Computational Electromagnetics and Applications
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Lecture 03
Finite Difference Methods –1

Welcome back! So we saw in the early and modules basic way to approximate the continuous differentials. Either it is a first order differential or second order differential using certain difference equation. So let's use those differentials to model a simple onedimensional wave equation.

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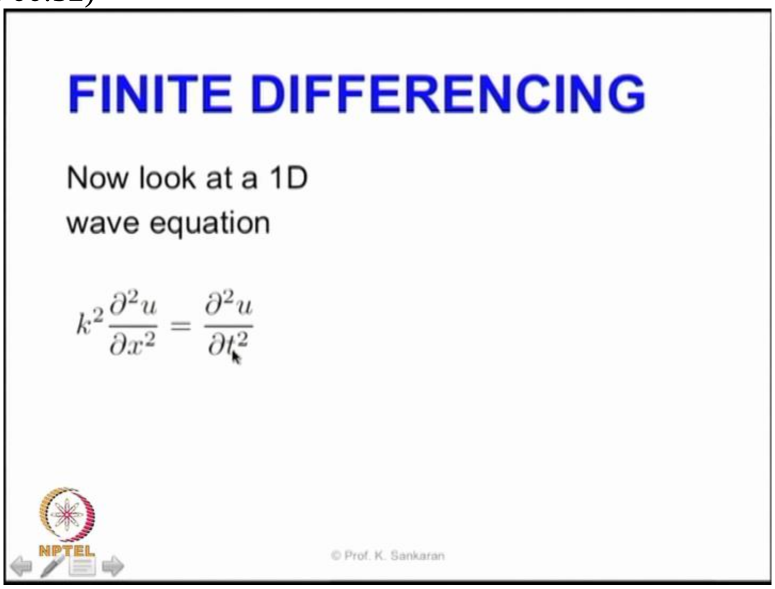

So to start with the one dimensional wave equation represented by this particular equation so what we have is we have a second differential with respect to x multiplied by a certain quantity let's say the constant is K square and is equal to the second differential u with respect to time we are having is a simple case of one dimensional equation. It could have a physical meaning we are talking about the value that is changing only in x with respect to time.

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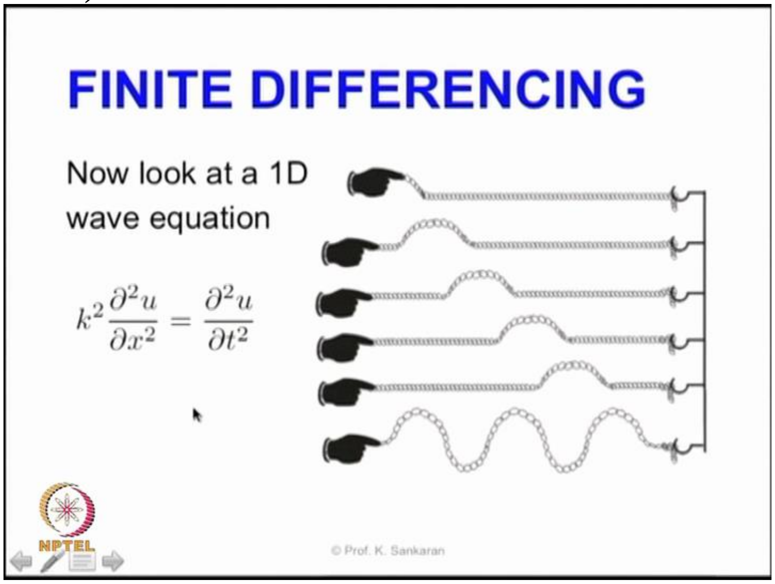

It is a one dimensional form here you say time equal to, 1 time equal to 2, time is 3, time equal to 4 and so on and so forth. And obviously we have to give certain boundary conditions. Before Talking about the boundary conditions; let us only look at the differential equation and try to transfer this differential equation into finite difference equation, because we have got already the approximation for d square you by dx square and also for d square you by dt square.So using that approximation which you s a w in the earlier slides. So we will use those approximations here and we can write them directly in the form which is given here.

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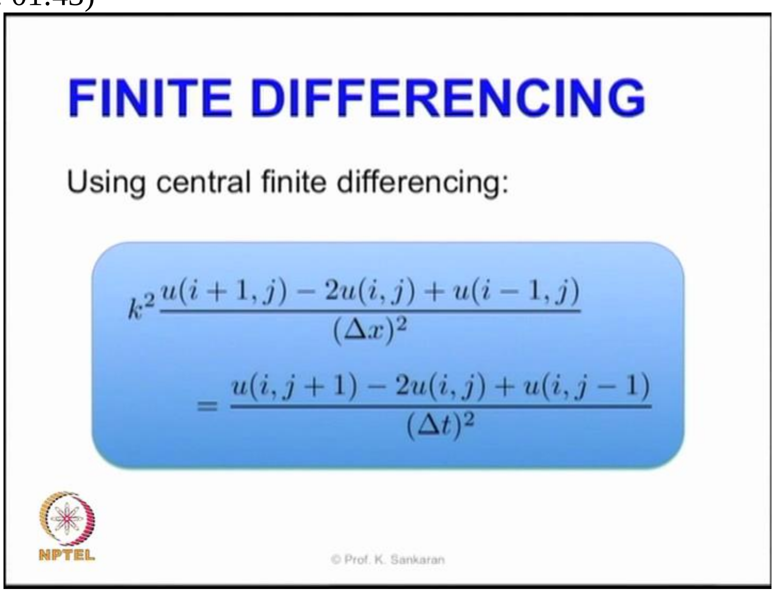

So we have the k square and we have the approximation the central differencing approximation with respect to x for the second differential is equal to the central differencing approximation with respect to t. So what you have got here is a Central finite difference equation. And this is when you are rearranging the terms so here I have the value written in terms of time the j components represent time so I want it to compute the value at j plus one.

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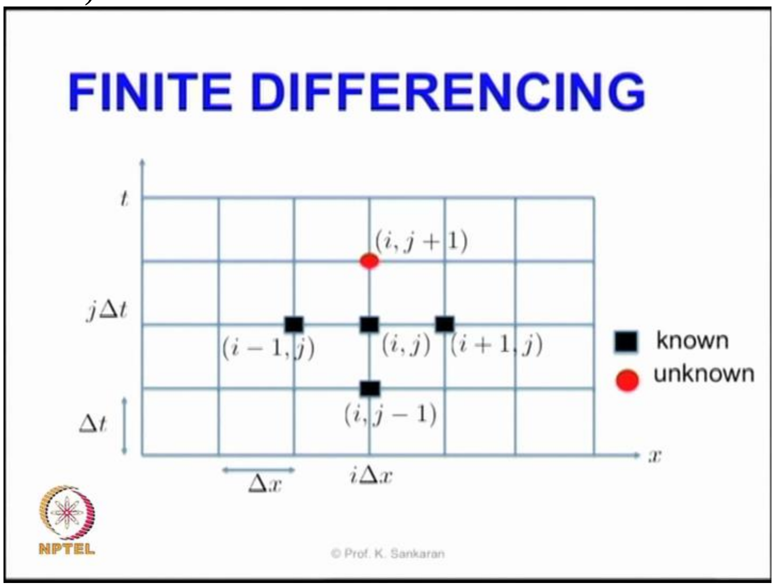

Remember in the slide before we said that we are interested in computing the values unknown at time j + 1.So I need to get simple equation which will give me the values of the variable at time i, j + 1 using the values which are before.

