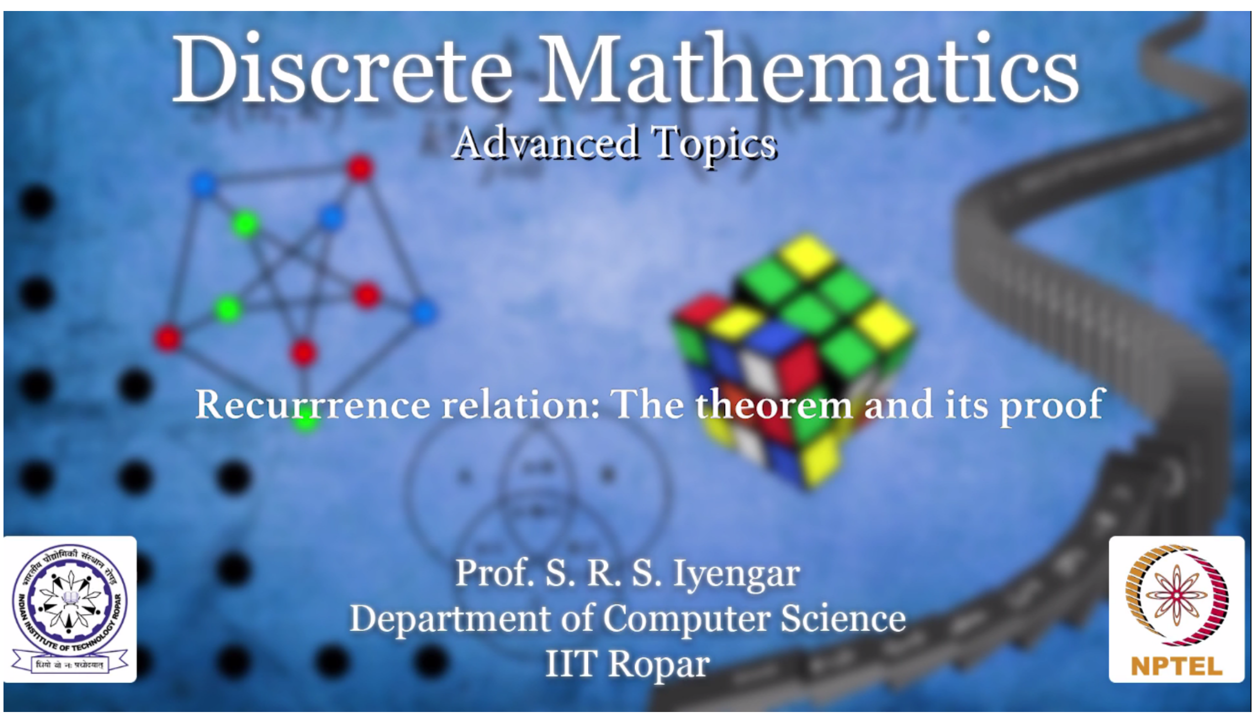
Discrete Mathematics

Functions

Advanced Topics



Recurrrence relation The theorem and its proof

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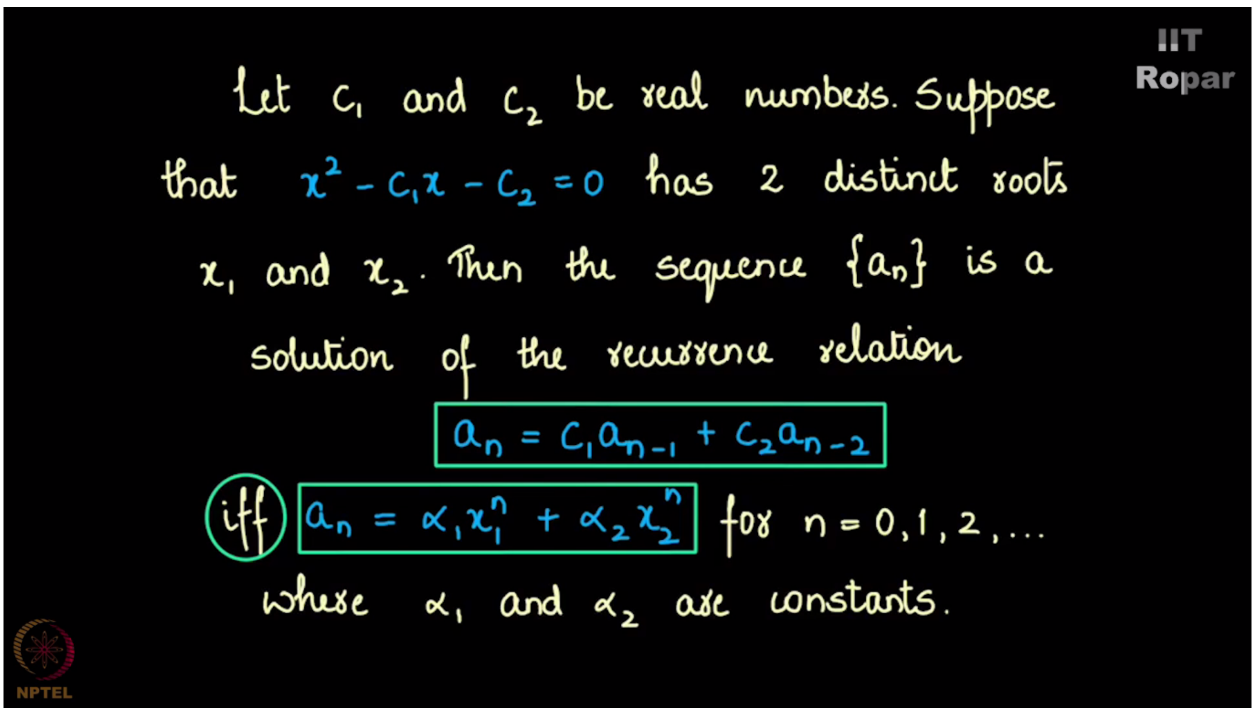
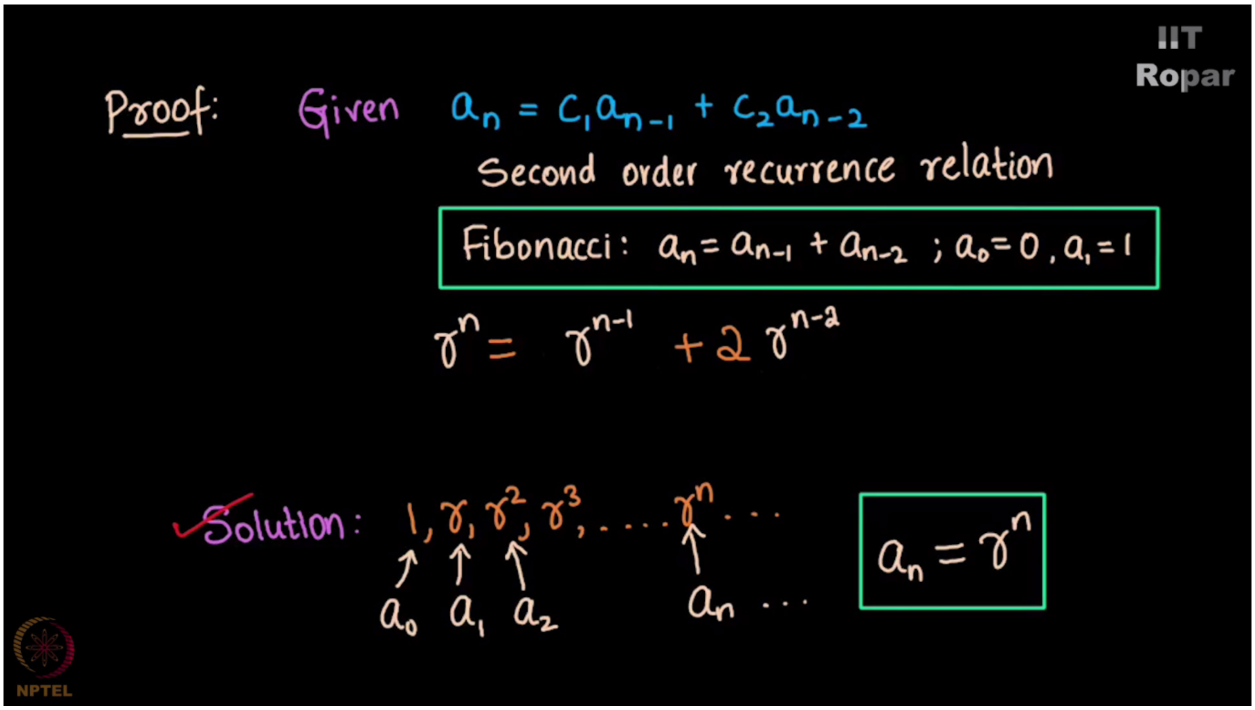
If you remember it right we did state this result. The result that goes something like this. If you have a recurrance relation defined by AN is equal to C1 times AN minus 1 plus C2 times AN minus 2 then the solution can be written as AN is equal to alpha 1 times X1 to the N plus alpha 2 times X2 to the N for summed X1 and X2. The point is it's if and if only result. You will soon understand what we mean by the if and only if result.



