this is I have to multiply this particular PDE with a particular variation that is what I am doing to do in the step 1.

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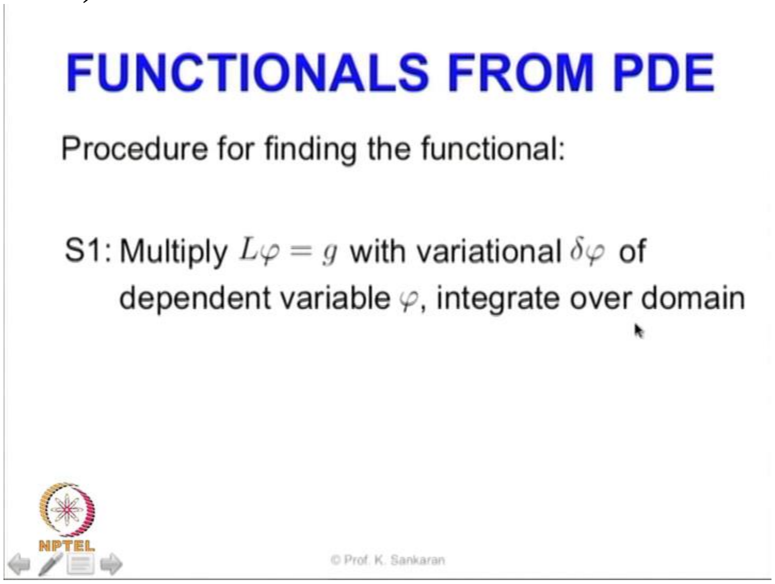

So step 1 as you can see in the slide multiply this particular PDE with the variation written as Delta Phi of the dependent variable Phi and I am integrating it over the domain.

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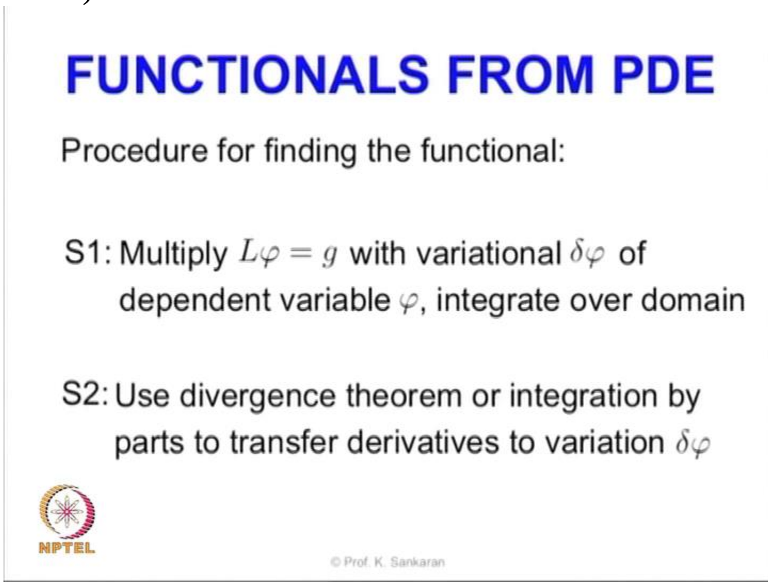

And the second step is I will use the divergence theorem or integration by parts, either of them because they both lead to transferring the derivatives to the variation itself. Because I do not want the derivatives to sit on very difficult expressions. So I wanted to transfer it them to the variation itself. So it becomes mathematically itself.

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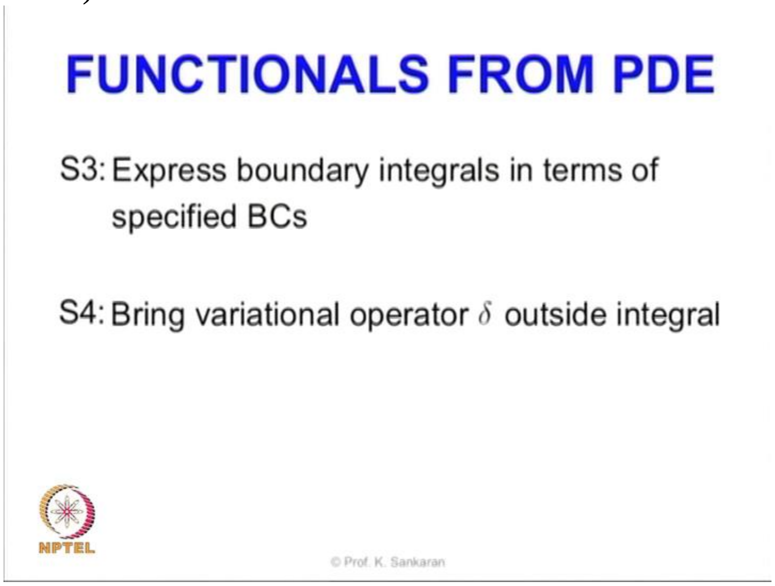

And then the step 3 is I apply the boundary conditions on those terms and I bring out the variation operator outside so as to get the final functional. So I am going to do this on this particular expression.