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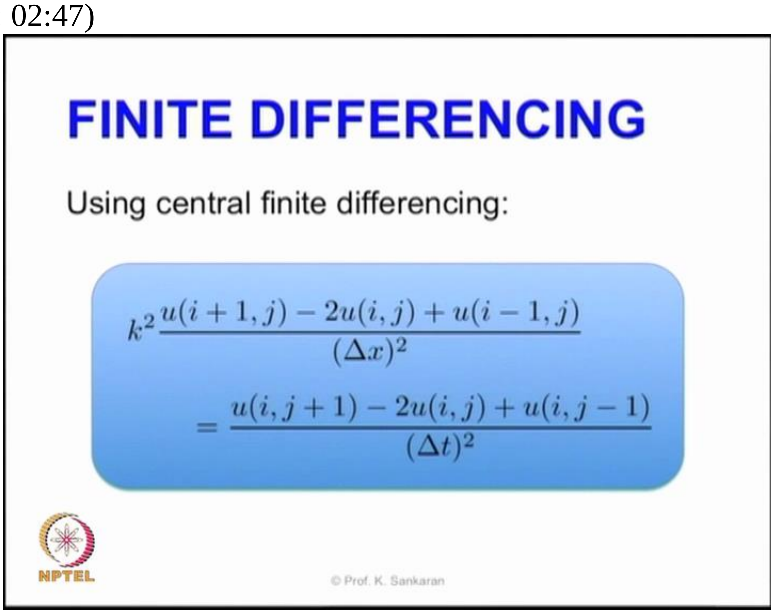

So what we have got now is Finite differencing formula using the central differencing scheme.So using this particular expression we can reorder these terms search that we can bring this i, j + 1 on the left hand side and right the other terms on the right hand side.

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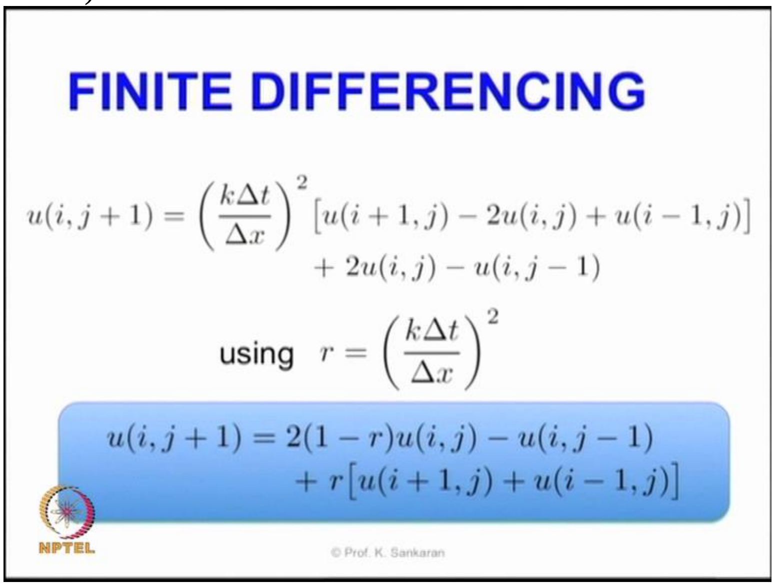

What we get here essentially is a term an expression for u of i, j + 1 is equal to certain value on the right hand side. So this value totally depends on the value of you at time j or earlier terms. Remember j is the time co-ordinate. So if we substitute the value r all the term here which is a constant k Delta t by Delta x square so what we get essentially is a compact form of writing u of i,j + 1 as a function given here so a term that is r sitting here and we have certain multiplications. So these are nothing but weights. So what you see as the coefficients are nothing but weights. So here the weight of u of i, j is 2 (1 - r) the weight for the component i, j -1 is -1, similarly the weight for both the terms i + 1 j and i - 1 j is r. So if you take a simple and straight forward condition where we assume the value of r is equal to certain values.So obviously the value of r cannot be any value it has to be certain values in between certain bounded quantity.So what that bounded quantity is going to be is something that we will see.

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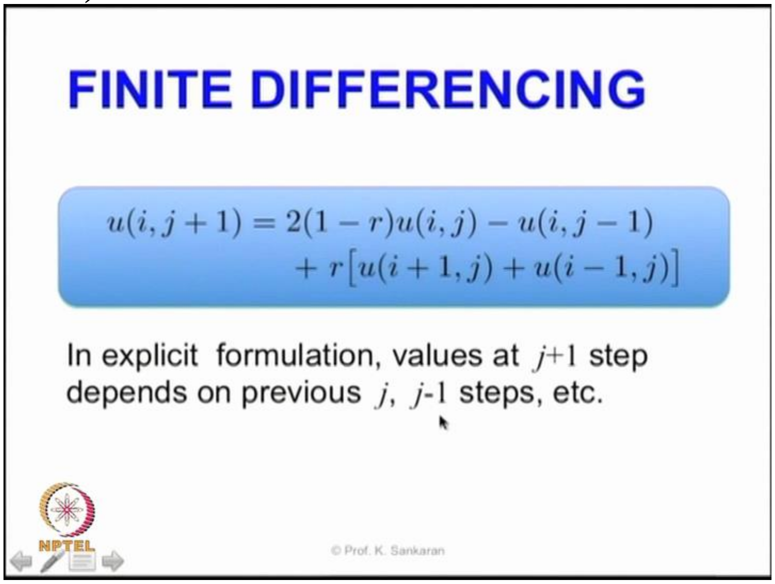

We will see here the formulation what we here is an explicit formulation as I told before we are computing the value of j plus 1 purely based on the value at j or j minus 1 steps.

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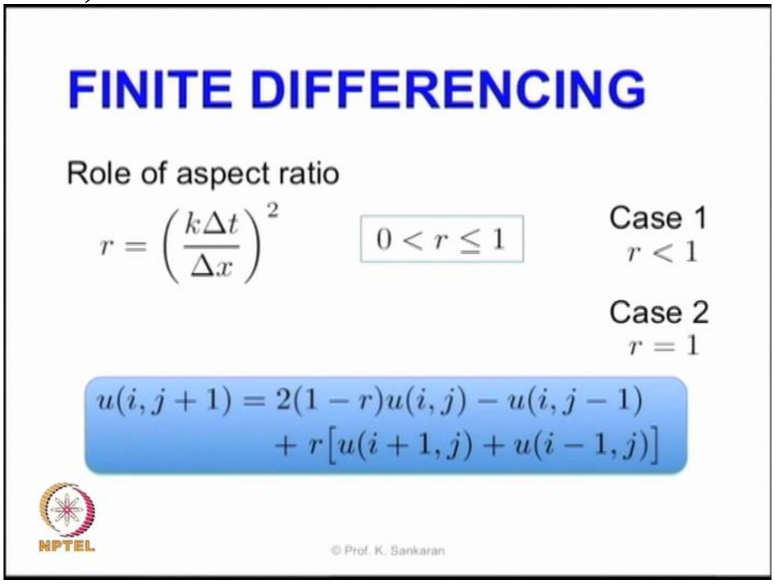

So the role of r is quite interesting to notice. So the value of r is bounded between certain two values. As I said it is definitely more than 0 but it could be either 1 or less than 1.So based on this bounded value we can have two cases. Case number 1 is r is just less than 1 and it is more than 0 and the second is r is exactly equal to 1. So when we have the value or r exactly equal to 1, what we see is certain terms that are getting cancelledthe term which is here 1 minus r all these terms will get cancelled for r is equal to 1. Because 1 minus r will become 0 so this term will completely disappear whereas the other terms will stay.