Now let us try seeing a proof for this theorem. So what are we expected to show here? What is given here? AN is equal to C1 times AN minus 1 plus C2 times AN minus 2. This is called a second order difference equation or a second order recurrance relation. You see Fibonacci sequence is exactly like this. AN is AN minus 1 plus AN minus 2 where A not is given to be 0 and A1 is given to be 1.

now let me play around with this equation. AN is C1 AN minus 1 plus C2 AN minus 2. So I keep trying what could be the solution. I go on and on. I spend day and night thinking what could be a solution for this. Let me in particular take this recurrance relation. Look at this recurrance relation AN is equal to AN minus 1 plus two times AN minus 2.

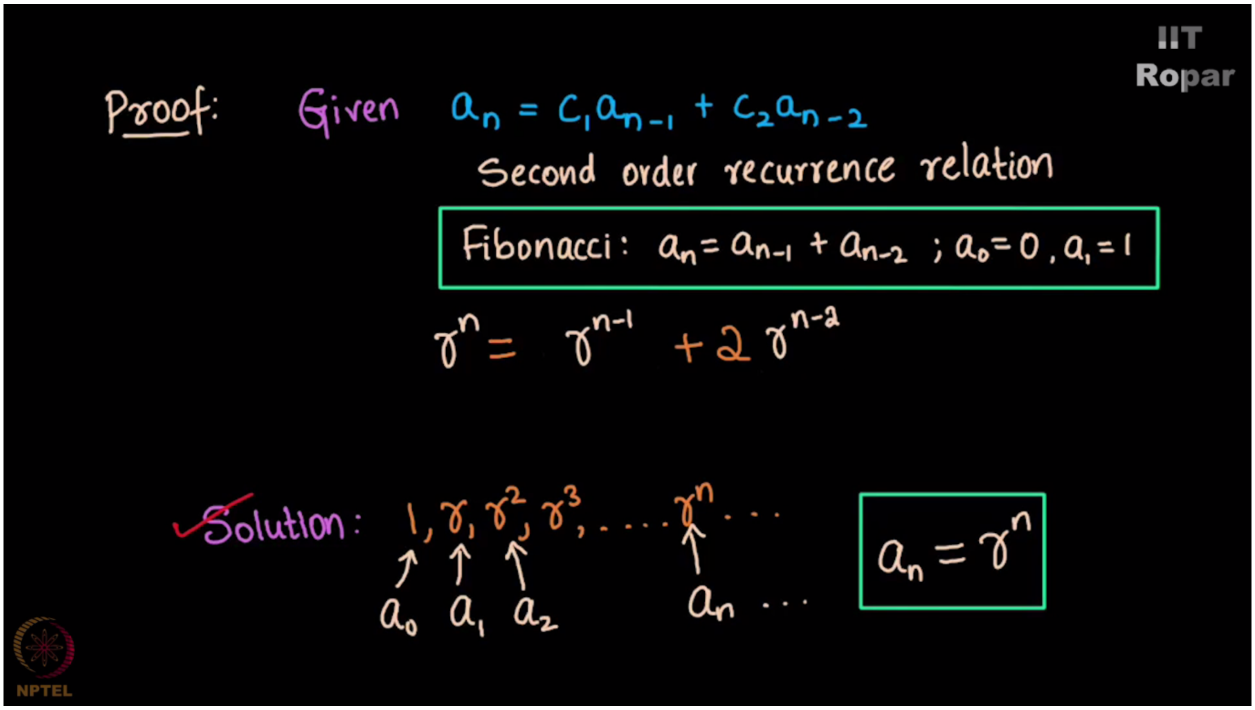
Let me just suspect that the solution could be of the form I, R, R square, R cube, and so on. Maybe a solution to this sequence could be of this form. How do you know? I don't know. I am just guessing. With a lot of effort I realized it could be this let's say. But then when you look at this you will realize that A not is 1, A1 is R. A2 is R square and AN is R to the N and so on.