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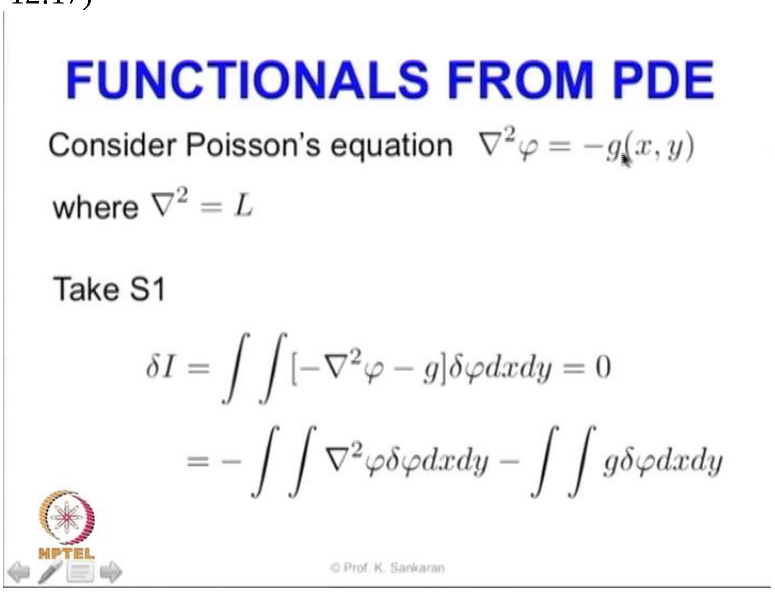

I said I am going to set the value of L as this and the minus sign is taken on the other side. So I am doing the first operation. I am going to multiply by a small variation. So and then I am integrating it over the entire domain. So let us do this step by step so that it will be clear how I have arrived to this point. So now we will take a simple Poisson Equation and do a little bit of mathematics. It is going to be a little bit tough, but if you follow and stay with me I think we will get to the final expression.

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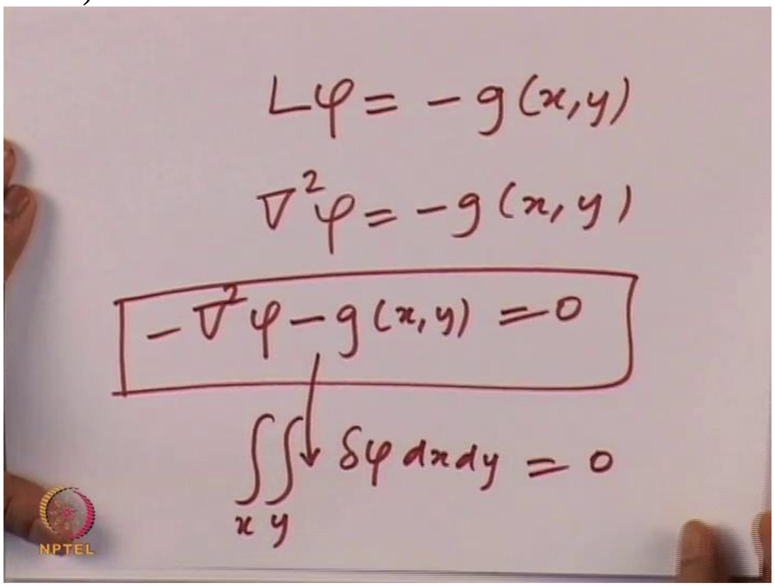

So let us start with this expression given by L Phi equal to minus g (x,y) and L her is the Poisson operator so we say Del square Phi equal to minus g (x,y). I take it to the left hand side. And I say minus Del square Phi minus g(x, y) is equal to 0. So this is my starting point and I am going to multiply this expression with a small variation of Delta Phi and I am going to integrate it over, so I am going to put this expression here and then I am integrating it over the surface. So this is a two dimensional surface. I have a two dimensional integration. So x, y and then I will have dx and dy and the left hand side will be equal to 0.

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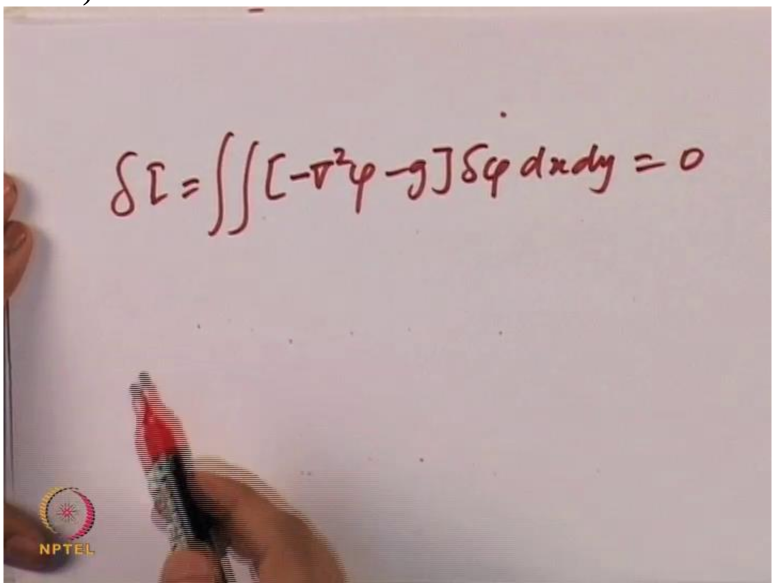

So when I put this expression here you will get to the point where we have the form when I say that is a small variation in I is equal to double integration [minus Del Square Phi minus g with a small variation in Phi dxdy is equal to 0. I say this is this expression of Del of I because I have a small variation. My assumption is this particular expression when I do it step by step and take out the Del out of the expression I will get to the functional what I am interested in and I have assumed that the functional will be called as I so I have an idea.