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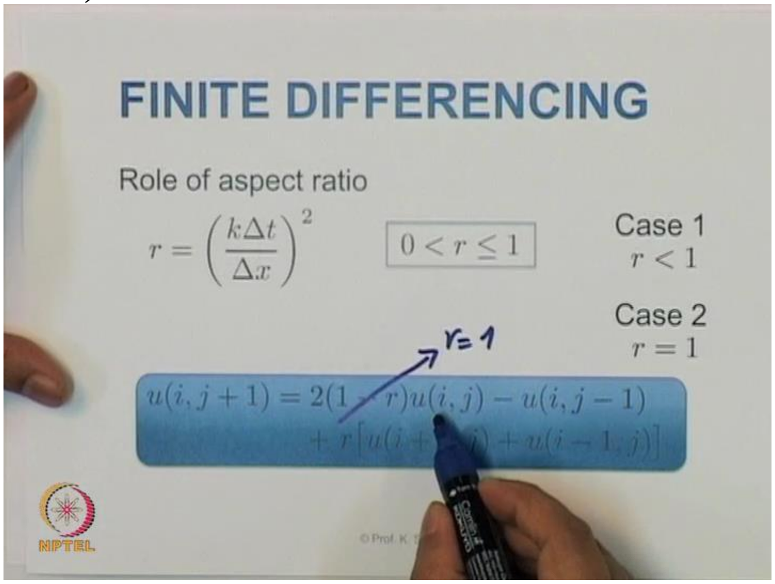

So that is what you will see what is means is if you see the 1 minus r term here is purely based on the weight that is for the coordinate point i,j. so if I put r is equal to 1 my value of the function at u of i, j is not important but if I put r is less than 1, my values at i,j is also important.

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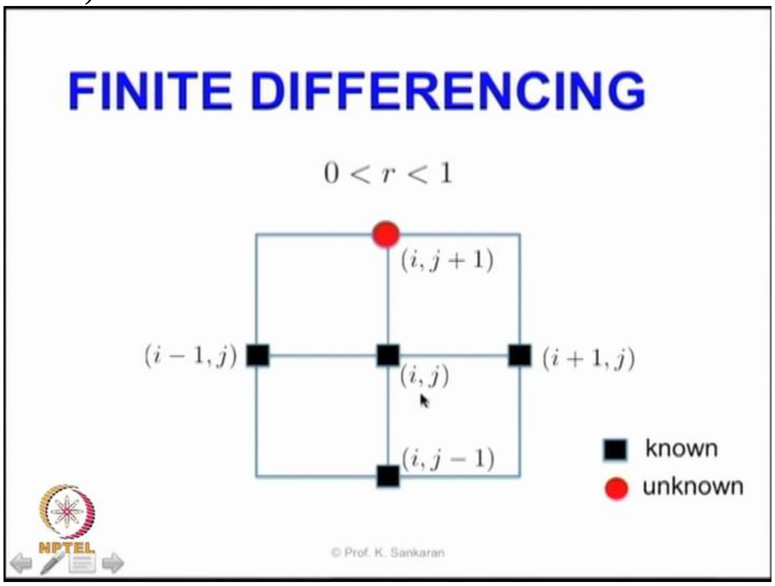

That is what you will see pictorially in this particular graph. If I wanted to compute the value of various time steps and if I set the value of r is between 0 and 1, and strictly less than 1 what I say is my value at this point can be computed using the values at all these neighboring points including i,j.

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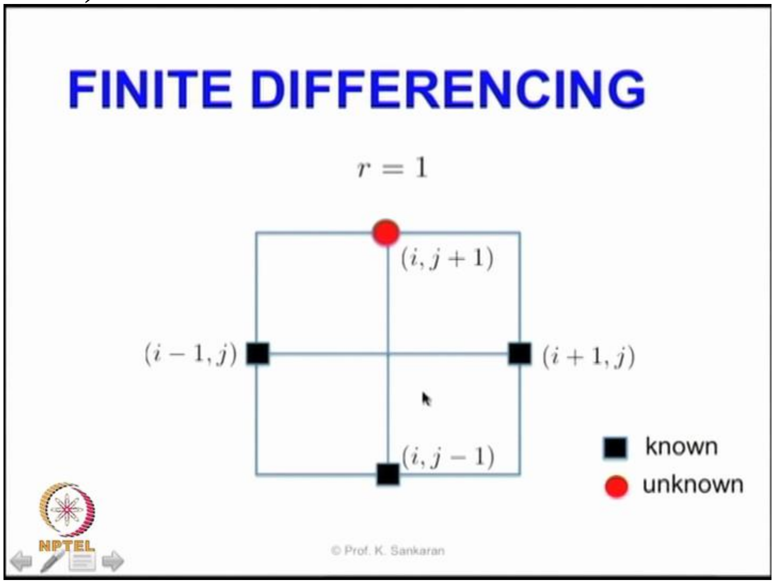

But the moment I put r equal to 1, the value at the center which is the waiting of 1 minus r will become 0 it is not important it will be purely based on the values here.so this is the way we are going to compute the values of u at various time steps,and we are going to take a very simple problem represented by this particular expression given in the slide.

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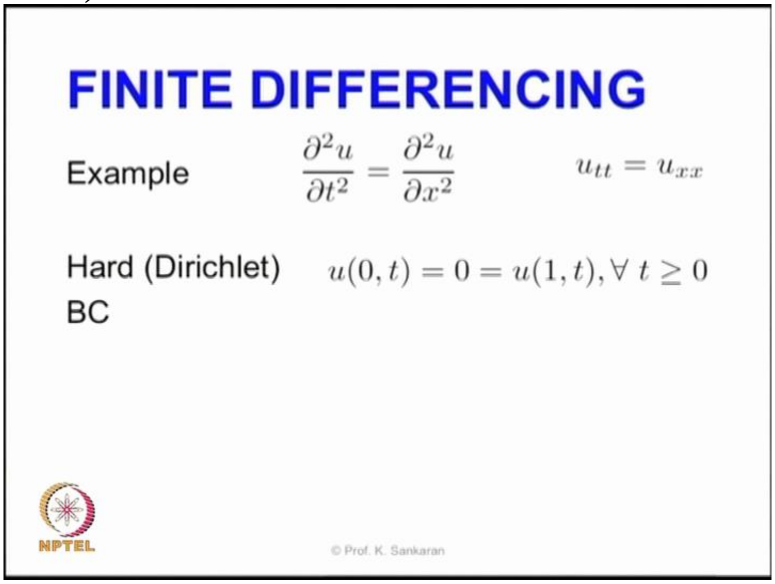

So we are interested in a problem where it is nothing but the same problem but what we have done is we have put k equal to 1, partial differentiation with respect to time, second order partial differentiation with respect to time is equal to the second order partial differentiation with respect to x. and we can write this particular expression in a much more compact simplified notation like here. And what we said before is in order to run the simulation we need to set certain boundary conditions. So first let us look at the boundary condition, so the boundary condition says the value of u at time t any time t but the boundaries are 0 and 1. Remember in the problem we had before we set the value of x goes from 0 to 1.

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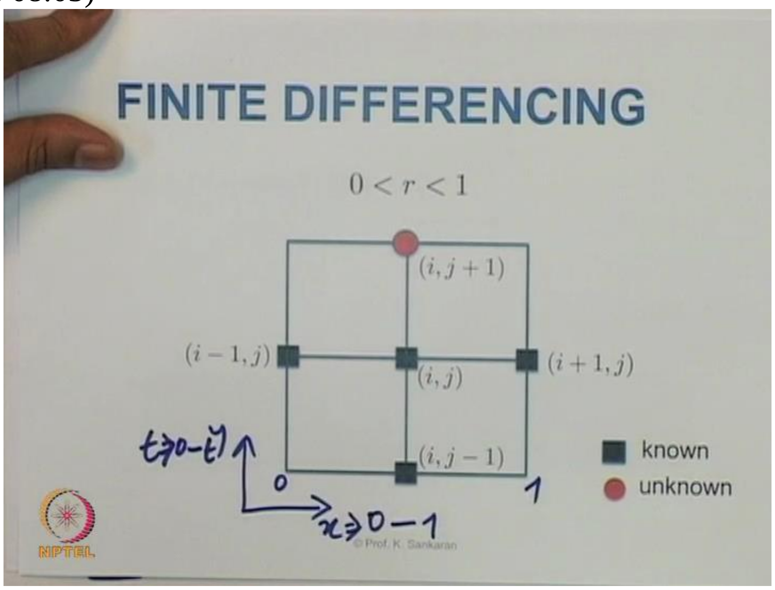

So in this case if you see this is the X axis and this is the Yaxis. and the x axis is the value that goes from 0 to 1. So this is 0 this is 1. And similarly the value the Yaxis is a time axis. And it goes also from 0 to certain value, let us say t. And we are talking about certain space discdizaton and also time discretization which we will see now.

