Computational Electromagnetics and Applications
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Exercise No 6
Finite Difference Methods-II

We looked into forward in time, centered in space scheme for solving the adjection equation. We saw that some of the inherent limitations of using such schemes for practical problem where we quickly run into instability and time stepping should be very very small and this puts a lot of stress on computational requirements.

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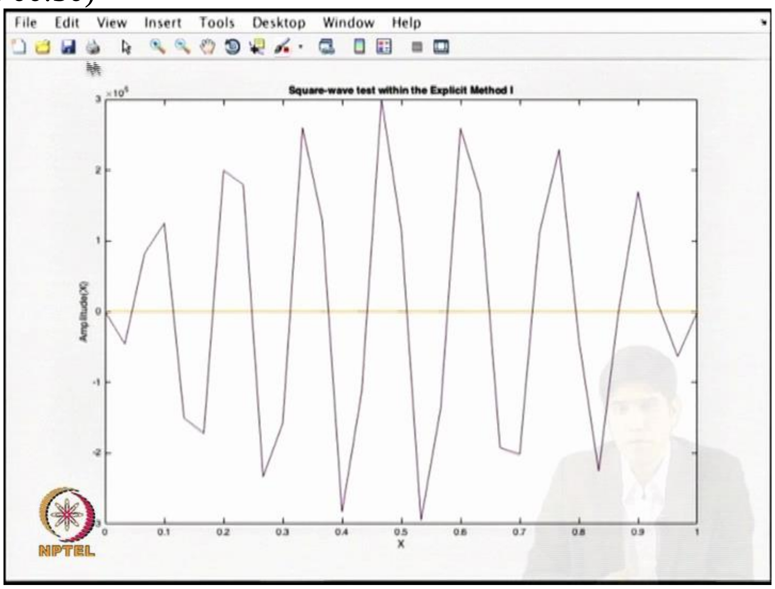

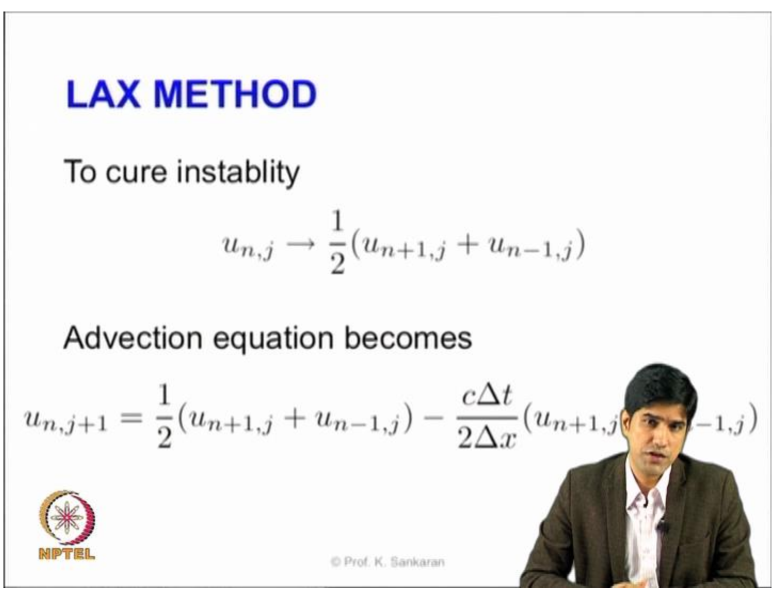

So with that we will talk about a cure for such instability issues using the Lax method and that is what we are going to look into this particular exercise.

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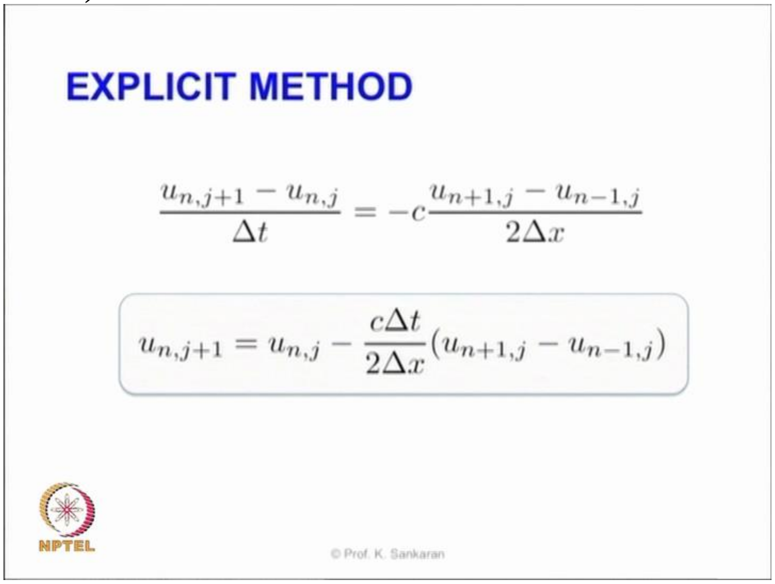

Before going into the Lax method we will spend a little bit time in the forward in time, centered in space scheme. And if that is essentially given by this particular equation what we have forward in time, centered in space scheme. c delta t by 2 delta x, the value should be much much much lower than 0.5 when we are going closer and closer to 0.5 we run into some instability issues.

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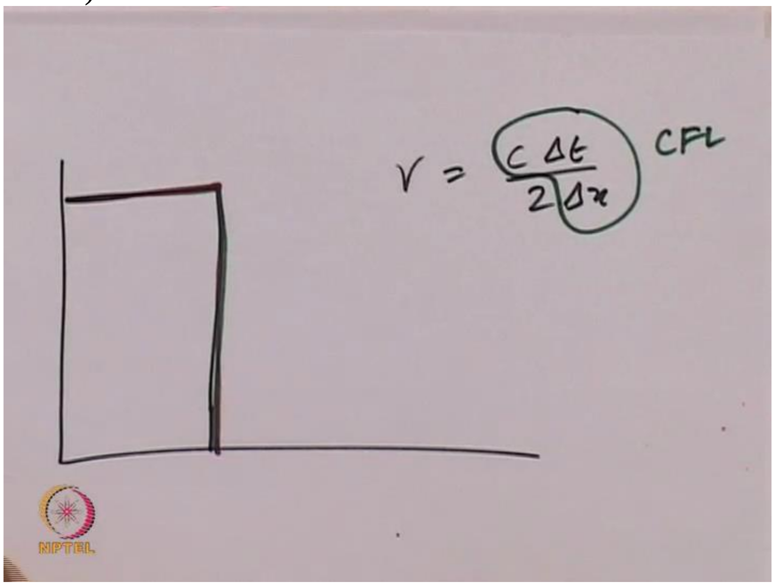

So in other words r equal to c delta t by 2deltax which is half of the CFL requirement remember this term is a CFL term.

