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NPTEL ONLINE COURSE

Discrete Mathematics

Let Us Count

Binomial theorem

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Remember (A+B) square is A square + B square + 2AB, this was taught to us in our school days. We will now see a generalized version of this result called the celebrated binomial theorem.

We are now going to discuss what is called the binomial theorem which stands for simply this. It looks very complicated but bear with me there are a lot of interesting consequences of this theorem and in fact the proof of this theorem itself is very interesting. And it has numerous applications.

So we will go very slowly and try to motivate you to understand what entails by binomial theorem. What is A+B the whole square? You are probably wondering what is chap teaching. This is something that we learned in our primary school days. A+B the whole square is A square + 2AB + B square. As you all know some of the most interesting and most important aspects of math starts from a very elementary example. And the motivation for this elementary example is simply a question such as this. You will now observe that how a bunch of questions that how a bunch of questions starting from A+B the whole square will lead to a magnificent concept in this date.

Fine. So how do we get A+B the whole square expansion, we write A+B next to each other and then we pick a A here and then a B here. We get AB. When we pick a B here and then an A here we will again get AB, BA rather, we write it as AB and then you pick a A here and a A here you get A square. Then you pick a B here and B here you get B square. Again very elementary. But what I want you all to observe is what happens when we take A+B the whole cube. You could write it as A+B into A+B into A+B, thrice.

How does the multiplication actually happen is the question that we are posing here and I want you all to observe how exactly we multiply A+B into A+B into A+B. What do we do? You pick a A from the first cell, I call this a cell. And in the second cell, you can pick whatever you want. You pick a A or a B. Let me pick a A here. Third cell, again let me pick a A here which gives me A cube. This is one way of getting a term in the product. Another way is pick something here, let me choose A, let me choose B here and let me choose A here. I get A times B times A which is A square B. correct. And then so on and so forth.

The point to note here is not the answer on the right hand side. The point to note here is how exactly is the multiplication done. The answer is the multiplication is done by picking exactly one element from these individual cells and you have three cells. What are the total possible ways in which you can pick one element per cell and obtain a term on the right hand side. Correct.

So we all know the answer for this if you do continue doing. It is A cube + 3AB square + 3BA square + B cube. You can verify this. Your AB square comes thrice. Your BA square comes thrice. B cube comes once. A cube comes once.

Now let us go forward and try to see what happens if we were to take A+B five times. What would happen? Think about it. If I were to ask this question what is A+B whole to the n it's slightly confusing. You see A+B whole square was very easy. A+B whole to the cube was also very easy. A+B whole to the five you also observed what is happening there but in general what is the answer for n?

Now I want you all to visualize in place of n, a number ten. You are probably wondering why to go from two, three, five and then ten, it is easy on your minds. Okay. Generalization and arguing with n as the term anywhere in mathematics is actually slightly confusing. We should always use replacement for the number n and then try to think of what would happen if the number is replaced by n. So my choice of a number is 10 right now. So what is A+B whole to the ten? Let me observe this carefully as and always you have A+B, A+B ten times. How on earth will I get A to the 10 here. I will only get it in one way. Why? In individual cells, all these cells if you pick A, A, A without picking a B anywhere you will end up having A to the 10. Now you have only one way to do that. And then if I ask you this question in how many ways can you get A to the 9 the answer is no way. How can you get A to the 9? you are forced to pick either A or a B from every cell, first cell, second cell, up to 10^{th} cell. Correct. And how can you get A to the 9 from ten cells you will be forced to pick a A in the last cell or a B from the last cell. So what you can get is

A to the 9 times a B. Correct. But in how many ways can you get this. In how many ways can you get A to the 9 times B? That's an easy question to answer actually. Observe carefully. You can choose a B from any of these ten cells and call it as the cell from where you are picking your B. From the rest of the cells you declare that you are going to pick only As. Now this question boils down to you picking a B from ten cells. You can pick that in ten ways. Correct.

So this ten will be the coefficient of A to the 9 times B. Think for a minute. So A to the 9 times B you get ten ways in the expansion. Correct. And then my next question in how many ways can you get A to the 9 which means from 8^{th} cell you pick A and from the rest of the cells you pick Bs which is two Bs. In how many ways can you do this?

So basically you pick from two cells and declare that only from these two cells you will be picking your B. and from the rest of the cells you are going to be picking As only. In how many ways can you do that? You can do that in 10C2 ways. You all now know what 10C2 means. I need not explain why it's 10C2. Why is it? That's because given ten cells you touch some random two cells, some two cells in how many ways can you do that, 10C2 ways. Every time you touch this you declare my Bs are going to be from these two cells only. Once declared, rest of the cells you are only going to pick As.

So you are going to pick 8 As and 2 Bs and there are 10C2 ways in which you can do this. Similarly, 10C3 ways in which you can pick 3 Bs and rest of the cells you will pick As. Let me complete this now.

Finally, we have 10C10 and no As are picked which means out of 10 cells all 10 cells are Bs which means this will be B to the 10. Now you see the point I made long back. It is not a good idea to write to think anything in terms of generalization, in terms of n. Now you did it for 10, you now know how to do it for n. So what is A+B whole to the n? It's going to be A picked from all the n cells plus A picked from all cell except one cell which is declared to be a B cell from where you pick. This can be done in n ways to be precise, nC1 ways, you see it was 10 in the previous case. Actually there was 10C1 plus in how many ways can you pick two Bs into B square nC2, the remaining is A to the n minus 2. The next term will be nC3 A to the n minus 3 and I am obviously not explaining. I am hoping that you all are following what I am saying so on and so forth upto nCn A to the 0 times B to the n. What we have obtained just now is called the celebrated binomial theorem with enormous applications.

Let us now see a few applications of this result.

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