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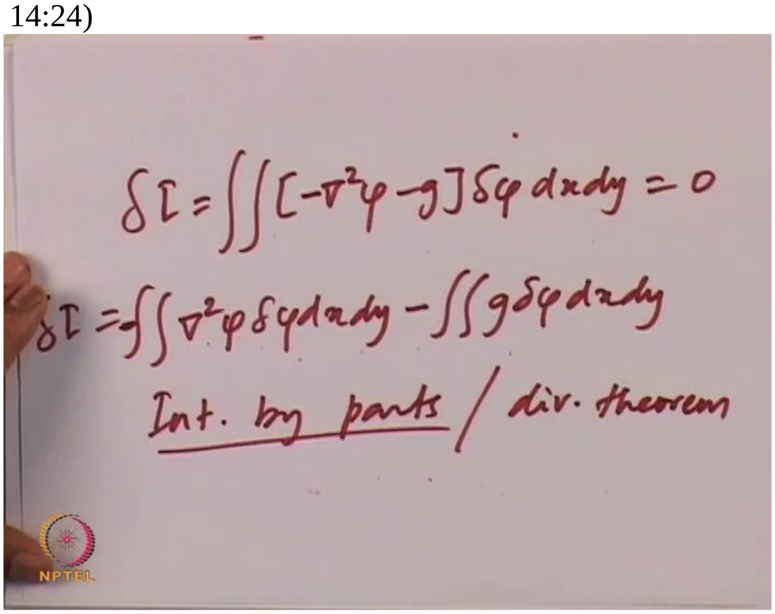

So this I can expand equal to the double integral we will take the minus out Del square Phi Doe Phi dxdy minus double integral g of Doe Phi dxdy. So this is the basic expression and this expression is equal to the value I have given. So this is the step number 1. So in the step number 2 what I am doing is I am going to apply either integration by parts or I can apply the diversions theorem. So I we will use the integration by parts. So as to transfer the derivatives on to the small variation itself. So my idea is to change the derivative from the higher order derivatives directly on the small variation. So I am doing the integration by parts to get to the point where I say x equal to.

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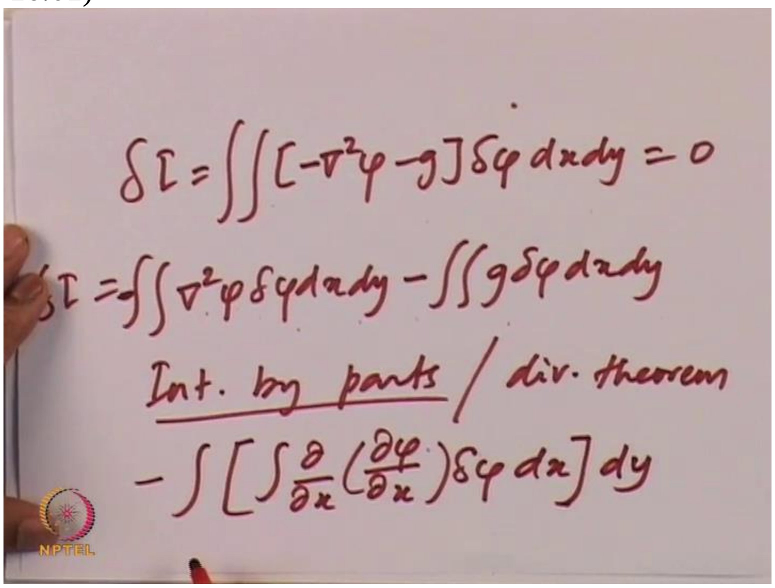

Let us say my first integration by part will be minus integration [integral d by dx (dPhi by dx) small variation times dx] dy. So this is I am just changing the way it is being written.

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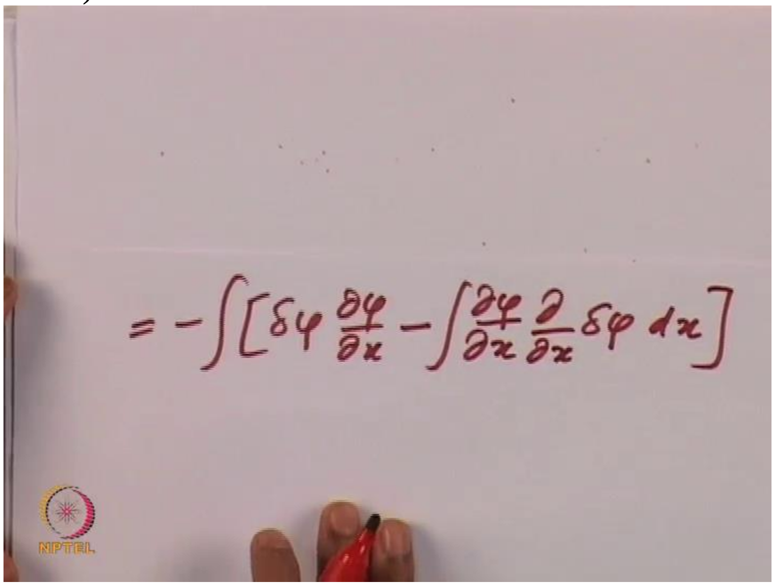

And what I am going to get is the next expression which is this equal to minus integral [small variation dPhi by dx minus integral partial differentiation with respect to x dPhi by dx small variation times dx]. So in the step 2 what I have done is I have done the integration by parts to split the term from the initial point. So in order to take the higher order derivatives from here directly on the Phi. So I have arrived to a point of writing it like this.