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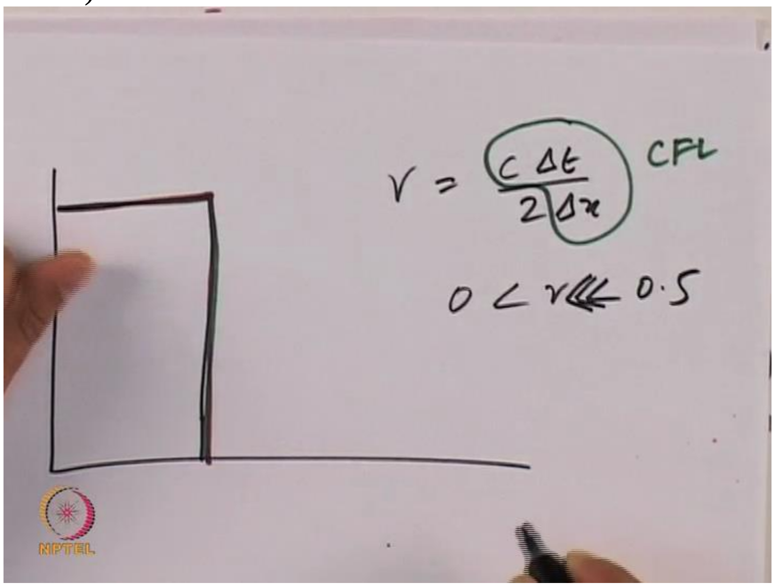

And since it is half of the CFL term we have to be in the range that is 0 less than r less than 0.5. As when we are going closer and closer to 0.5 we run into instability issues. In fact we have to be very very low compared to 0.5.

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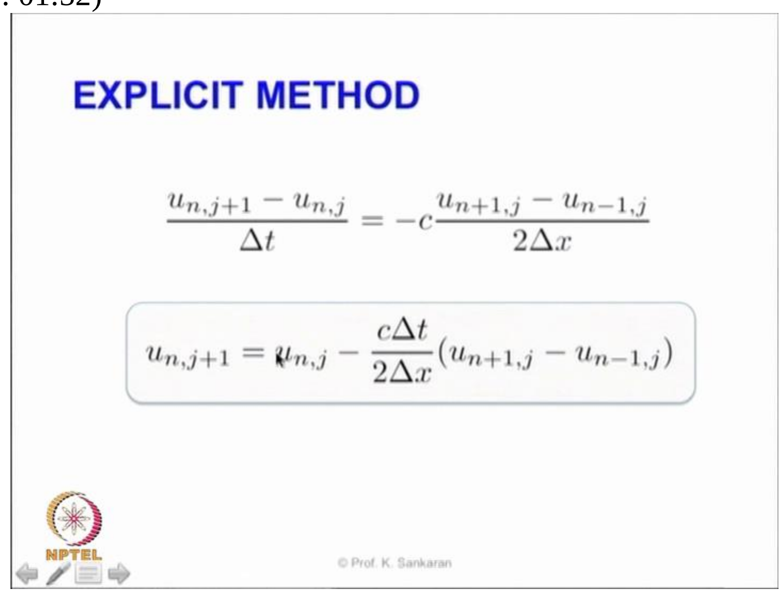

So now with this background we are going to go into the Lax method, so the Lax method is the clever way of fixing this problem. So in other words what we are going to do is instead of having the value of U of n, j we are going to substitute average of the values.

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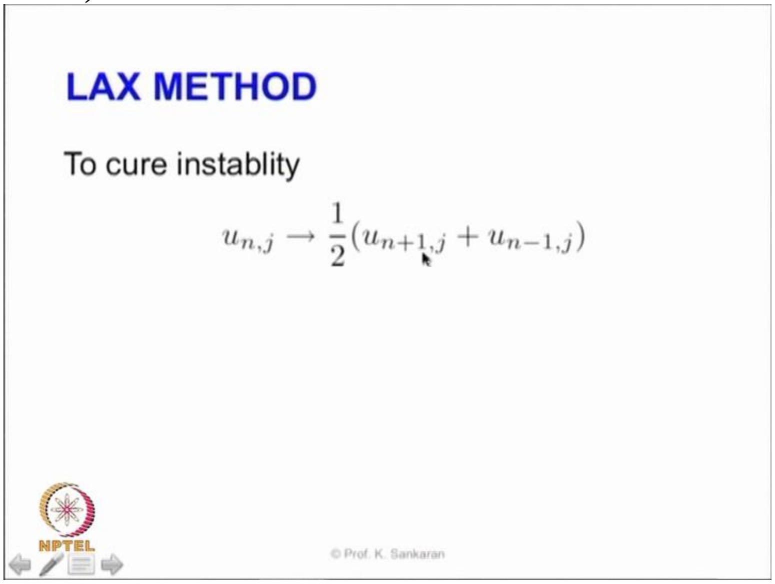

In the average of the values what we are doing is we are taking the value at time step j but n plus 1 and n minus 1.

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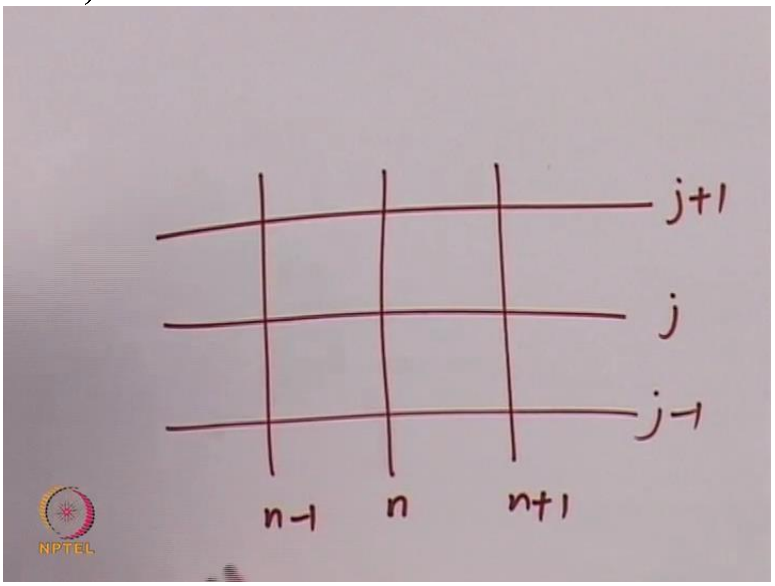

But remember our stencil is going to look like this. So this is time step j, j plus 1, j minus 1 and this is n, n plus1 and n minus 1. So we are going to compute the value at n, j plus 1.

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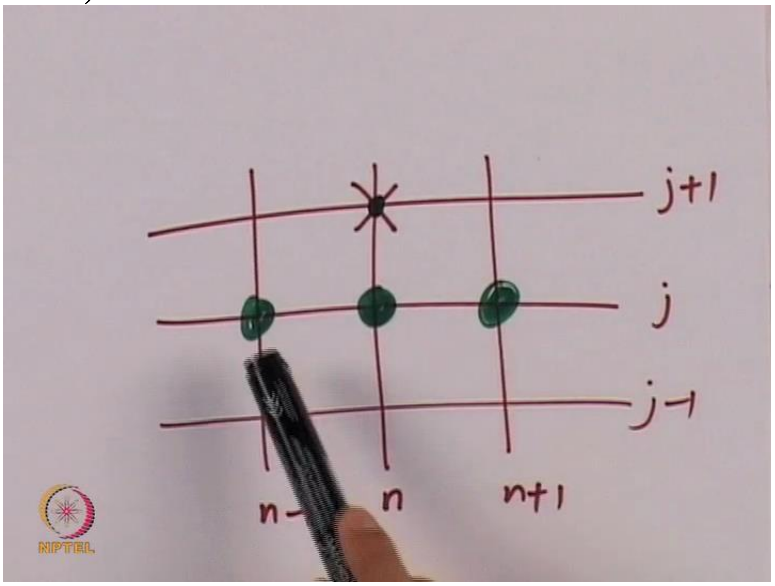

So this is the value that we are interested in. So initially in the case of the forward in time centered in space scheme what we did is we used a value at time step n, j. So n, j i am going to mark them in green. So we use this value and then we use the value at time step j n plus 1 and then we use n minus 1, j so we use this value. to compute the value let us say at this point.

