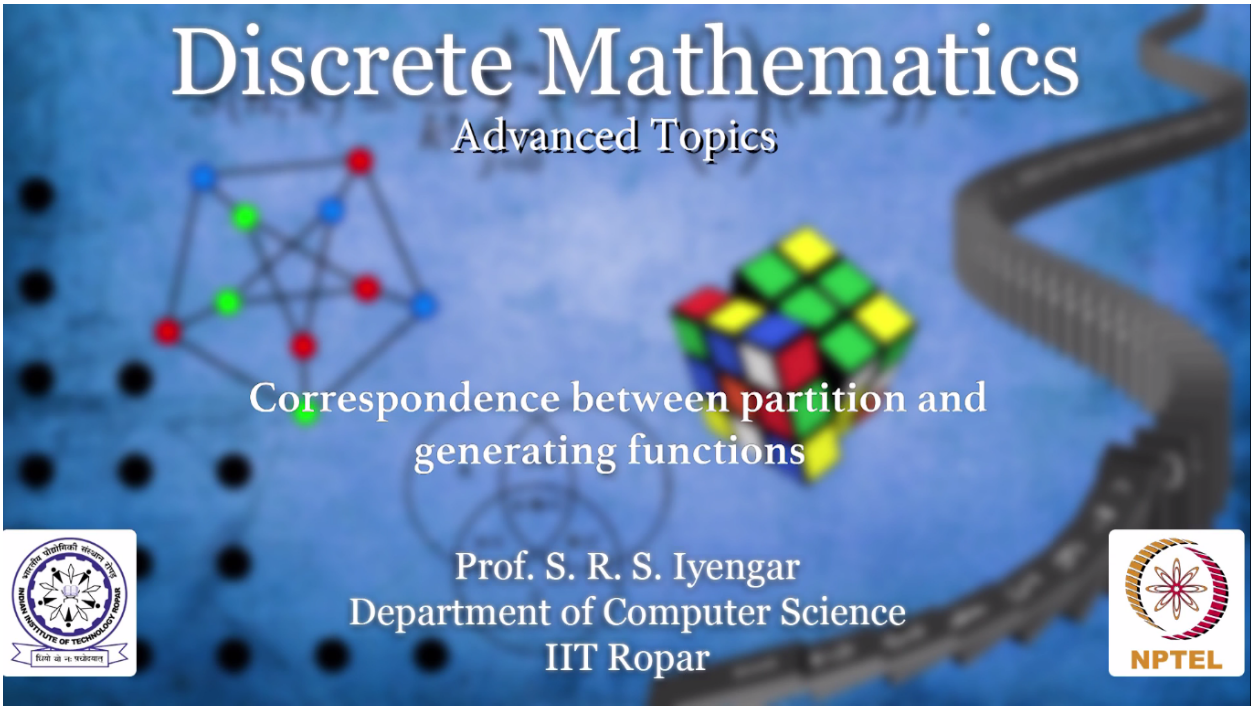
Discrete Mathematics

Functions

Advanced Topics

Correspondence between partition and generating functions



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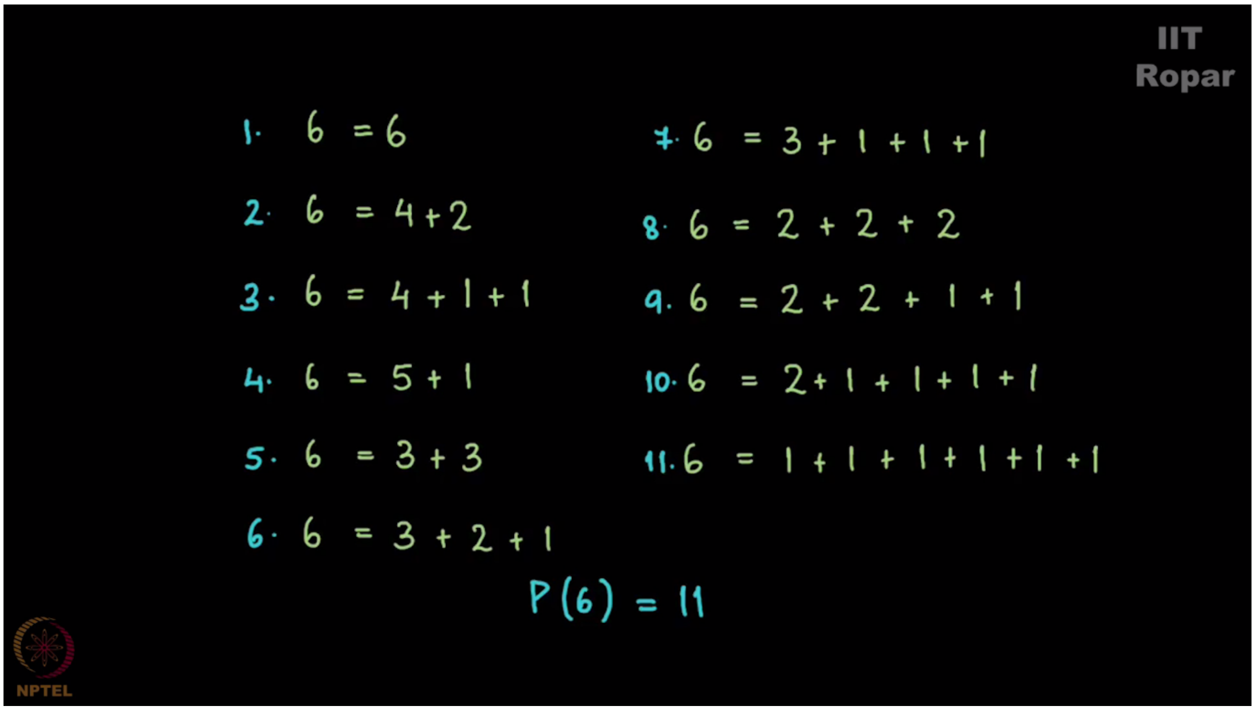
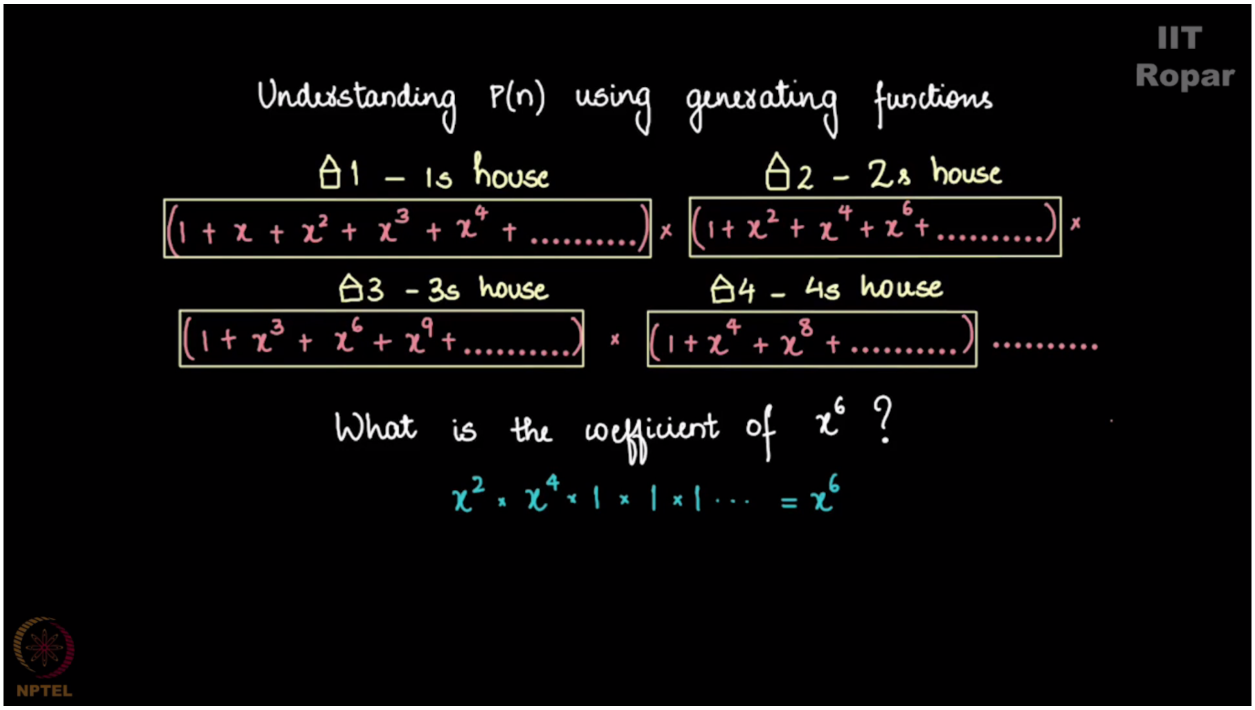
Look at this. In how many ways can I write 3? what do I even mean by this? It will be self-explanatory just observe. 3 equals 3. 3 equals 2 plus 1. 3 equals 1 plus 1 plus 1. There are three ways in which you can partition 3. In how many ways can you partition 4? Before that in how many ways can you partition 2? 2 equals 2. 2 equals 1 plus 1. So the number of ways in which you can partition 2 is 2 ways.