Now if this is true what happens to the recurrance relation. AN is R to the N which is equal to look at this thing, AN is equal to AN minus 1 plus two times AN minus 2 translates to R to the N is R to the N minus 1 plus two times R to the N minus 2. But then you see this implies tell me how, R square equals R plus 2. I am simply dividing throughout by R to the N minus 2. And this again implies R square minus R minus 2 equals 0.

This is a quardractic equation. From where did we start? Just think. Maybe you may want to write things down parallely as I am explaining. So what are the roots of this quadratic equation? A here is 1. B here is minus 1. C here is minus 2. minus B plus or minus square root of B square minus 4AC which is 4 into 2 that is 1 plus 4 into 2 that is 9 divided by 2A which is 2 which is 1 plus or minus 3 by 2. 1 plus 3 by 2 is 2 and 1 minus 3 by 2 is minus 1. 2 and minus 1 are the roots. Now what have we just now realized? When R is 2 and minus 1 R square minus R minus 2 equals 0 is satisfied which implies that R square equals R plus 2 is satisfied. That is true because R square minus R minus 2 is 0 and that is true implies R square equals R plus 2 and by multiplying R to the N minus 2 entirely you get R to the N equals R to the N minus 1 plus 2 times R to the N minus 2 which means AN equals RN is satisfying my given recurrance relation provided my R is 2 or minus 1.

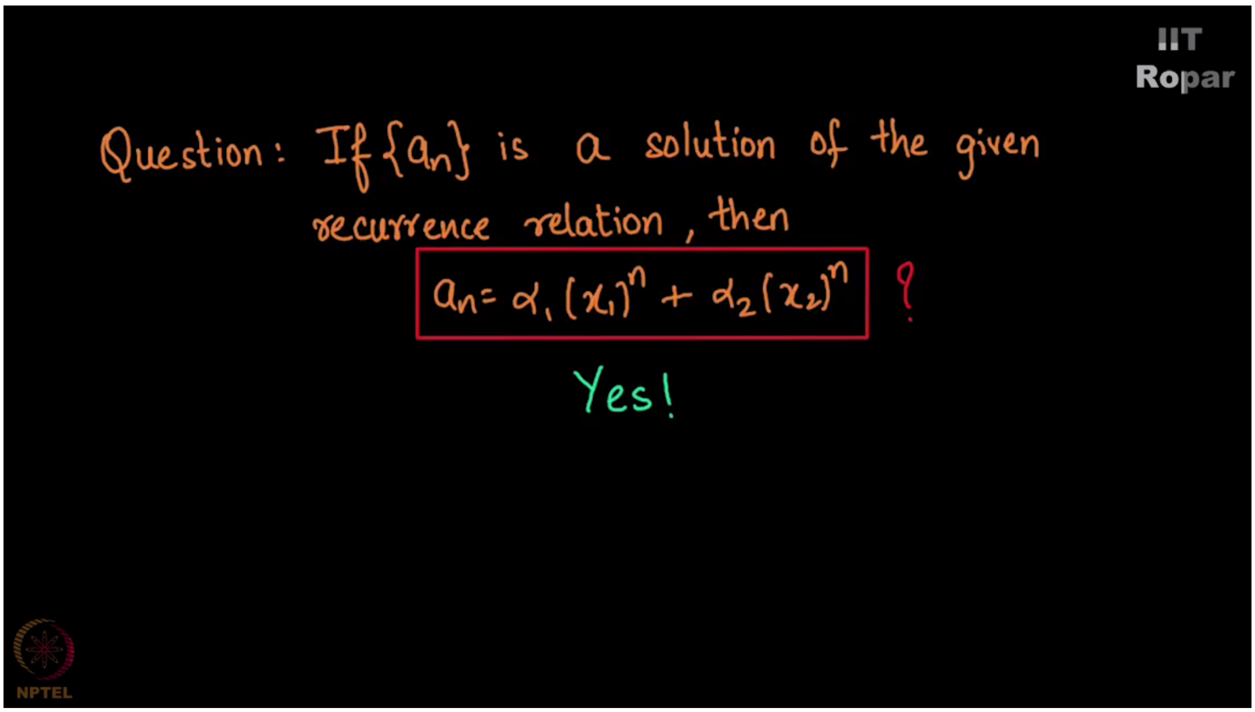
so you see let us try this out. Look at the sequence 1, R, R square, R cube. Let me replace R by 2 so we get 1, 2, 2 square, 2 cube, 2 to the 4 and so. Does it satisfy AN equals AN minus 1 plus 2 times AN minus 2. let's see is 2 cube equal to 2 square plus 2 times this element 2 let's verify. 2 cube is 8. 2 square is 3. 2 into 2 is 4. Perfectly right. So this seems to be satisfying our recurrence relation.