Now what we are going to do is instead of taking this value at time step n, j we are going to take the value at j. But we are going to take the value that is average of these two values n minus 1 and n plus 1.

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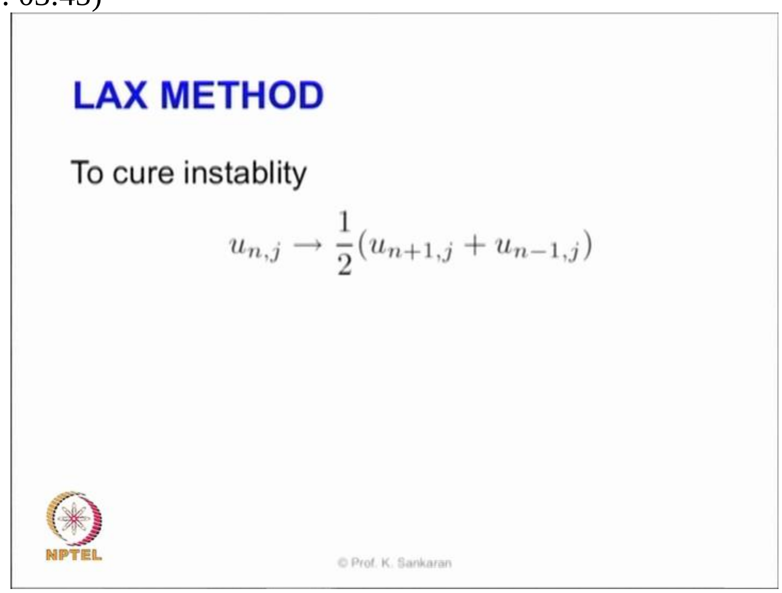

And that is what this equation is about, So what do you see is we do not take the value at n. So that way the Advection equation itself becomes as follows

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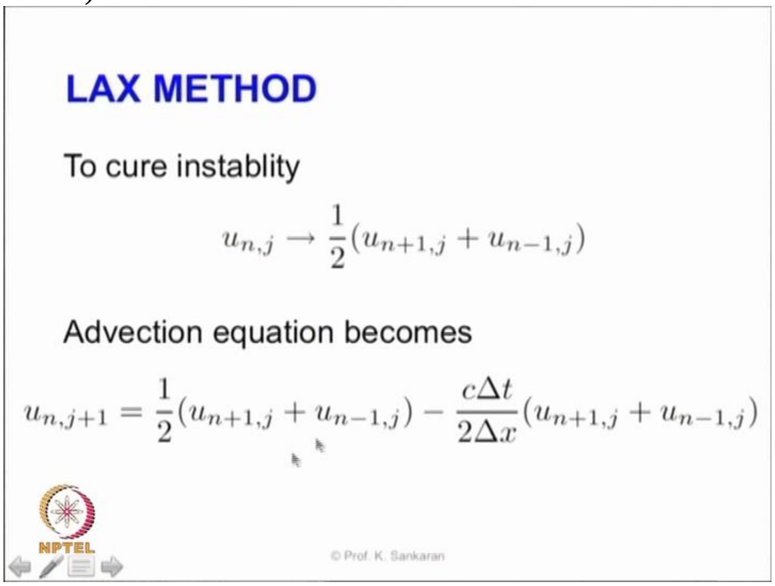

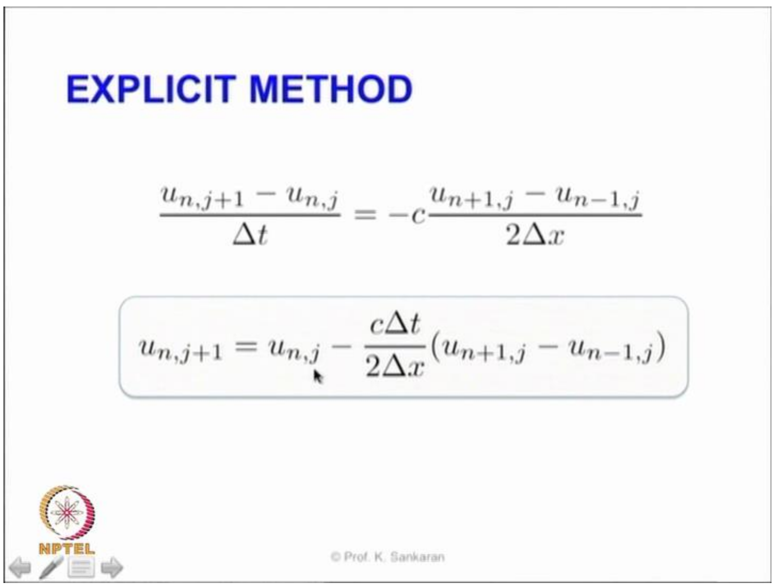

Remember in the previous case you had n, j instead of n, j I am going to use this value and then the equation will become like this

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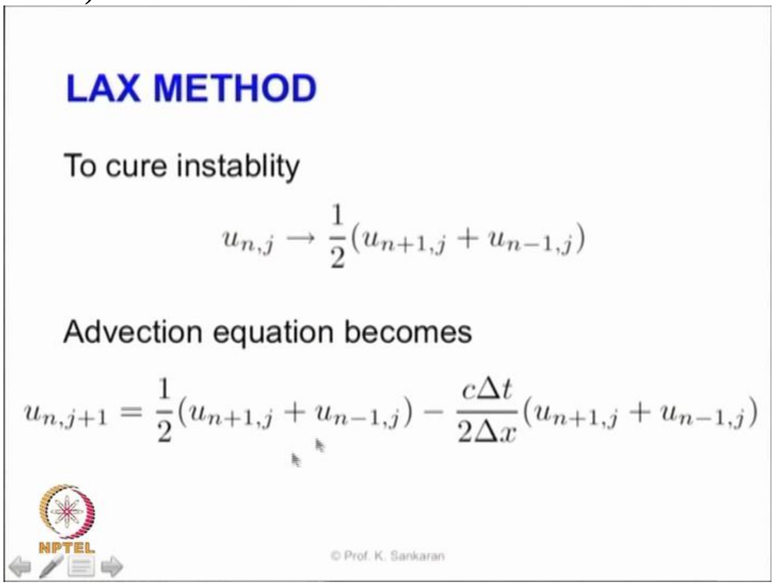

So once we have that we are able to somehow improve the quality of the scheme and it is essentially goes to the CFL condition which we talked about.