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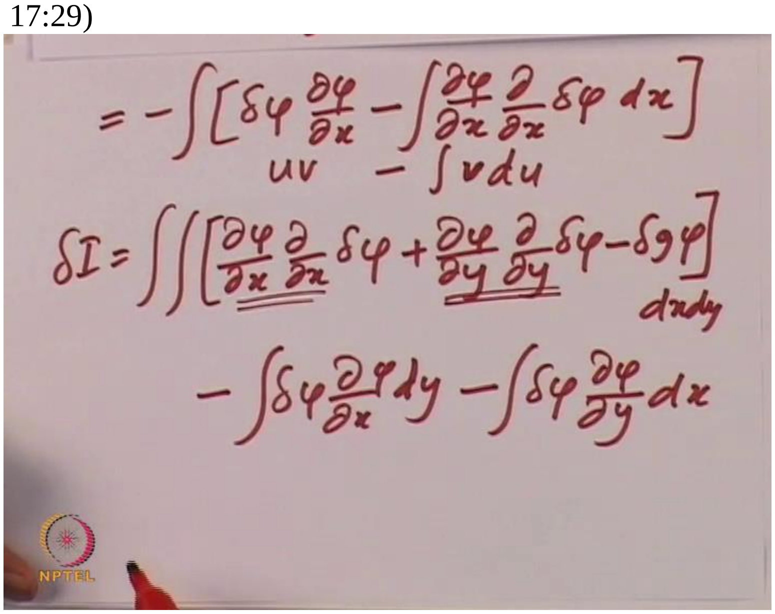

So in the step 3 and step 4 what I am going to do is I am going to apply the boundary conditions but for that I need to derive to the point but what I will have is dPhi of I is equal to two derivatives so remember that I have applied on integration by parts to get this two expressions this expression is the way to get from integral udv is equal to uv minus integral vdu. So this is the integration by parts I am applying that integration by parts on my initial expression of the small variation on this particular integral.

So what I will get is applying the integration by parts I will get an expression which is similar to this partial differentiation with respect to x partial differentiation with respect to x again of Phi plus dPhi by dy d by dy Phi minus dgPhi] multiplied by dxdy minus integral a small variation multiplied by the partial differentiation with respect to x dy minus same thing with respect to x. So remember this is the terms directly coming from the Del square term. So these are terms that are directly coming from the Del square term. Since we are only in two dimensions we have only two differentiation second order differentiations and applying the integral udv is equal to ud minus integral vdu we are getting to this point. Now what we can do is we can take out the Del outside of the equation to write this entire equation in a simplified form.

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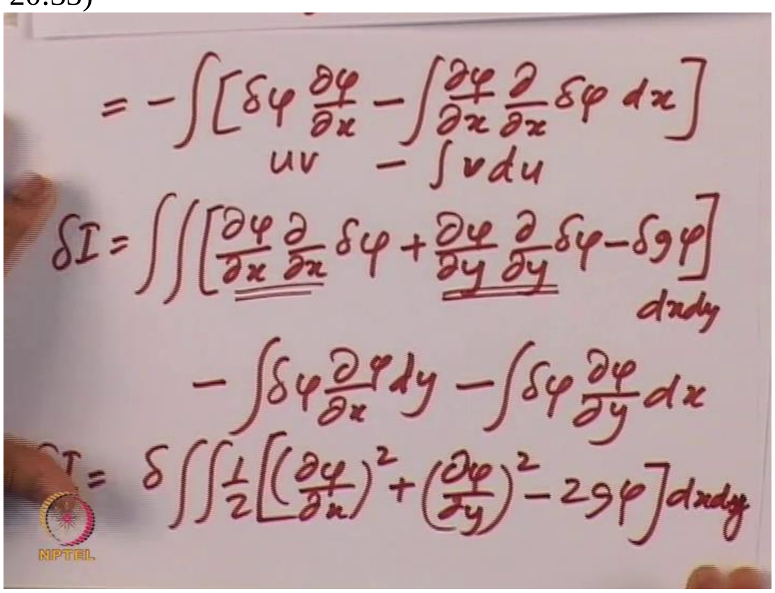

So what we will have is Del is equal to so I will take out the entire equation Del outside and then I will have an integration where I will have 1 by 2 [(dPhi by dx) square plus (dPhi by dy) square minus 2g Phi] dxdy.

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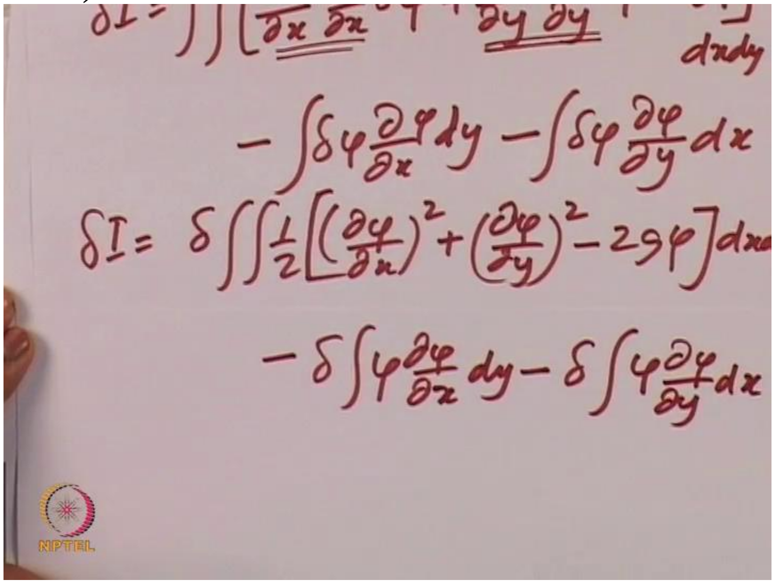

And we will have the term we will have these two terms coming into the play which will be minus Del integral Phi d Phi by dx dy minus Del integral Phi d Phi by dy dx.