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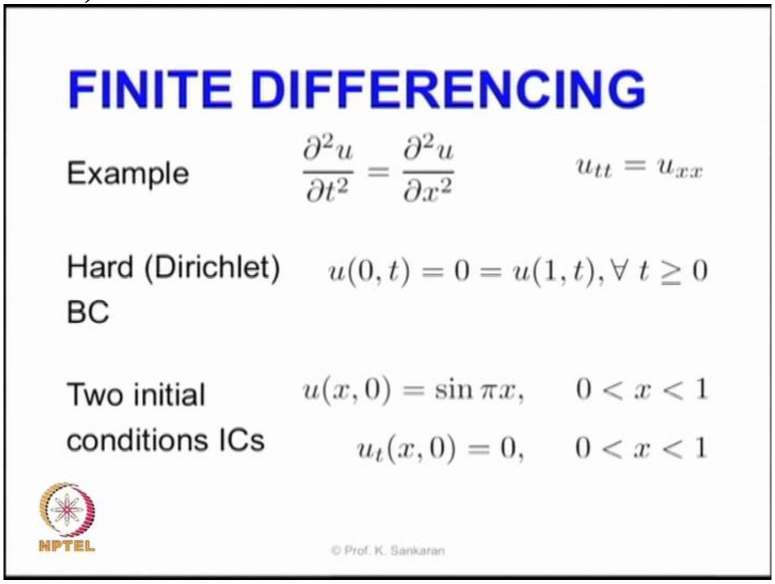

So what we are saying now in this particular condition, it is called Dirichlet boundary condition because it is named after a scientist Dirichlet. So when we say the value at x 0 is given for all time steps as 0, and the value of u at x equal to 1 for all time steps is also 0 and that is what we have represented mathematically in this particular expression.

And , Since it is a second order differential equation we need to set two initial condition and those two initial conditions are for all time stepswe set the boundary condition but for all special steps at time equal to 0 we set the initial condition, it is initial because time is equal to 0 here. So at time equal to 0 at X coordinates the u value is given by sin of pi x.

Similarly the first differentiation with respect to time for all special steps but time equal to 0 is given to be 0. In other words the value of u at time step equal to 0 is given by sin of pi x and the value of the differentiationwith respect to time at time step equal to 0 is equal to 0.

And these are the two initial conditions which we have to satisfy. Without worrying too much we can get the analytical solution of this particular expression which is given by the value here.

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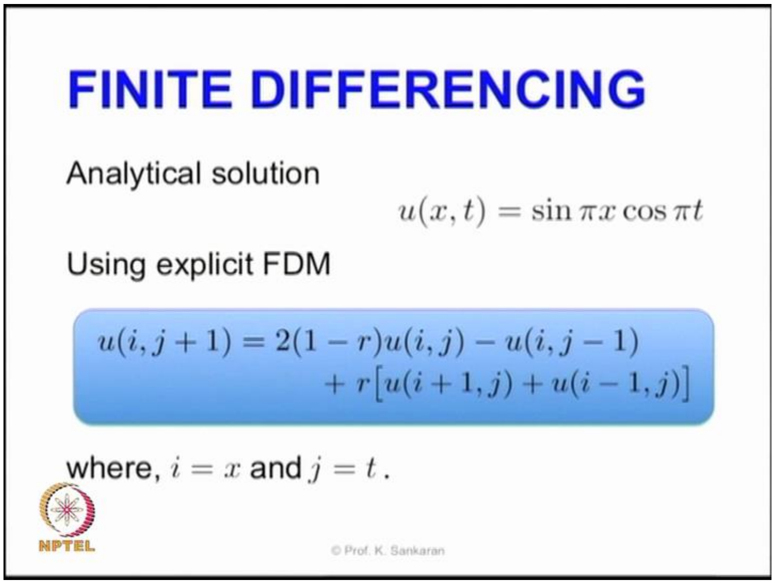

So the analytical solution for that particular expression is given by the value sin pi x and cos pi t. so it is a multiplication of sin pi x and cos pi t. And what we are seeing now is we can use this as a reference solution because we know the analytical solution but we can also use the finite difference formula to compute the value of the expression in a compact form as given by this particular expression. What we have got now is the value of u plus i, j+1 is equal to the value given by the right hand side.And obviously we said we have the value i belongs to the x coordinate and j belongs to the time coordinate.

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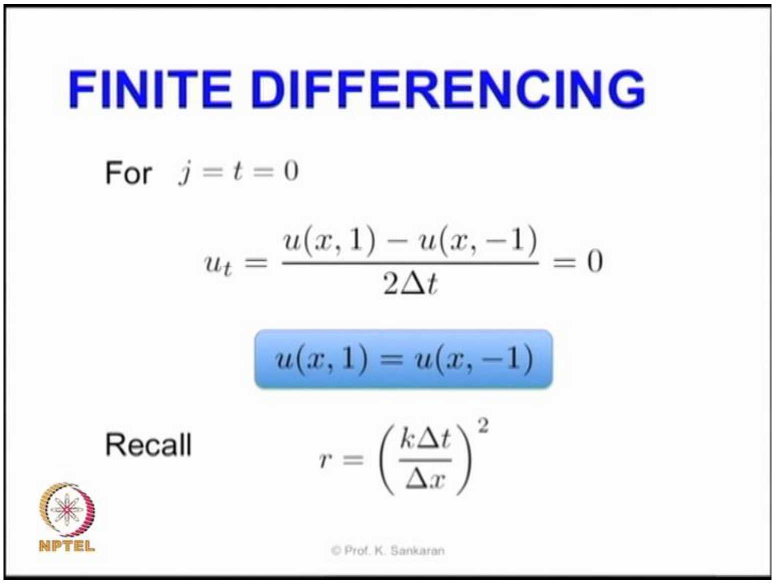

So if we put i and j certain values we can already get the values for the initial and the boundary condition that is what we are going to do now. We put j is equal to t equal to 0.We can get the value for first derivative given by this expression and we know delta t cannot be 0, so only the numerator can be 0. And this is the initial condition which is given we set for u of t at the initial time t equal to 0 should be 0. So this particular expression should be 0, which leads to the condition that the numerator term should be 0 in other words u of x, 1 should be equal to x, minus 1. And recall that we use the value r is equal to k delta t divided by delta x square.