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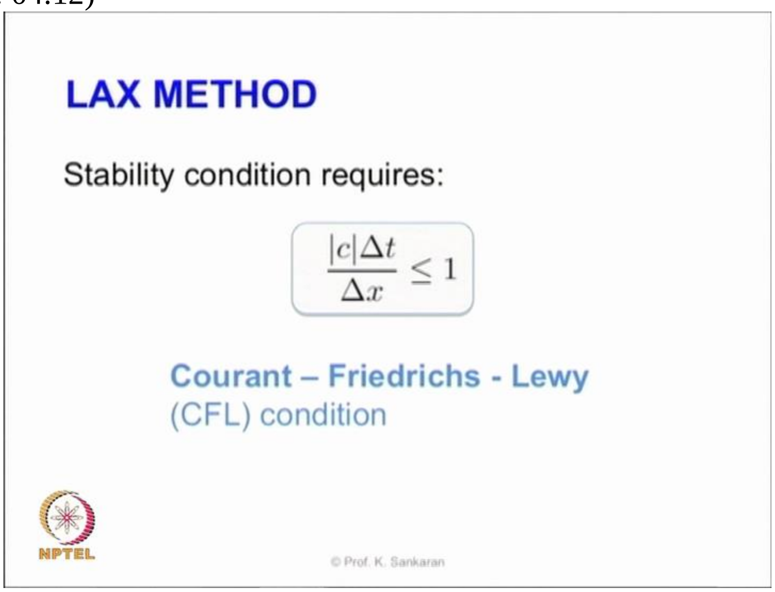

Which is due to Courant, Friedrichs and Lewy and this condition is umm pretty much going to be there in every scheme that we are to do. And some schemes at least in the explicit methods, implicit methods we are not worried about CFL Condition but for almost every explicit method CFL condition is going to be very very hard condition. With this Lax method we are able to improve the quality of the solution for a larger r value and that is what we are going to see in this particular example.

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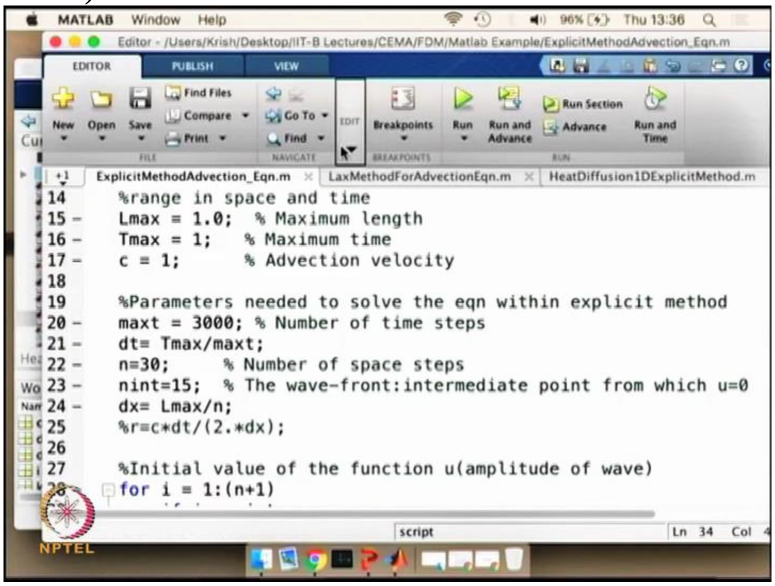

So let us go into the Matlab program itself. Before going into the main program of the Lax method, I am going to one more time look into the Advection equation using the explicit method which we did in the previous modules. So remember that the r value is going to be strictly lower than 0.5. And in fact we have to have r value that is very very low compared to 0.5.

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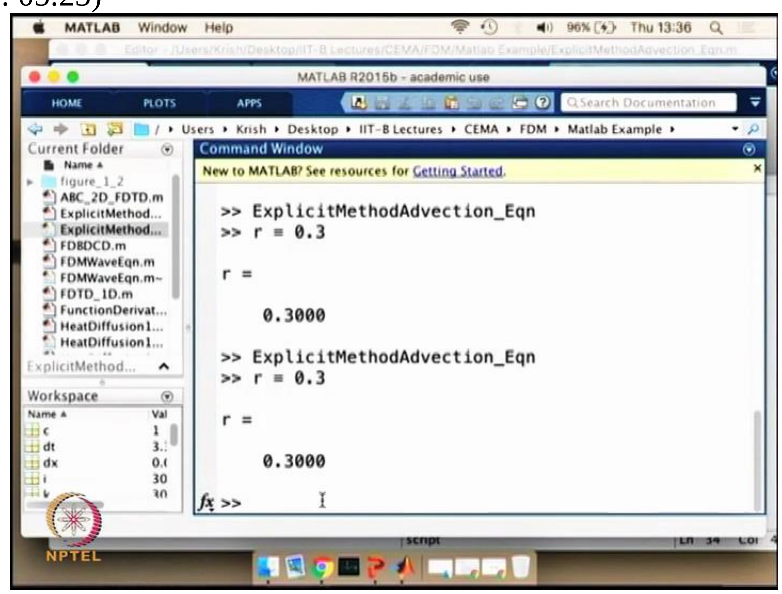

And let us run the program one more time for this explicit method and we will see how the solution looked how it is going to improve using Lax method. So remember our r value when it is equal to 0.3.

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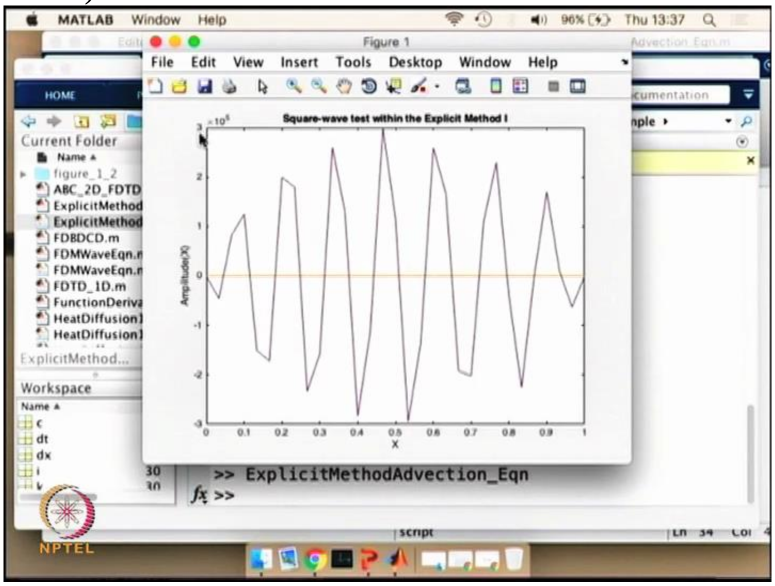

And when we are running the equation we are running into a huge instability issue. The value of the amplitude is going to 3 multiplied by 10 to the power 5 which is a very large value. So in order to solve for this issue remember that the we have to go very very low time stepping.