Let us denote this by P of 2 equals 2 and P of 3 was equal to 3 and what is 4? 4 can be written as 4. You see P of 2 is 2. P of 3 is 3. I suppose P of 4 should be 4. Let me see. 4 equals 4. 4 equals 3 plus 1. another way. 4 equals 2 plus 2 yet another way. 2 plus 1 plus 1 another way. The last way is 1 plus 1 plus 1 plus 1 as you can see there are 5 ways. So my guess was wrong P of 4 is not 4 P of 4 is 5. For 6 if you see 6 equals 6. 6 equals 4 plus 2. 6 equals 4 plus 1 plus 1. 5 plus 1. 3 plus 3. 3, ,2 1, 3, 1, 1, 1. 2, 2, 2. 2. 2, 1, 1. 2, 1, 1, 1, 1. and all 1s. If you can count these it will be 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11. So P of 6 is 11.



Let me observe something here. Something according to me is totally counter-intuitive and in a way even poetic poetic because it's so creative that you can now bring in a brand new concept to explain what is this P of n. What is that concept? The concept of generating functions. Let's see how. Look at this. look at this 1 plus X plus X square X cube X to the 4 so on multiplied by 1 plus x square plus X to the 4 position a 6 and so on multiplied by 1 plus X cube plus X to the 6 plus X to the 9 and so on and then 1 plus X to the fourth plus X to the 8 plus so on and so forth. Let's say we go up to infinity like this. Now here in this expansion what is the coefficient of x to the 6? Let me think what is the coefficient of x to the 6. In how many ways can you get X to the 6? Look at this from the first let's call this a house from the first house here within the first bracket is first house if you pick let's say X square and for the second house you pick X to the four and the rest of the houses you pick 1, 1, 1, 1, 1 you will get an X to the 6.

Now the first house I call it the 1s house. I tell you how rather why. The second one I'll call it the 2s house. The third one I'll call it the 3s house. Fourth one 4s house and so on. So what am I doing here?



You can get X to the 6 by picking X square for the first term 6 to the 4 from the second house one way correct. Now this denotes X square in the first house denotes look at the power two here it denotes 1 plus 1 is present in the partition. If you pick X to the 4 it denotes 2 plus 2 is present. So when you pick X square from first house and X to the 4 in the second house this corresponds to 2 plus 2 because of X to the 4 from the second house plus 1 plus 1 because of X square from the first house. Things will be very clear stay patient look at this what if I took an X cube from the first house 1 from the second house and X cube from the third house. What does this correspond to?