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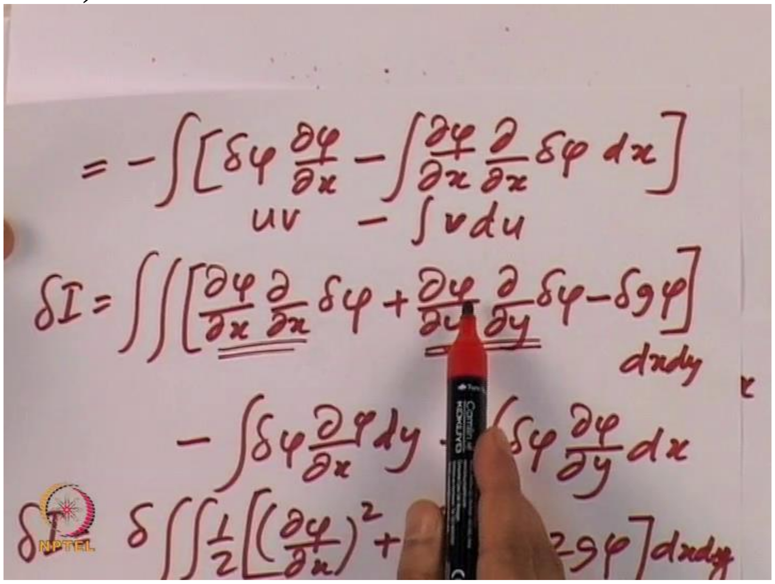

So what I have done here is I have taken out the Del out of this particular expression in the initial expression what I have got is the Del term was always inside and I have taken the del term outside the expression to derive the form which will be having Del terms out of the integration.

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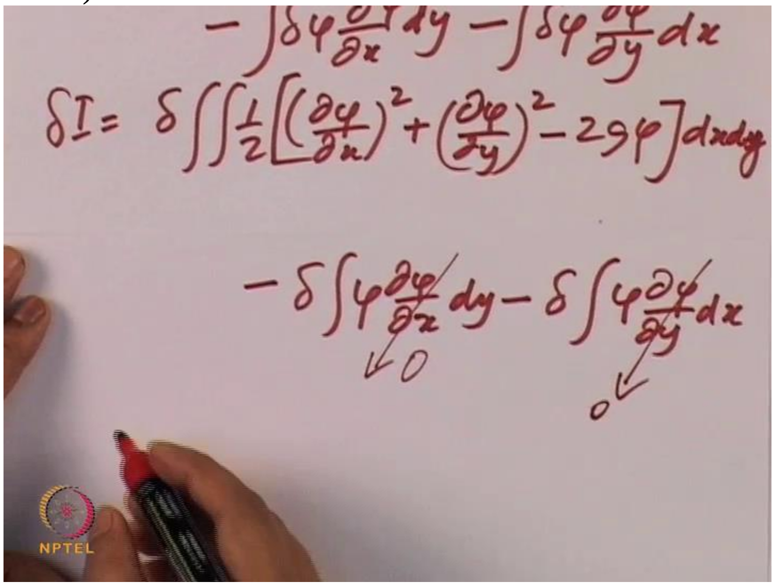

So now I can apply the Dirichlet and Neumann boundary condition. Remember these terms are the boundary condition terms and if we say that we are talking about homogeneous boundary condition, so these terms will be equal to 0. So the Neumann boundary condition will be fine. The functions normal derivative will be equal to 0. So we will have only the term which is here so what we will have is we will have the term which will only contain this particular thing.

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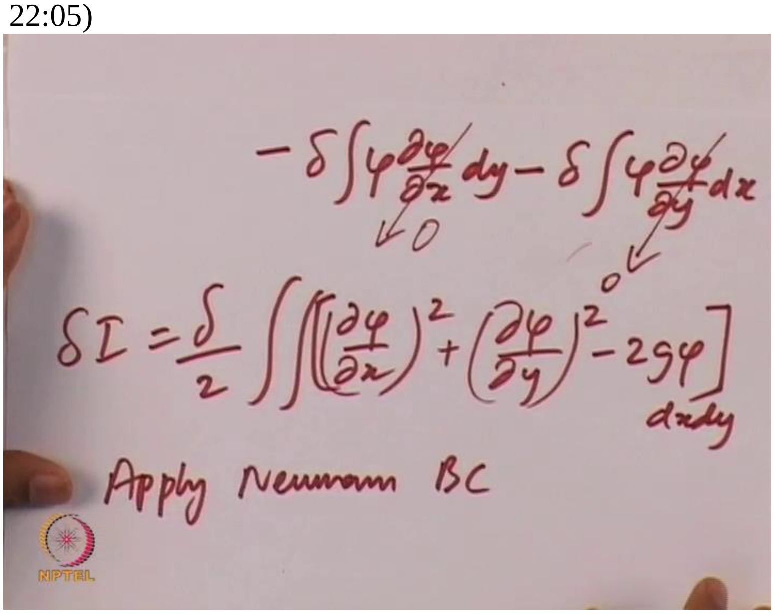

So we can write them as Delta I is equal to Delta by 2 two integration (dPhi by dx) square plus (dPhi by dy) square minus 2gPhi] dxdy. This is applying Neumann boundary condition. So these terms are going to 0 so you will have only this term. So I have taken the Del out so basically I have a functional so this Del and Del gets cancelled. So what I have is an I which will be the value of the I. So this is because of this.