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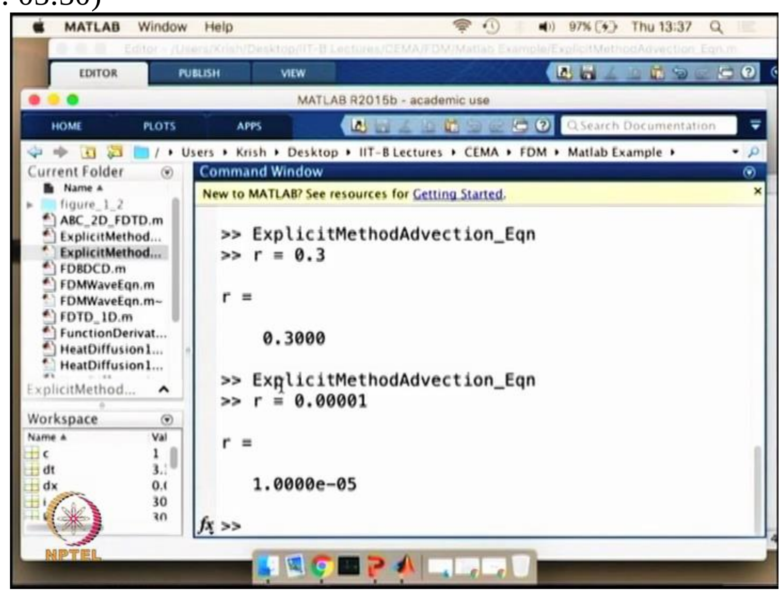

So r value should be equal to much lower so r value typically is in the range of 0.0001. Once you do that you start to see that the solution is coming closer and closer to the expected solution.

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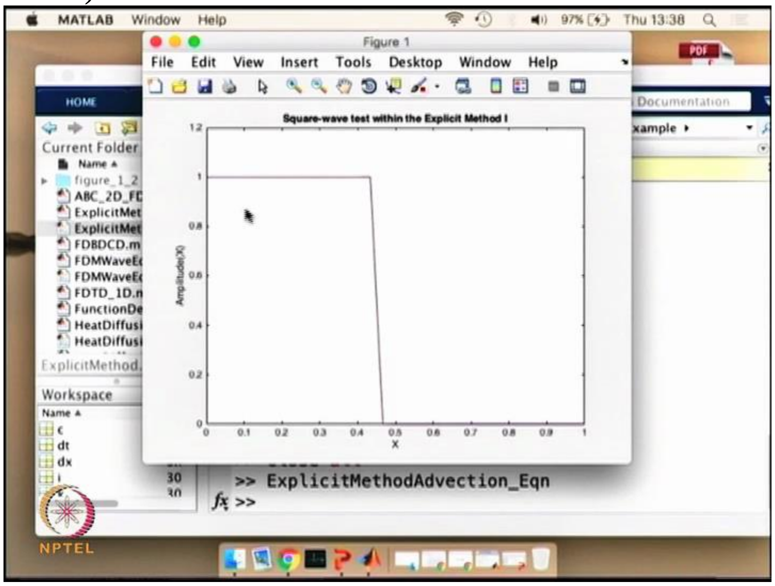

So now you are able to see the value is coming closer to the expected solution where the square wave is being replicated by the numerical method and the expected result is correct but of course we have to go very very low in r value.

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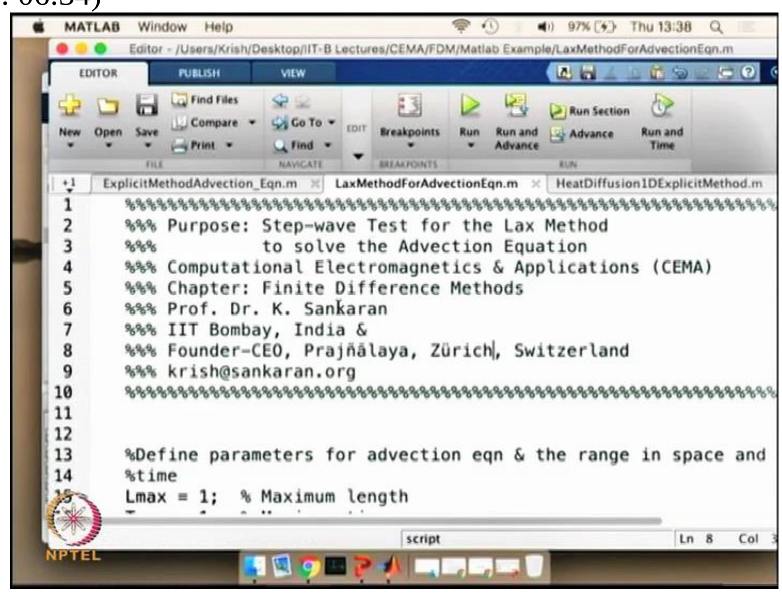

So as we said we are going to change this situation and use a different method using the Lax method. So let us introduce the Lax method itself in this particular Matlab code as before it is a step wave test function.

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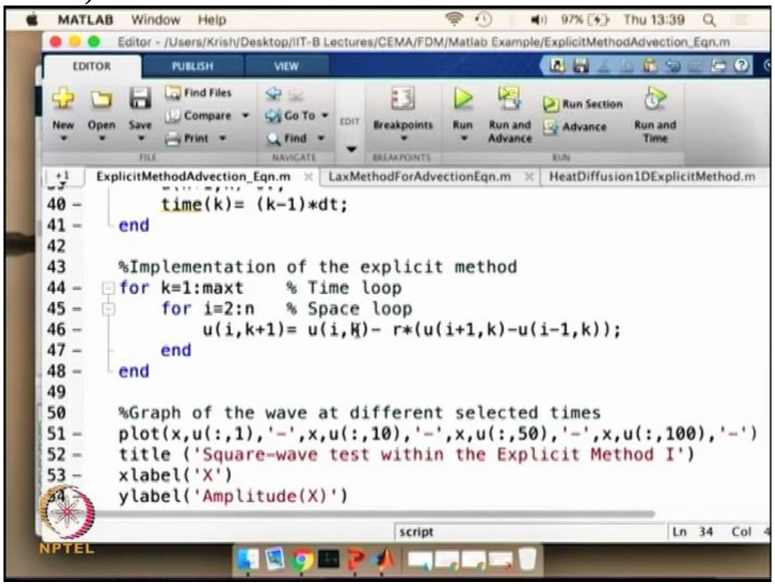

And what we are doing is we are trying to use the value instead of explicit method we computed the value using u of i, k in the Lax method.

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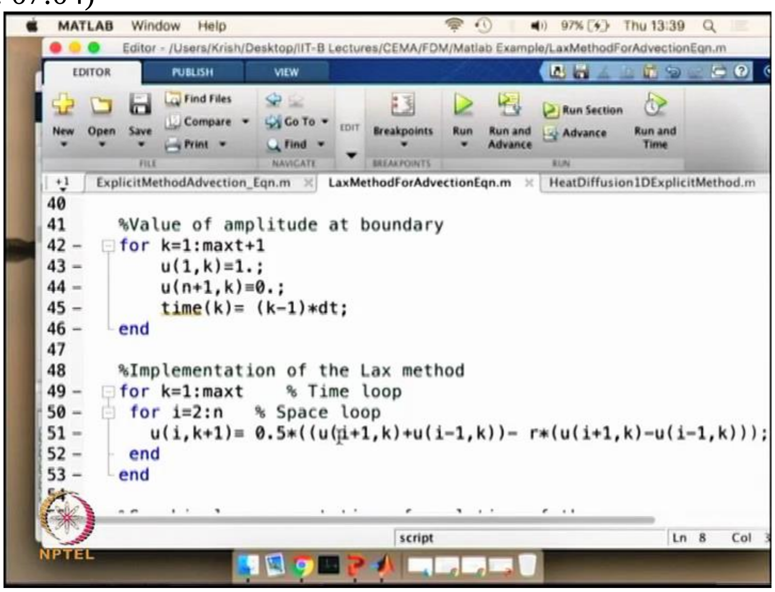

We are going to use 0.5 U of i plus 1 of k and plus i minus 1 of k and we are taking the average of those.