So what we can do is we can always equate AN to be R to the N and look at the associated quadratic equation, solve it and we will get to know 1, R, R square, R cube is always a solution. Observation one. Observation one. Any second order recurrence relation gives rise to a quadratic equation whose solution tells us two different sequences that satisfies our recurrence relation. Not only that my observation two, a second observation says that whenever you multiply a constant to the entire sequence even that is satisfied by the recurrence relation.

If 1, R, R square, R cube satisfies my AN equals AN minus 1 plus two times AN minus 2 multiply throughout by constant let's say alpha, alpha, alpha R, alpha R square, alpha R cube this sequence also satisfies the given recurrence relation. Now we have two sequences. So let's say R 2 and minus 1 are two roots. R equals 2N minus 1 let me call 2 as R1. Let me call minus 1 as R2. Now my observation three is that any constant times R1 which is 2, 2 to the N, basically AN equals C times 2 to the N is also a solution. AN equals sum D times minus 1 to the N is also a solution. You can check, verify. In fact, AN equals sum of these two. C times 2 to the N plus D times minus 1 to the N is also a solution. We have just now shown that given any recurrence relation AN equals something times AN minus 1 plus something times AN minus 2 you can always write that as the recurrence relation solution always can be written in the form AN equals alpha 1 X 1 to the N plus alpha 2 X to the N for some alpha 1, alpha 2. In fact you can always write any alpha 1, alpha 2 and this will be a solution. Think about it.