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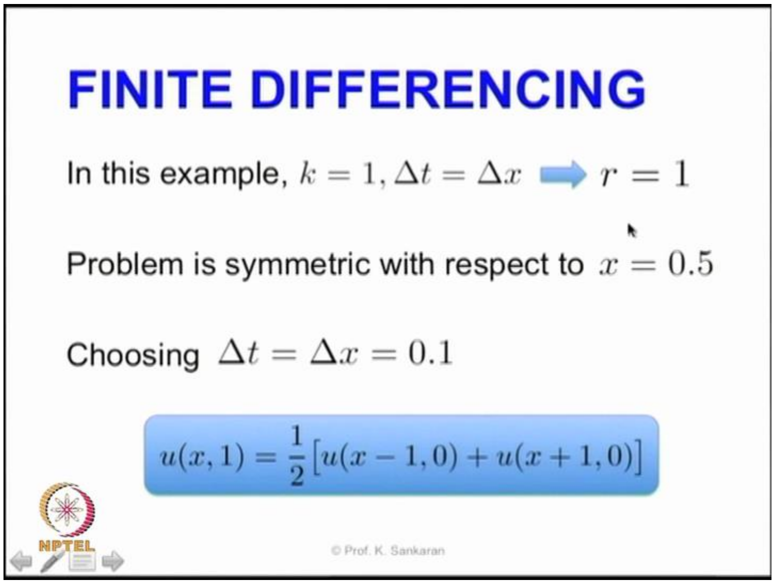

So in our case k equal to 1 and delta t is equal to delta x for simplicity, it need not be like that. for the sake of simplicity we said delta x and delta t are equal. So we get the value r is equal to 1. And this particular problem what we have in the slide is symmetric about the point x equal to 0.5 and that you will see when we do the simulation in the next slides. So what we set is we set certain values for delta x and delta t.

And since they are equal we say it is 0.1 for simplicity. Obviously we can change the value of delta x and delta t to different time step, and we can compute the values. So if we set it is equal to 0.1 what we will get is an expression for your effects and we can compute the values accordingly and umm we will use the value r equal to 1. Obviously when r is not equal to 1 we get a different set of expression. This particular set of expression is only valid for r equal to 1. And now we will go into the numerical simulation of this particular problem.

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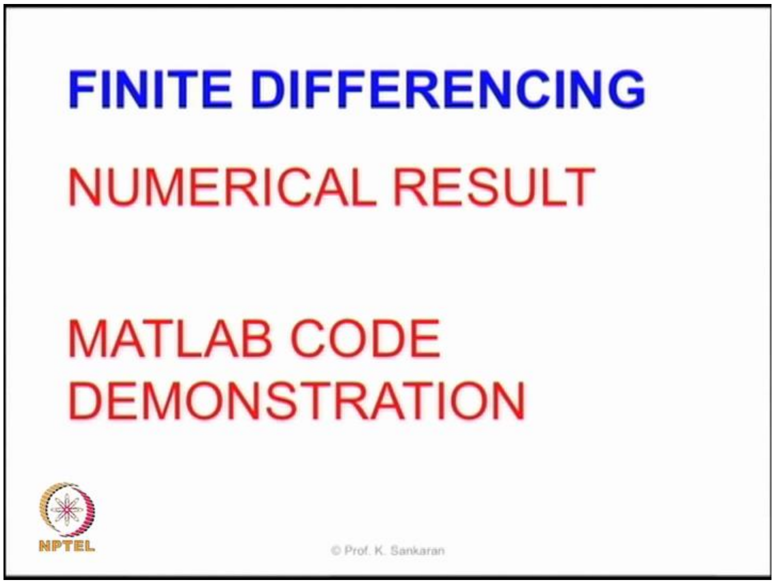

So what we will see now is the Matlab code and we also will compare numerical result and throughout this course we will be using Matlab to showcase or demonstrate certain proof of ideas certain numerical schemes.

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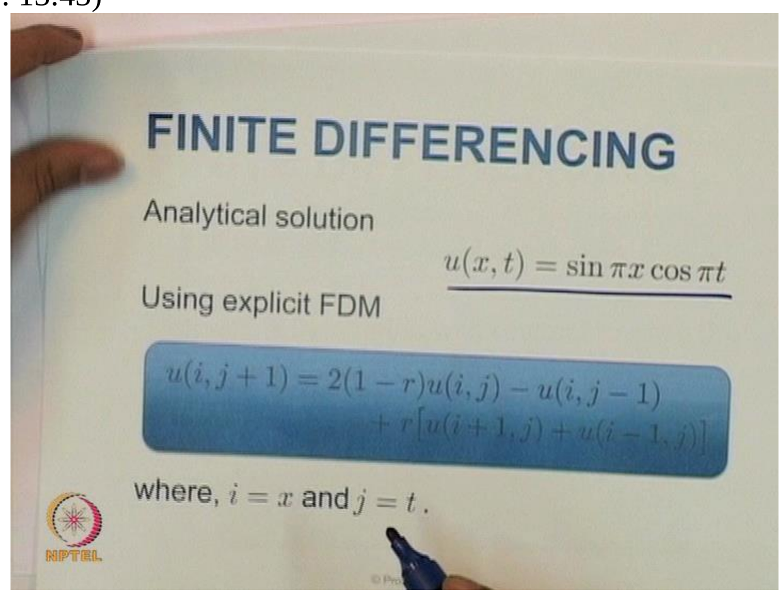

So now we will focus on certain problems using Matlab. We know the explicit formulation is given here and we also know the analytical solution. So we are going to set certain values here.

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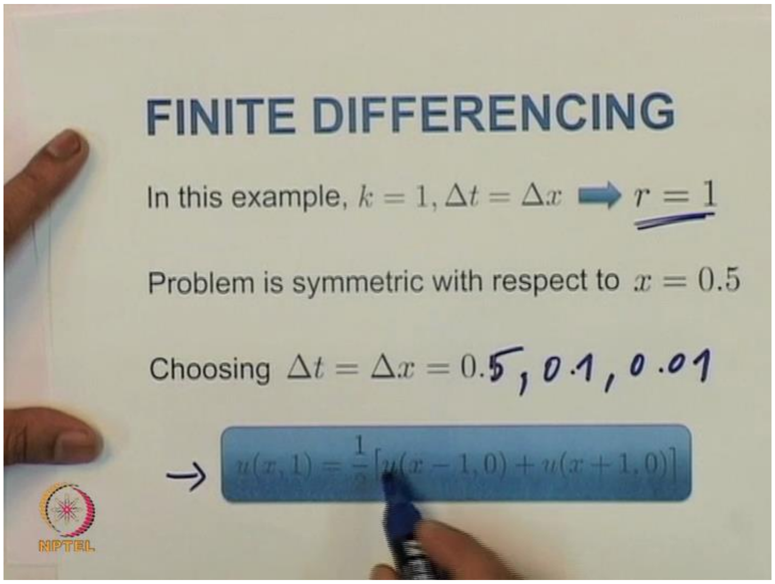

The values why I am setting here is, I am making sure the r value is going to be 1 so and I am going to change the value let us say from I am going to go from 0.5 to 0.1 to 0.01. I am going to refine the mesh from 0.5 to 0.1 to 0.01, and for this I have an finite difference expression using the central differencing scheme and this is what we are going to see in the Matlab code.

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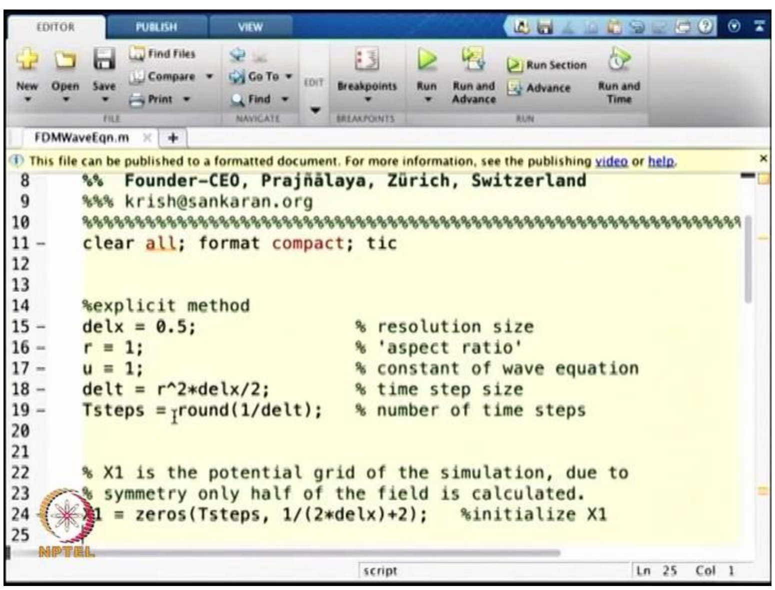

So let us look into the matlab code itself. So what we have got now is a matlab code for solving the problem represented here. utt is equal to uxx and this is we are setting certain values and let us say we are starting with delta x is equal to 0.5. And internally we have set delta x is equal to delta t. so based on the special resolution we are choosing for delta x our value of the resolution in time step is also going to be changed because it is related through this expression

And we set r quality to 1, and we have the value of u a constant to a. so we say the value of u will be 1, the maximum value.And we are computing the values at different time steps using the equation here. so we are initializing the grip in the step and we are initializing the conditions and the reflections so what are interested is in computing the value at different time steps and special steps.