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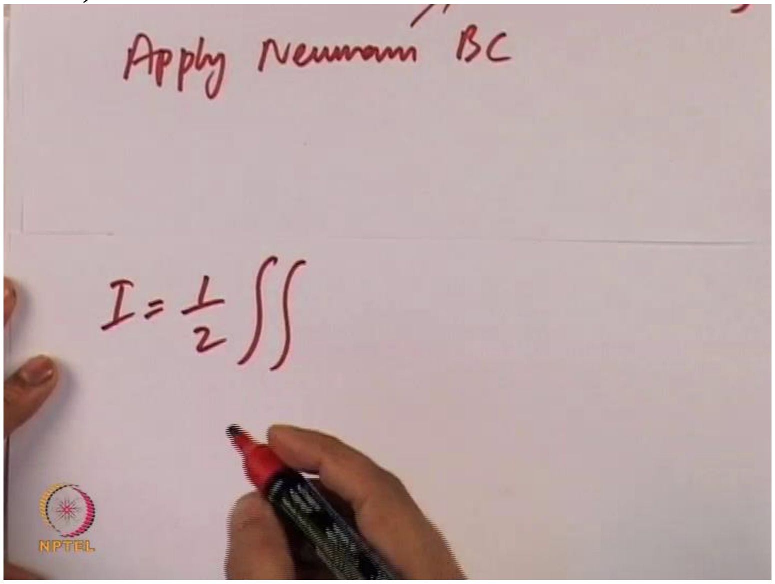

So when I apply the when I look at the symmetry of this left hand side and right hand side what I will have is I is equal to half double integral.

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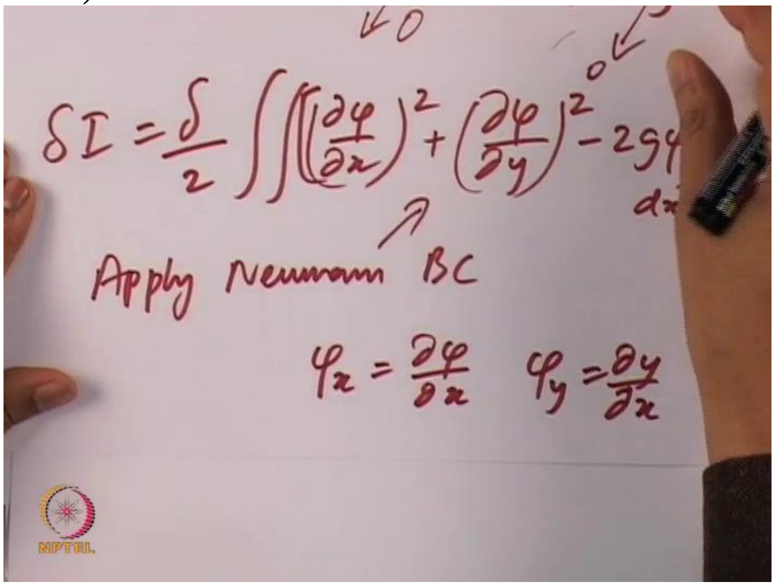

So instead of writing the equation like this I can simplify it by saying dPhi is equal to and dy is equal to dy by dx.

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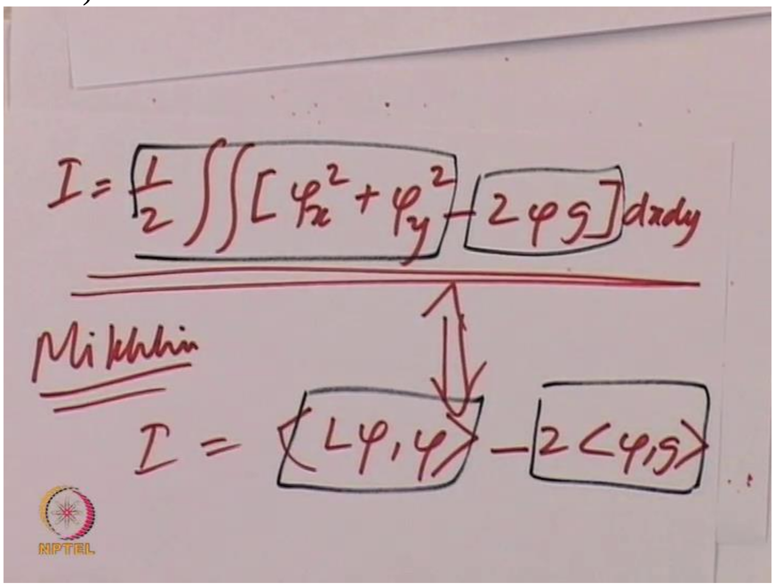

So when I apply that I can write the expression in a simplified notation as [Phi x square plus Phi y square minus 2 Phi g] dxdy. This is the final expression of the functional. Remember in the initial case when we talk about work of Mikhlin which gave us the expression of I as <LPhi ,Phi> minus 2<Phi,g>. So this is a very simple similar case where we you got inner product term which is basically given by this term and you have got this term which is basically given by this particular term.

So what we have arrived now is basically we set we start with the simple PDE and we have arrived to the functional. it has been mathematically a little bit too complex. but I think step by step we have seen how to get to it. So you can follow it and do it also on other PDEs.