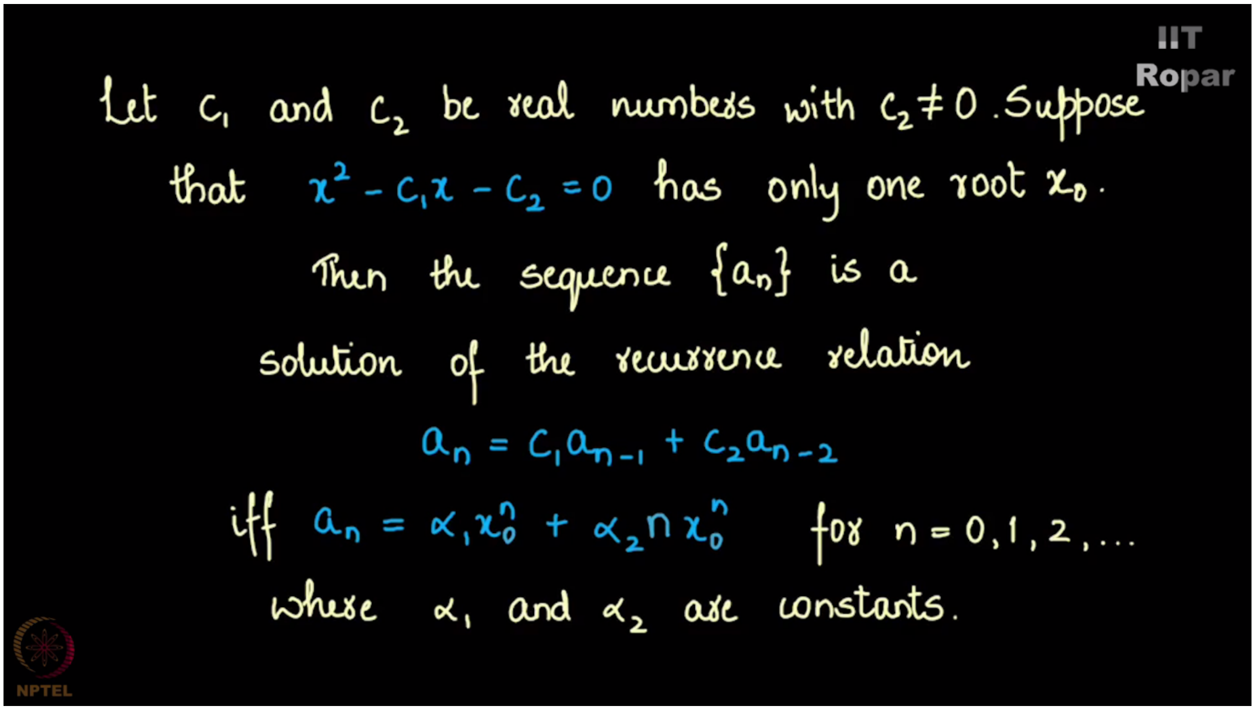
What has happened so far? You were given a second order recurrence relation by hook or crook I observed that a geometric progression of the form 1, R, R square, R cube could be a solution for this. In fact, I pinned it down properly. I even told you for what are well a sequence like that be a solution for this recurrence relation but then a question in math is always if a problem is given to you, you should not give us a solution. You must give us all possible solutions. In that connection if I ask you a small question don't break your head much. This require some patient thinking for you to even understand what I am saying. If you don't get this in the first reading of this concept or first watching of this video don't worry much. Let me go ahead. If AN is something and it is acting as a solution to the given recurrence relation. Do you think it's of this form that AN is alpha X 1 to the N plus alpha 2 X2 to the N? Is it always of this form is the question.