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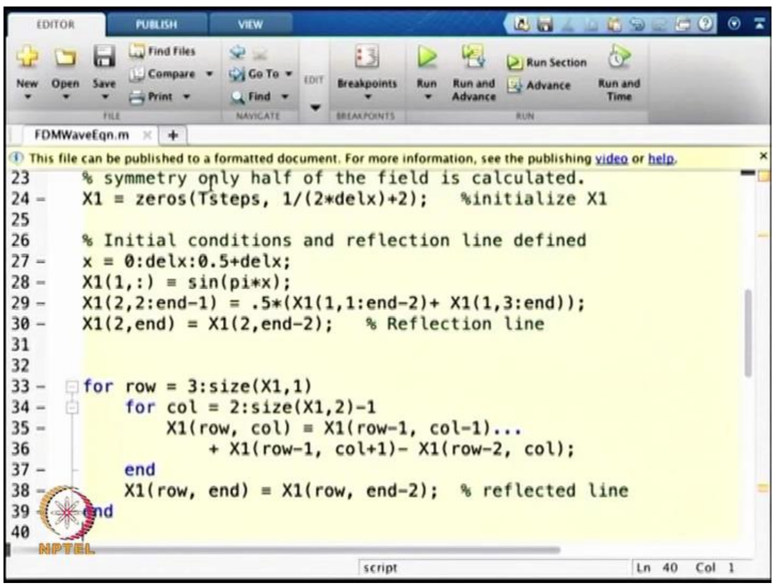

So the value of the x1 is only computed until 0.5.As I said this equation itself is symmetric so we are only computing the value only half of the field.So that is why we are having the expression here divided by 2 since the problem itself is symmetric we are only computing half of the field.And this is simply because we need to reduce the computational cost.

This is a very simple one dimensional problem the computational cost is not that important.But as you go in higher order such simplification based on the geometry and based on the problem itself is going to make our computation time shorter.So we are initializing the conditions and we are computing the value and the value here is x1 is the same expression whatwe have got in the finite and differencing form. So here we are using the value to compute the value at different time steps and the special steps.

So the rows and columns are the time steps and the special steps respectively.And we have the value of x1 which is due to symmetric condition it is only half of it.And once we computed this we are using the symmetric condition to recreate the entire field.So x2 is basically the value that we computed before and it has the other term which is due to the symmetry condition will also come into play.

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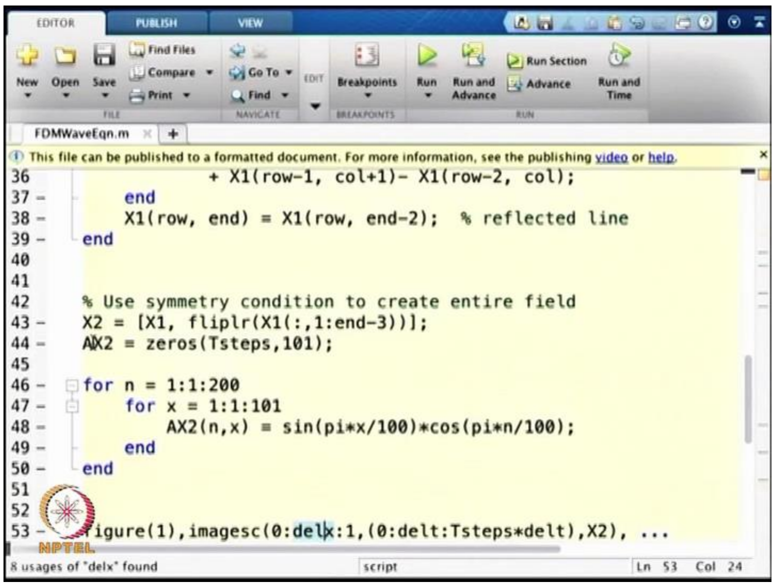

The value here AX2 is the analytical solution which we know which is computed based on the value which we know.So this is the (sin pi x) and (cos pi x) term so we are computing the value of AX2 here.So the reason why we are computing this is we can see what is the value that we are getting through numerical value and what is the value we are getting from analytical expression.

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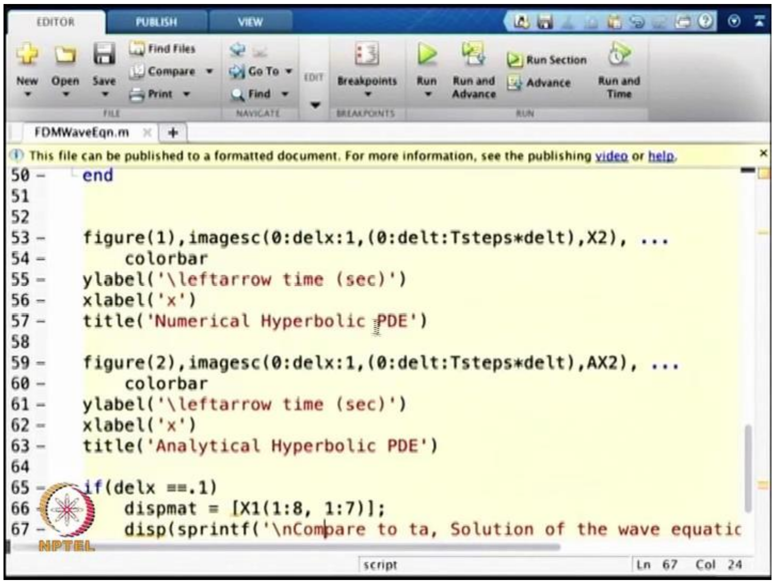

So here we have we are plotting two figures; the figure 1 is a numerical result for the hyperbolic problem.So the kind of problem we are simulating is called hyperbolic, we will describe this at a later stage it is enough to know that it is a hyperbolic problem.And value we are plotting in the figure 2 is an analytical solution.And once we run the simulation we will see the value is going to change as a function ofdelta x and delta t.

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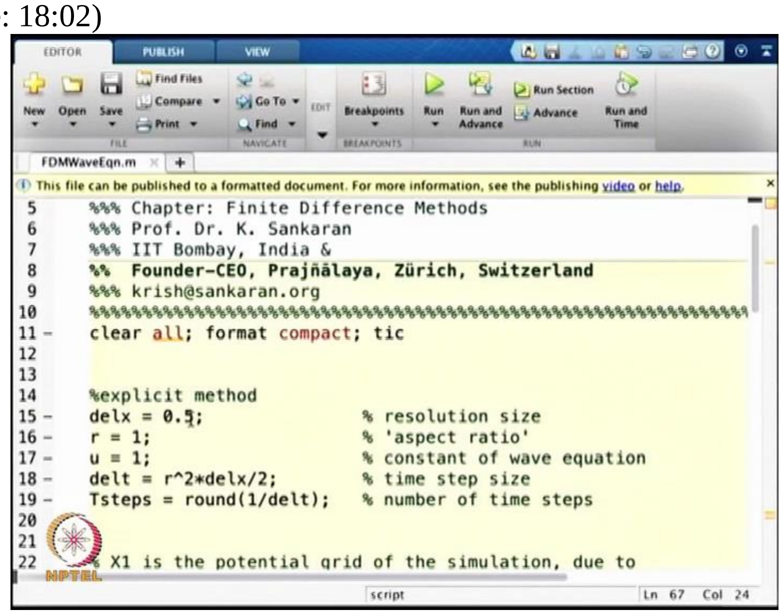

So the delx I am choosing 0.5 and the value of x goes from zero to one and this is a very very very course grid.