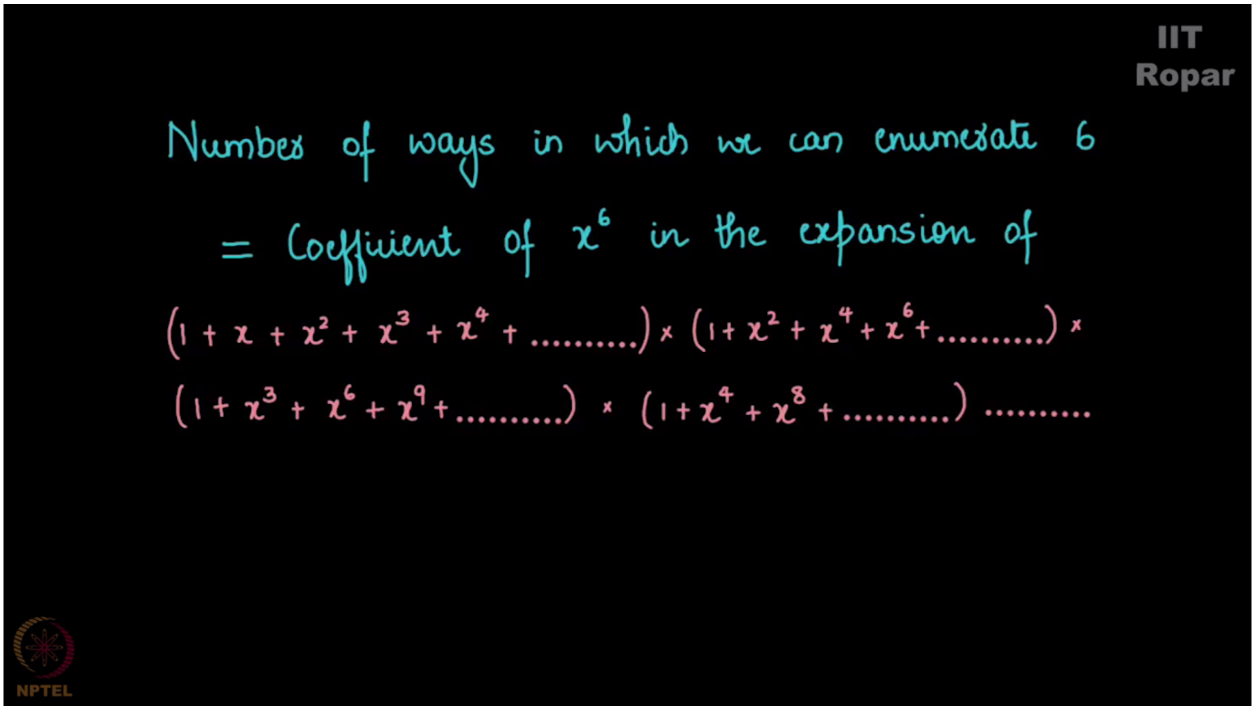
First house X cube means 1 plus 1 plus 1 is present. Second house 1 means X to the 0 which means no 2s are present. X cube from the third house means 3 is present. Okay. Now what if I pick X from the X X square from the second house and X to the 4 from the fourth house and 1 from the first house and 1 from the second house? So I repeat 1 from the first house X square from the second house. 1 from the third house and X to the 4 in the fourth house what does this correspond to? There are no 1s you see because you're picking X to the 0 which is 1 for the first house there are 1 number of 2 which is just 2. 2 is present only once because this corresponds to X square from the second house and you're picking one from the third house is X to the 0. Third house which means there is no 3. fourth house you are picking X to the 4 which means there is one 4 so this corresponds to 4 plus 2.

So the point is very simple. Depending upon how you are getting X to the 6 each way in which these entities multiply and give you X to the 6 represents a partitioning of 6 in a unique way and give me any partition of 6 for example if I take 3 plus 2 plus 1 this will correspond to X cube from the third house because there is one 3. X square from the second house and X from the first house. So there's a one-to-one correspondence with all possible ways in which you can get X to the 6 and all possible ways in which you can partition 6. Please note in partition we always write the numbers in descending order. By descending I mean non-ascending order it never increases it decreases or stays equal right.

So if you look at in how many ways can you write three you should not say 2 plus 1 and 1 plus 2. That's not allowed. Okay if you simply put in this rule that a number can be represented as let's say 6 can be represented as 3 plus 1 plus 1 plus 1 or 2 plus 2 plus 2 or 3 plus 3 or 2 plus 1 plus 1 plus 1 plus 1 so on the number of ways in which you can enumerate 6 is precisely the coefficient of x to the 6 in the expansion of these entities here in brackets right.

There's a correspondence between this and that right. Think about it.

Now this is a little difficult concept to digest to begin with. It again requires two or three rounds of thinking and watching the video repeatedly. We have been telling this from the beginning of the course right. It's very important for you to watch a few videos once more or maybe twice only then will the concepts be clear. Now an important frequently asked question.



Why on Earth should we even care if P of n is the coefficient of x to the sum is a n in the expansion of these entities? Why should we even care? We should care because some of the questions in discrete mathematics are such that they can be very easily solved with generating function approach. Here is one such example where I will ask you to make a note of this startling observation and then I'll show you how generating functions will come to your rescue.