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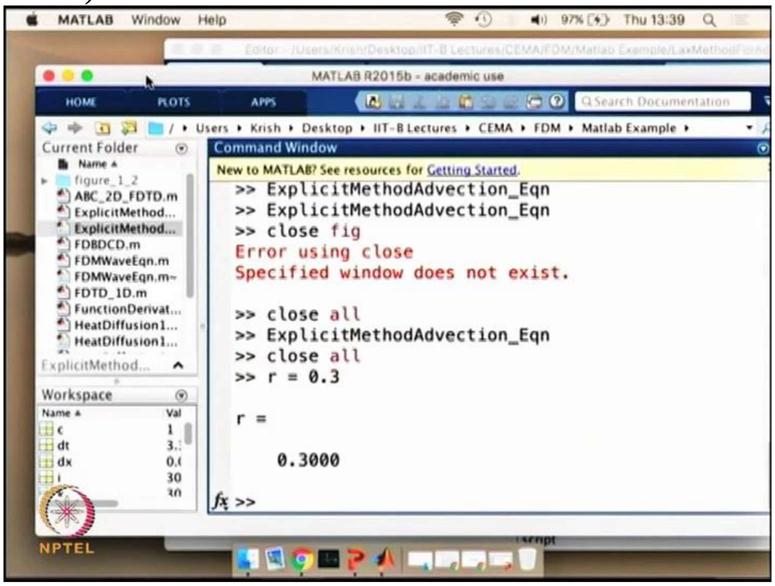

Let us try running the program for various values of r. So let us say we are going from, now I am going to keep r value as equal to 0.3. I am going to run using the Lax method so I am running it now.

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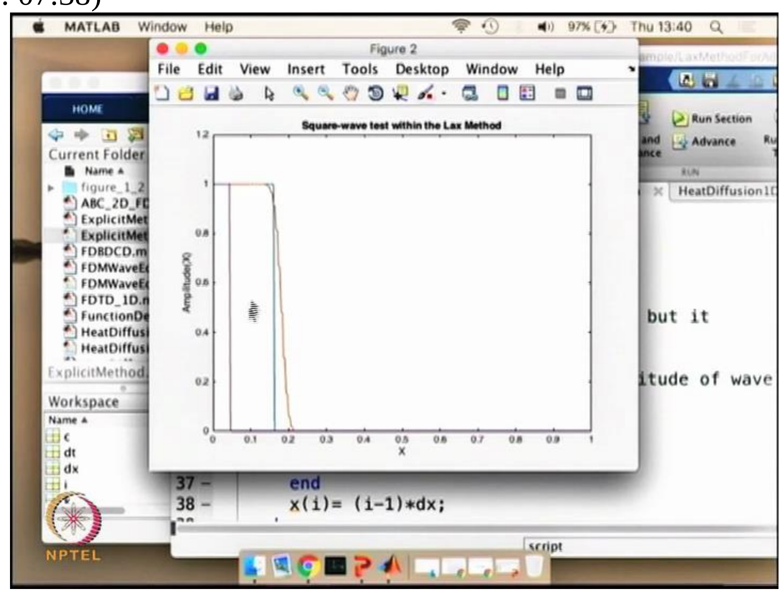

And we will see the value is replicating even for r equal to 0.3. The value is coming closer to the initial condition. As I said different colors are for different time steps and the initial condition is given y this particular line here the purple line and as the time step goes forward the solution itself is propagating in the forward direction. That is what we are expecting and that is how it should be.

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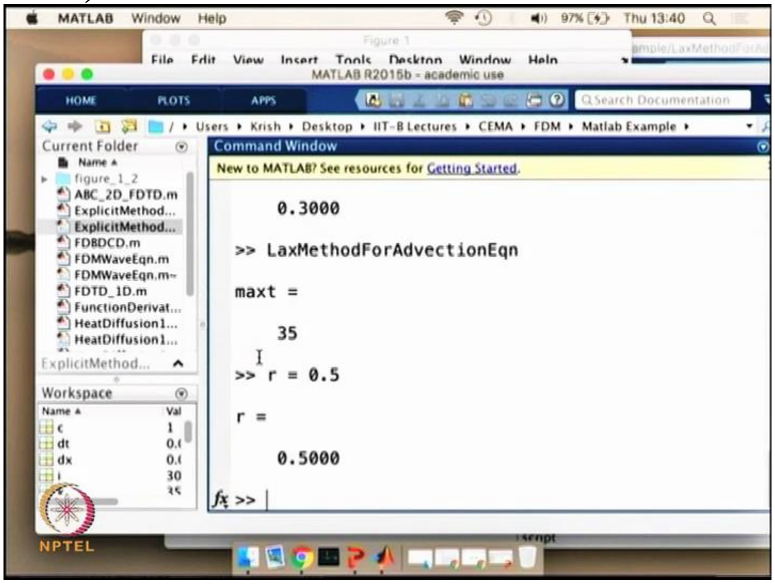

And now we will see what happens when we go r equal to 0.5.

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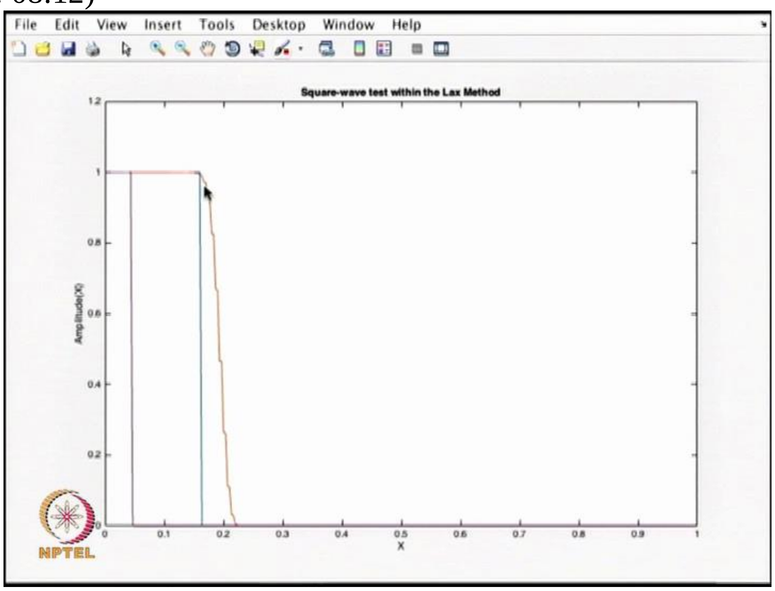

You see that still the solution is within the expected range because there is kind of some errors that are coming in the square wave function, and these are due to the numerical errors of the scheme itself it is still first ordered and second order in umm time and space respectively. But for that order this is very good replication of the differential equation.