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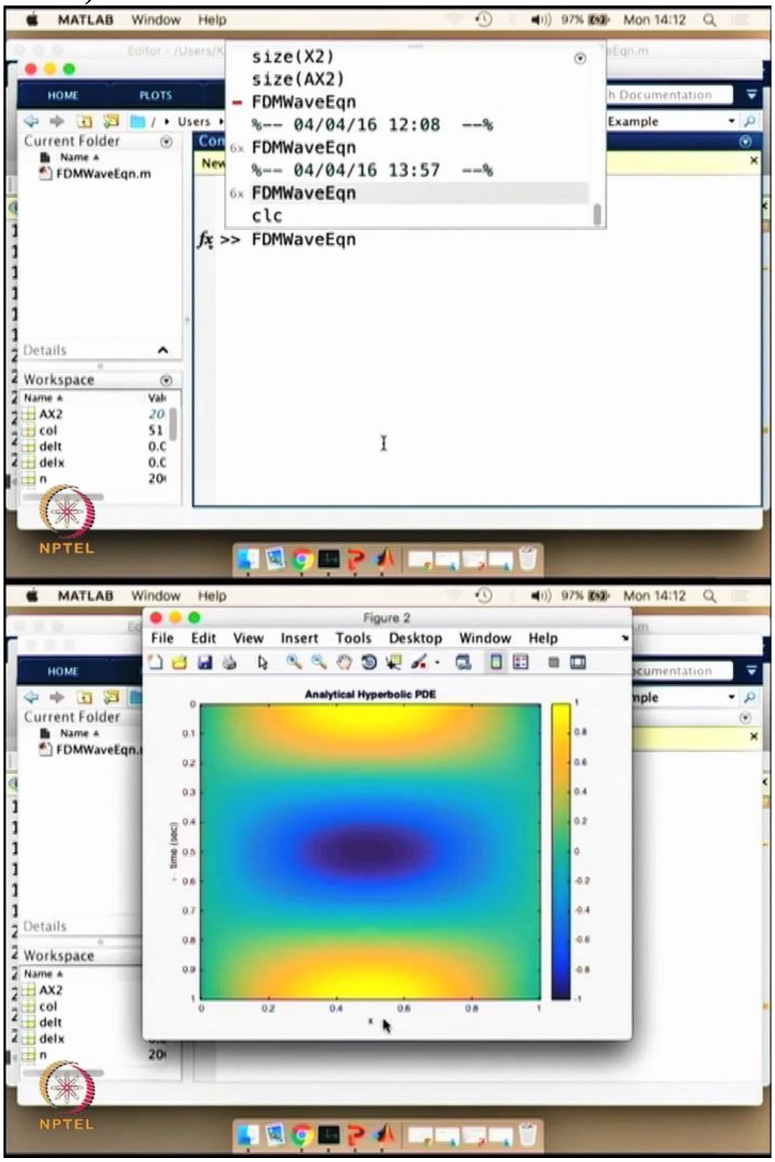

So when we run this code particularly the way we have structured it we get to figures.The first figure will be the value of the numerical result and the second figure will be the analytical result.So it is important to know what this analytical result means.So that we can compare the analytical result with respect to the value that we compute using numerical methods.So this is an analytical result.The analytical result is based on analytical problem.And the value is symmetric across X equal to 0.5.

You see that the value whatever is on the right hand side to X equal to 0.5 is the same as the value that is on the X equal to 0.5 left hand side.So that is why we only computed the value until one side and then we replicated the value accordingly.

Similarly it is also interesting to see the value is also symmetric across the time step but it is not important.And the numerical simulation should be something that is similar to this but obviously with certain errors.And it is important to see the analytical result has the value goes from minus 1 to plus 1.So plus 1 is the dark yellow and minus one is the dark blue and we will compare this these results for various time steps, various time steps and also various special resolution.

So initially you will complete this value X equal to delta x equal to delta t equal to 0.5 and we will compare it with the result of analytical solution obviously we expect the result to be not good because delta x and delta t is equal to 0.5 is a very veryvery course grid.And will improve this by refining the grid 0.1 instead of 0.5 and then we will compare the result of the solution numerical solution with analytical solution.And we will also do it for delta x equal to delta t equal to 0.01.

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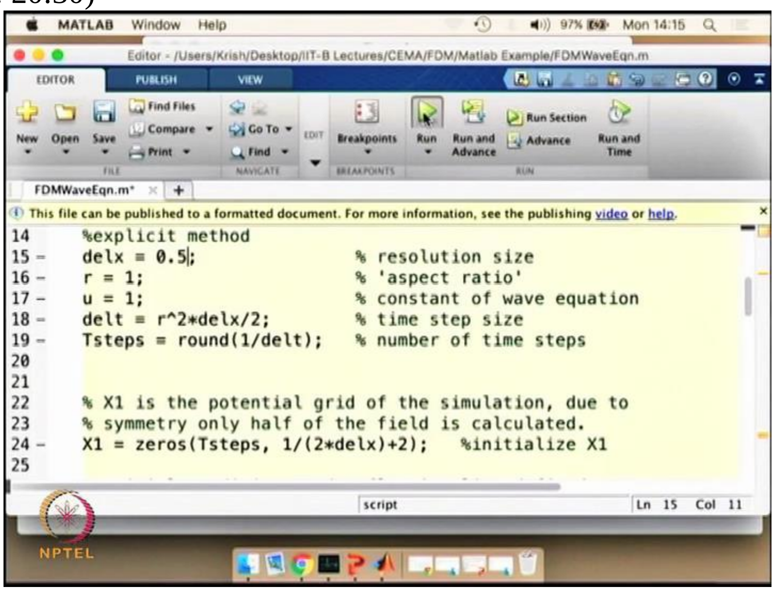

So let us take now the example of delta t equal to delta x equal to 0.5.So here I am going to use the value 0.5.And when I do it 0.5 and I am running the simulation what do you see here are two figures.

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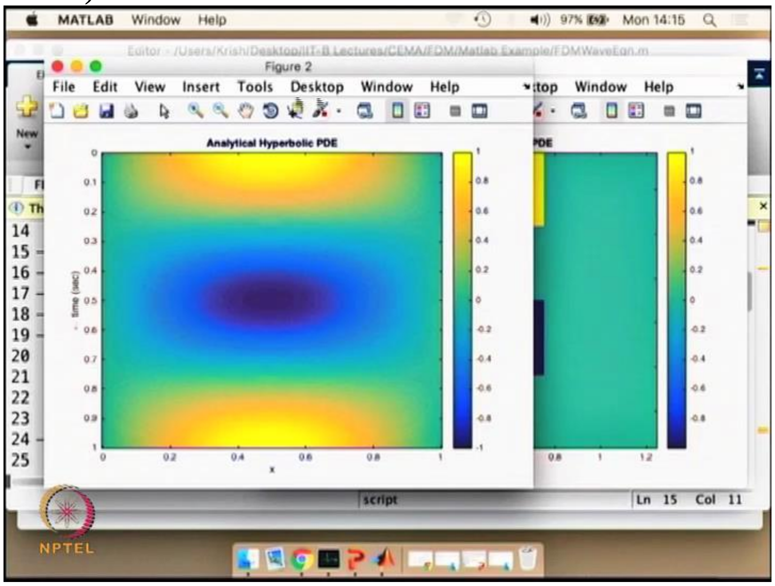

The first figure is the numerical result,and the second figure is the analytical result.As you can see the numerical result is a poor representation of the analytical result.You see the values on the left and right having some similarity but not exactly.You see that there is a maximum at the extreme point on time axis and and there is a minimum at the middle but again it's not very good.