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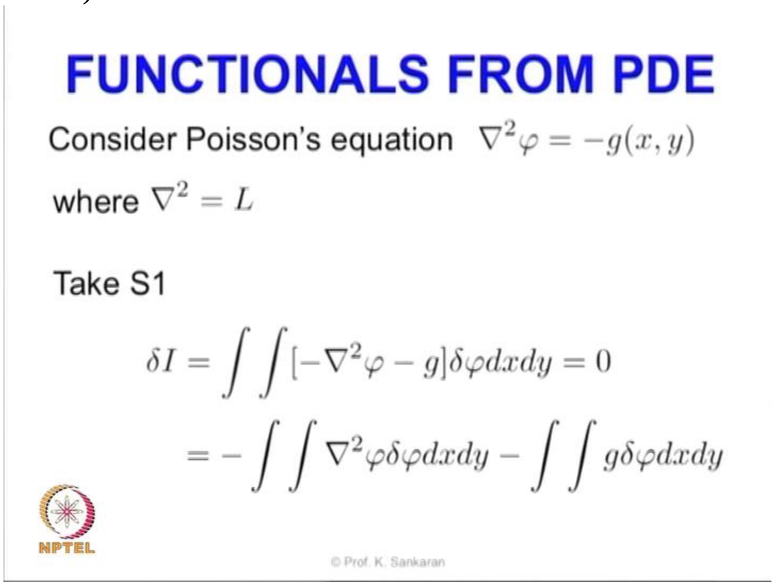

So let us summarize what we have seen now in the form of a slide here. So we started with the Poisson equation and we said we will do the step 1 where we are multiplying with the small variation which is given by this particular term and this is equal to 0 and we call this particular integration as Del I. And we are splitting the term taking the bracket out.

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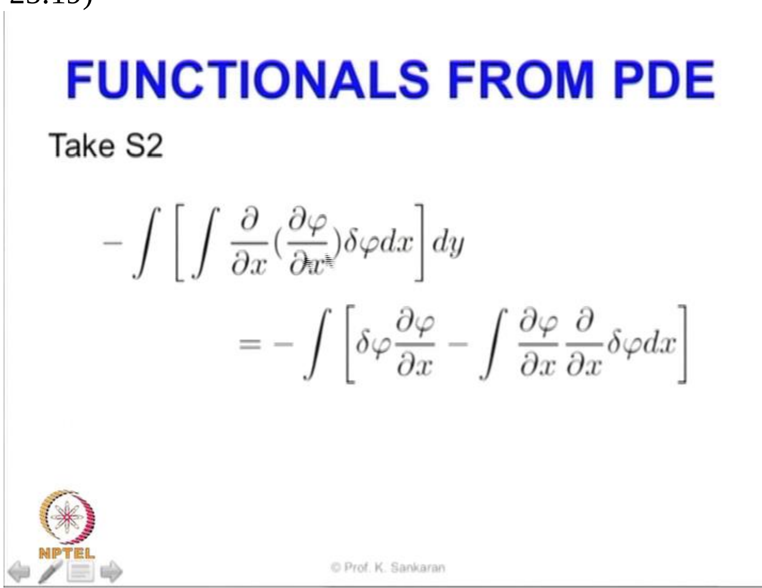

And then we are going and applying step 2 where we are using integration by parts. Remember I said this is integral udv equal to minus integral vdu.

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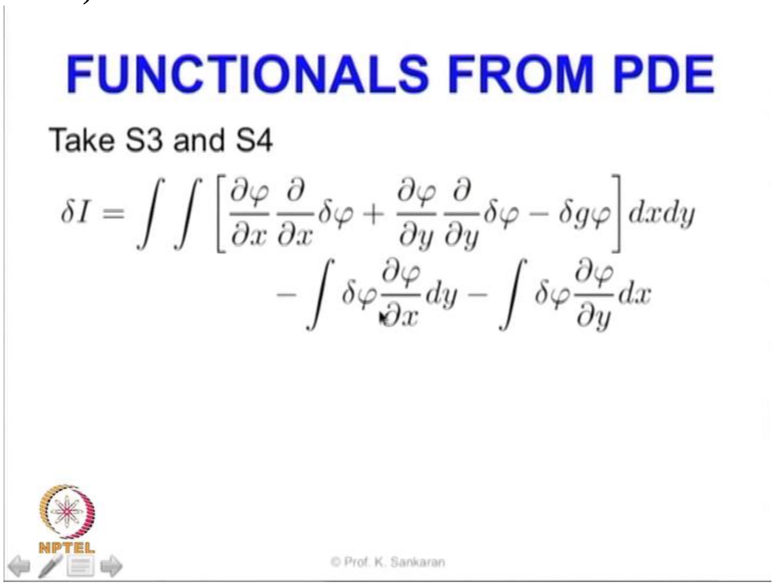

And this is exactly what you are applying here on this particular expression and we get terms which are the main terms and the terms which are related to the boundary terms.