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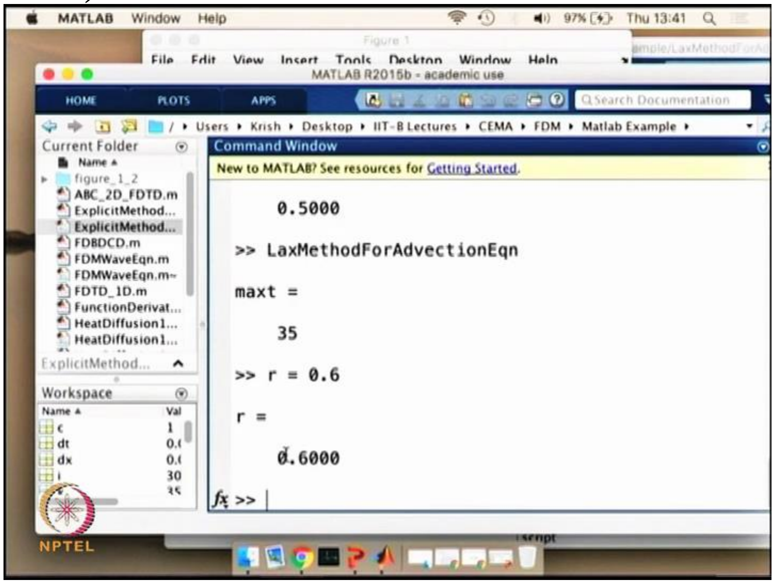

And as you can see we are able to run the program for r that is even higher than 0.5. So when we run for values that are higher than 0.5 we are still able to run the program. So what we have demonstrated using this particular Lax method is instead of taking the value at U of n,j when we take the average at a broader stencil we are able to improve a stability issue.

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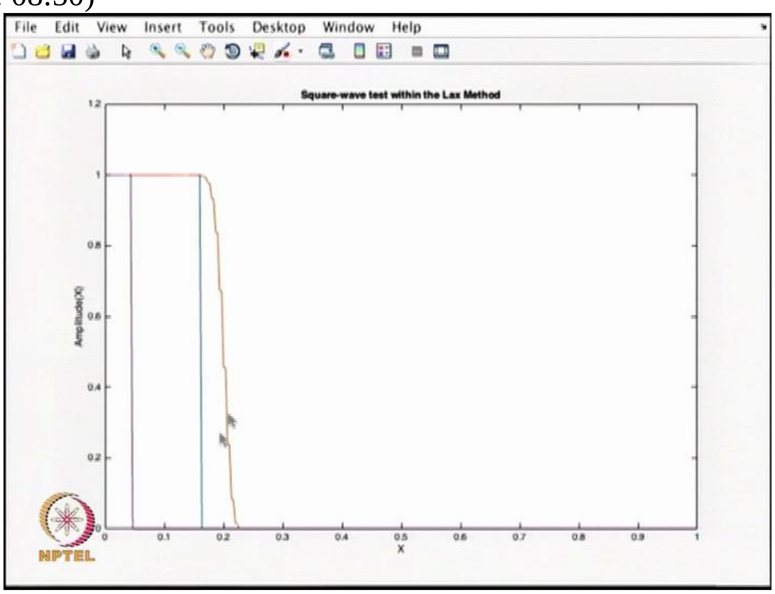

The program is still not accurate but still it is stable enough for us to run a program to see some simulation results. Obviously there are some numerical issues that comes because of the first order in time and second order in space. And of course this is something that we should definitely improve if you wanted to do any numerical problem solving using these methods.

But what we wanted to demonstrate using these examples is there is definitely some numerical issue coming from the forward in time centered in space scheme. We can improve that numerical stability issue or the numerical instability of the method by taking the average.

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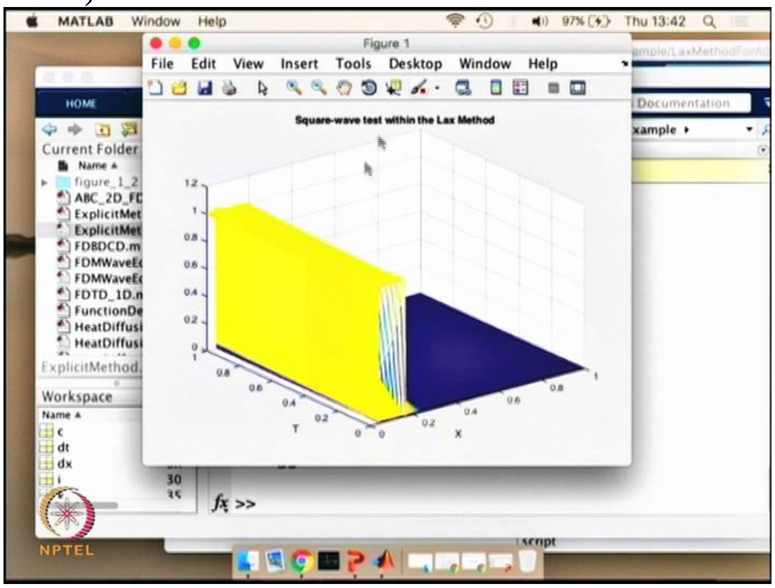

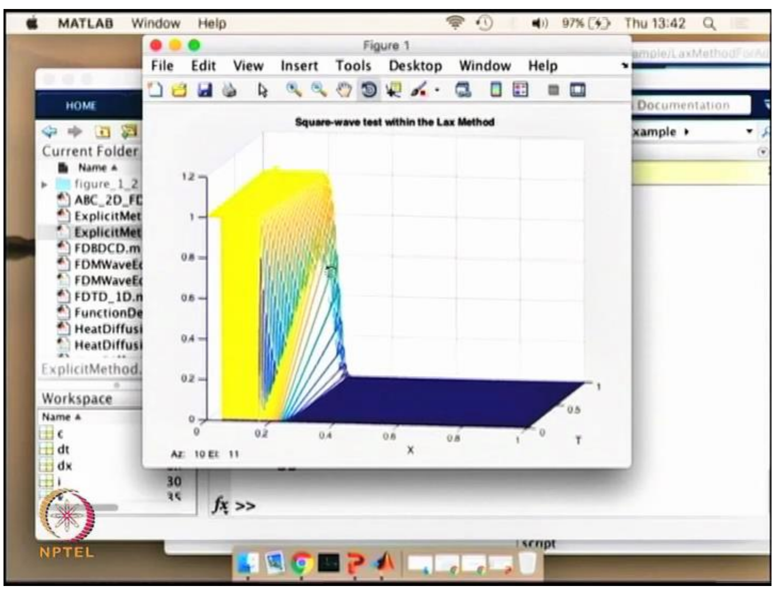

That is what we have done using this example. So what we are seeing here is solution itself is moving in the direction this is a 3D image of this particular solution. And you are seeing that there is an Advection that is happening and the initial step input has to go through the entire domain and it will propagate and that is what is the idea of the Advection equation.

We can solve the stability issue that is inherent in the forward in time centered in space scheme using some clever way of averaging the field variables at different points in space.