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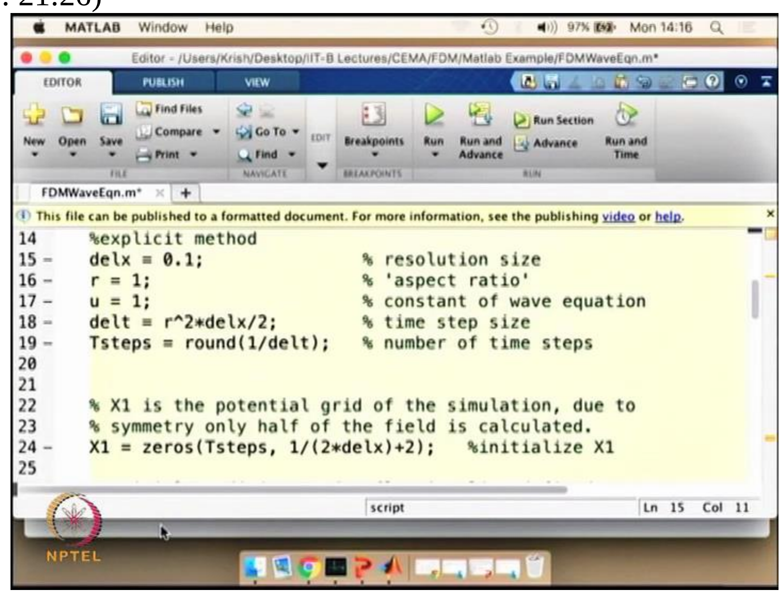

So let us see if I am improving this by going even final in the grid space.So I will go from 0.5 to 0.1. So when I do this I expect the value of the numerical results to improve.So let us see what happens.So this one should change and I as we expect this is changing.As you can see the values no longer simple to points but it is also showing certain nice behaviour.

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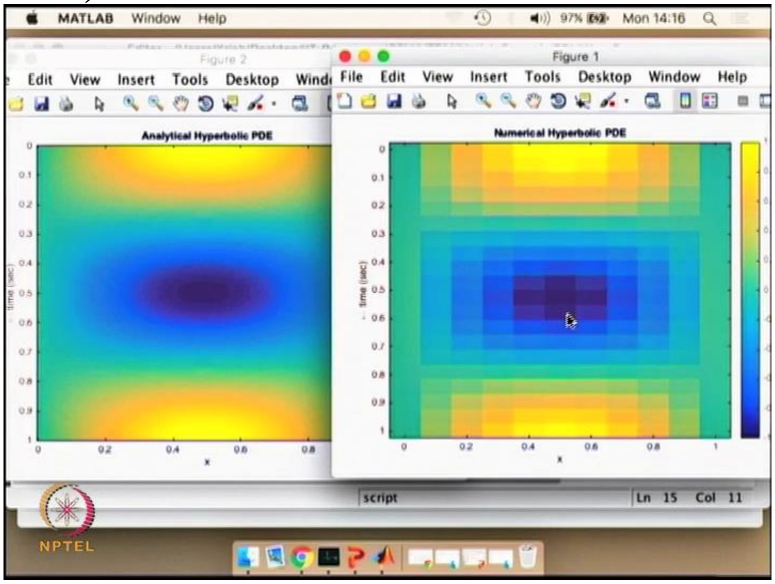

And obviously you can still see the numerical error and these are like what you see as pixel figures here.And let us know refine the grid even finer.

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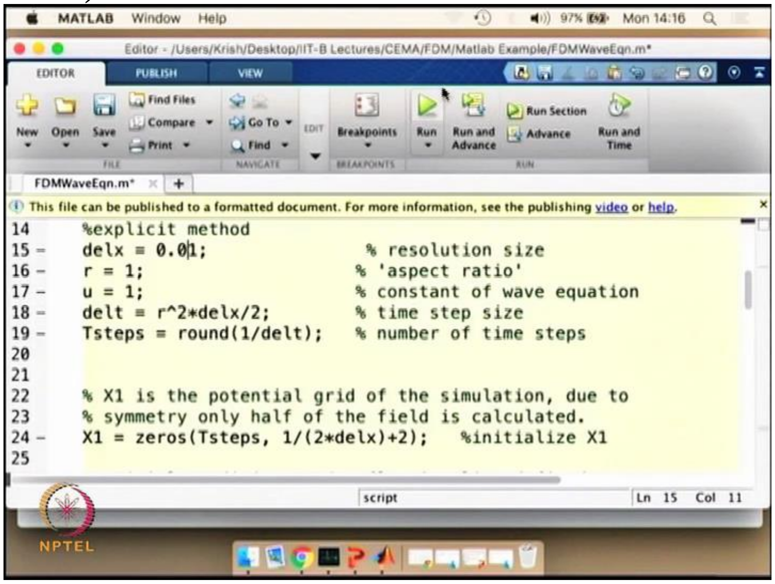

Let us go from 0.1 to 0.01 and we expect both the numerical results and the analytical result to be very similar.And let us see what happens.

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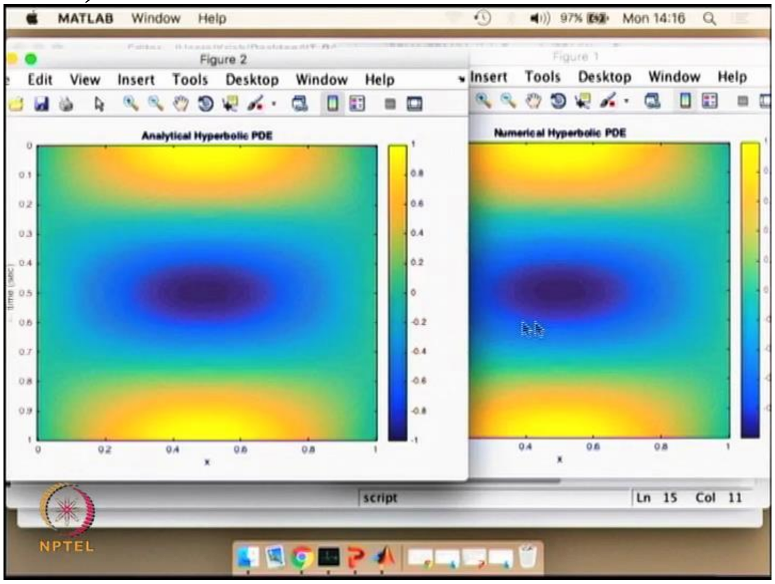

And as you can see the value of the numerical results which is on the right hand side is almost equal in fact very much the same as analytical result.And obviously there is still someerrorerror this error is so fine that it is not visible to our eyes.But if you compare the values to their decimal points you will still see there are some errors.

But this is more than enough for practical purpose.So with this we are able to compare the analytical result versus the numerical result.So we have come to the end of this particular module.We have started with a simple wave equation and we are showcased how we can do this stage by stage.We have set r is equal to 1 in this, case we can also set equal to any value between 0 and 1 and we are able to get a similar result.So what we have covered so far is, we have covered basic expressions for modelling finite difference one dimensional wave equation.

We have also showcased certain problems and next modules what we will do is we will look into accuracy and also stability of numerical method.

So with that being said we will come back again and we will focus on the further techniques in finite differencing.