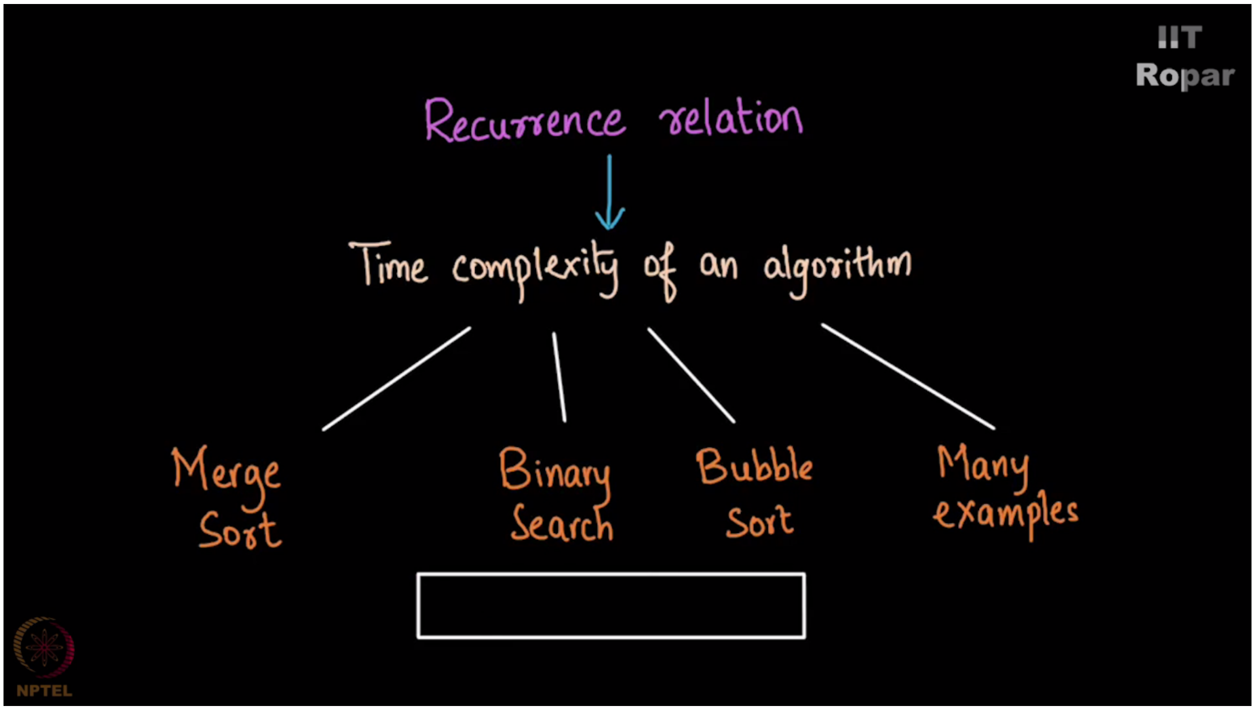
I repeat if some sequence is a solution for the given recurrence relation is it of this form always. What did we just now show? We showed that something of the form 1, R, R square, R cube, etc. can be a solution of this recurrence relation. In fact, we showed that a constant times multiplied by one of the root to the N plus another constant time multiplied by another root of the quadratic equation to the power of N will be a solution. Given some solution of this recurrence relation is it always of this form is the question. The answer is yes. This is very much true. There cannot be any solution which is not of this form. Every solution is of this form only. So this is actually a if and only if statement. If AN is a second order recurrence relation then its solution where X1 and X2 are root to the quadratic equation this particular thing will be a solution or if you give me any solution it will actually be of this form. Any solution is of this form is actually a small exercise problem. We will leave it to you.

Think about it. It's available in standard textbooks but what you need to know is whenever you encounter a situation where you are given a recurrence relation let's say like AN is equal to 9 AN minus 1 minus 20 AN minus 2 if you see its roots will be of the form look at this, I have solved it. Quadratic equation will be of the form X square minus 9X plus 20 equal 0 you solve it you will get 4 and 5 are the roots. So you know very well. Whenever you say AN equals some constant times 4 to the N plus constant times 5 to the N when you write it like this this will be a solution of this recurrence relation. That is all you need to know.

Whenever you spot a second order recurrence relation you should know how to solve it. Now a small advice rather an important warning message. This method works only when the roots are distinct. What do I mean by that? The quadratic equation that you get should not be of the form X square minus 4X plus 4. If it's of this form then the roots are repeated. If the roots are repeated you cannot write the solution like this. There is a different method for it and here is the statement of the result which tells you how to write the recurrence relation when the roots are repeated.



The roots can also be complex sometimes. There is a way in which you can solve a recurrence relation when the root is complex. So a different kind of recurrence relation has different types of recurrence relations have different type of solutions. Anyway in this course we are not going to go ahead very deeply into solutions of this recurrence relations. It is more like the theory of differential equations which some of you would be aware of. If we start spending more time on it you may not really see the connectivity between this and computer science. This is a humble opinion. So what we are doing is we are just motivating you people to think recurrence relation as something that gets originated when you are solving an algorithmic problem and you need to solve the recurrence relation in order to understand the time complexity of the algorithm that you are solving.



You see remember we solved the merge sort. The binary search. The bubble sort. And a few more examples that you saw. Recurrence relation just like that came. And we showed you how such recurrence relations can be solved. So in general when you see something like a Fibonacci sequence that is also a recurrence relation. You must know how to solve it. So this is just a first introduction to recurrence relations. You can refer it to any discrete math book which will give you a lot more advanced details about recurrence relations. But right now you all should be confident that whenever you see a algorithm you discuss its time complexity, if it looks like a recurrence relation you should look up the theory of recurrence relations and understand how to solve it.