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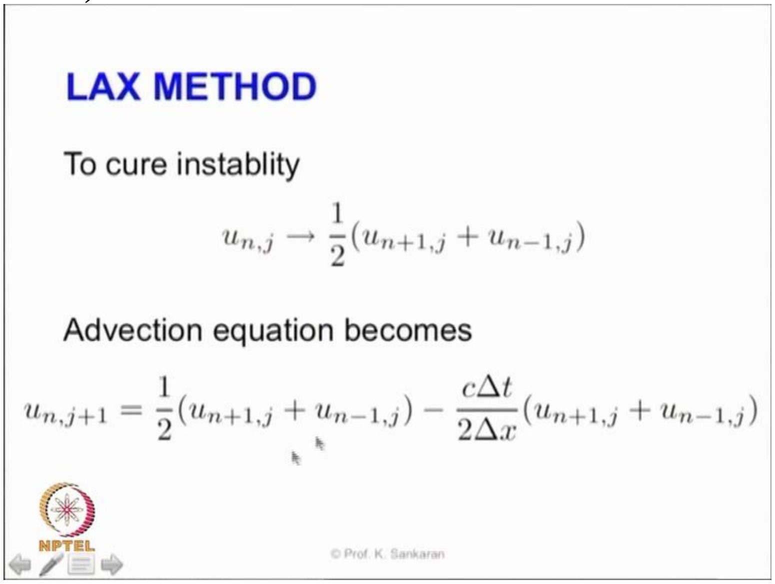

And that is what Lax method has shown us. So it enables us to run the simulation for even bigger values of r. And now I would like to leave this particular module with the question in mind.

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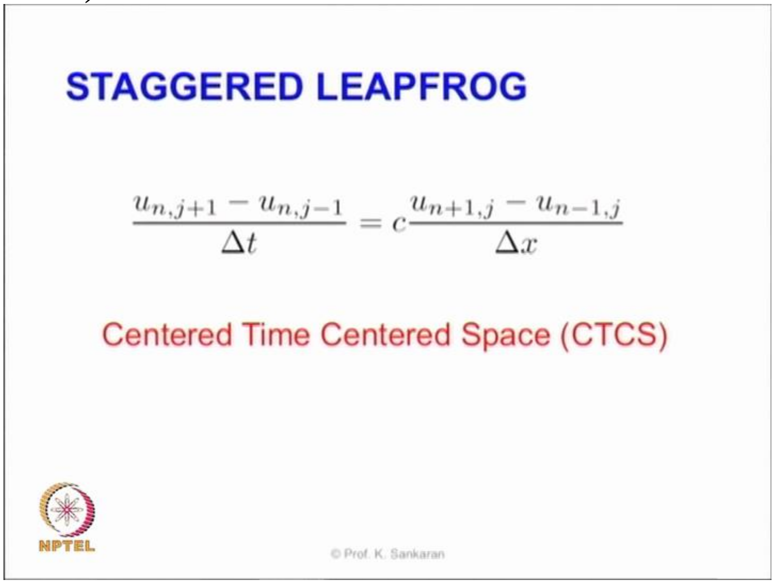

That is also something I wanted you to try it out is the Staggered Leap Frog technique. So remember in the standard Advection equation we did forward in time centered in space so we did simple forward differencing in the time variable and we did centered in space in the spatial variable. So what if we do both the space and time in Centered manner and that is what Leap frogging method is going to be about.

And the standard equation for that is going to be of this form and you can basically do the centered differencing both on the left hand side and on the right hand side.

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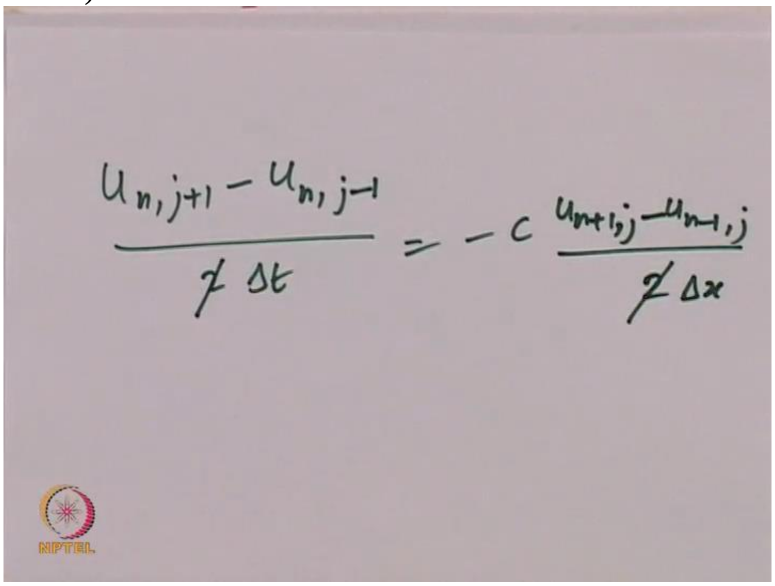

Remember that in centered scheme you will have the value of U of n, j plus 1 minus U of n, j minus 1 divided by 2delta t is equal to minus c Un plus, j minus U n minus 1, j divided by 2 delta x and this 2 and 2 gets cancelled and that is why we have only delta t in the equation.

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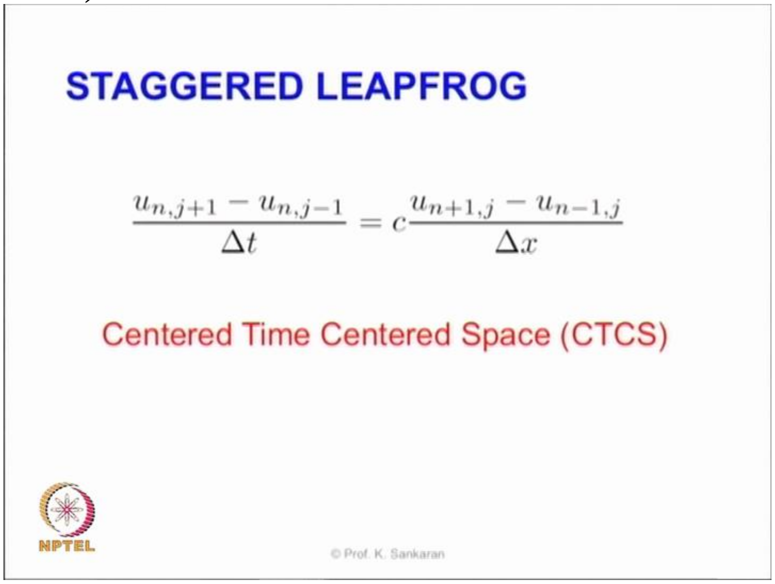

And that is what you see in the particular equation we call this as Centered in time Centered in space CTCS. It is nothing but the Staggered leap frogging technique. I would like to encourage you for doing this Matlab program for this particular scheme as well. Of course we will provide you with the Matlab code for Explicit method using the forward in time centered in space and also the Lax method. So once you can have those codes you can basically rearrange the terms and do accordingly for the staggered scheme. And you will notice that the scheme is also having certain properties for stability and which is good in some sense for you to try it out yourself before I say anything about it.

So that will be a exercise problem for people to look into it during the course of the week. So now with that being said we will come to a stop at this point and then we will take it forward in the next module.

Thank you