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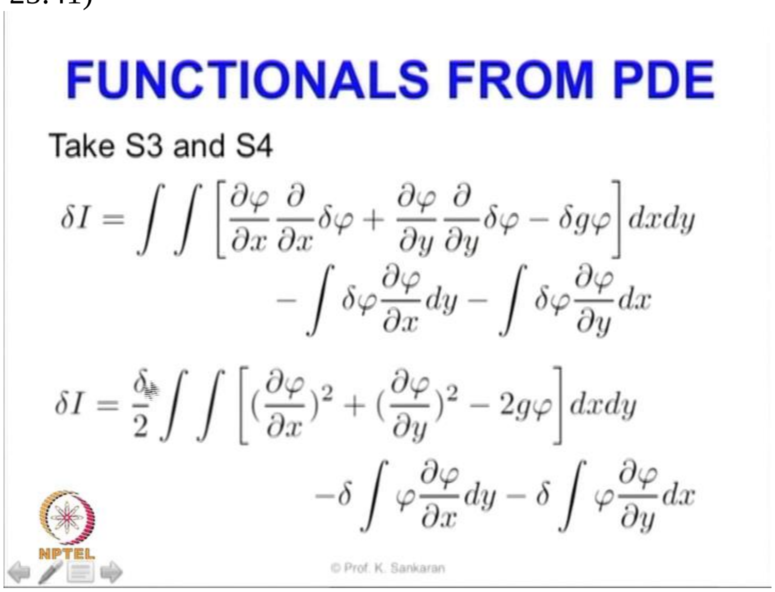

So when you apply the boundary conditions these later terms will disappear. So you can take out the Del out of the bracket and we write the entire expression in a form will have Del outside the integration.

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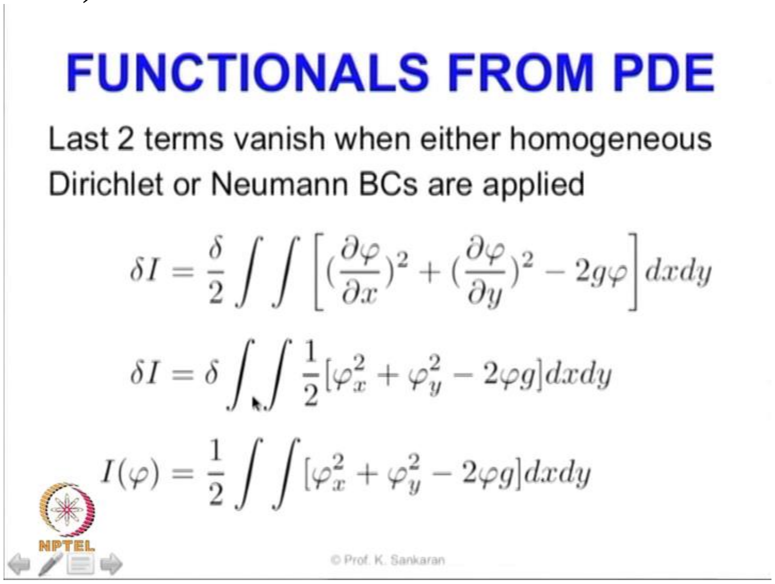

By doing that and applying the Neumann and Dirichlet boundary condition we can basically reduce this equation into only the main terms without the boundary terms and finally seeing the left hand side and the right hand side the Del terms get cancelled so you get only the I term which is given by this particular expression.

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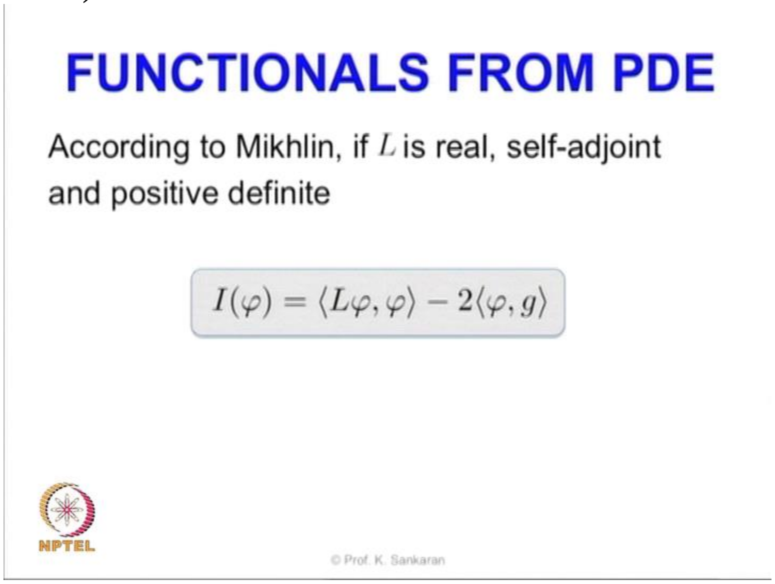

And finally we saw that this is very similar to what we have seen before. So in applying any of these methods to partial differential equation the boundary conditions are integral part of the problem itself because a problem will be ill post without a boundary condition.

In a boundary condition when we say it could be a simple Dirichlet boundary condition or Neumann boundary condition where we can say it has a value equal to 0 or some value. So it could be homogeneous or inhomogeneous condition. So the integral part of the problem is a boundary condition as well. So in this particular approach when we are applying step 1, step 2, step 3 to arrive to the functional from the PDE applying the boundary conditions we are able to simplify the expressions further.

Basically the boundary conditions are enabling us to reduce the terms and directly go to the functional term. And also the boundary conditions are given as part of the initial PDE itself . So when we use those boundary conditions we are able to get with the direct functional step by step. So we have looked into a very heavy mathematical journey I think it is worth it if you follow it and appreciate the mathematics behind it. So in the next modules we will look into finite element method and we will investigate more how this finite element method is applied for various electromagnetic problem. We will look into lot of simulations as well. So do not think it is only going to be mathematics. It is going to be lot of application simulation using Matlab.

With that I would say we will end here in this module and we will see you in the next module Thank you!