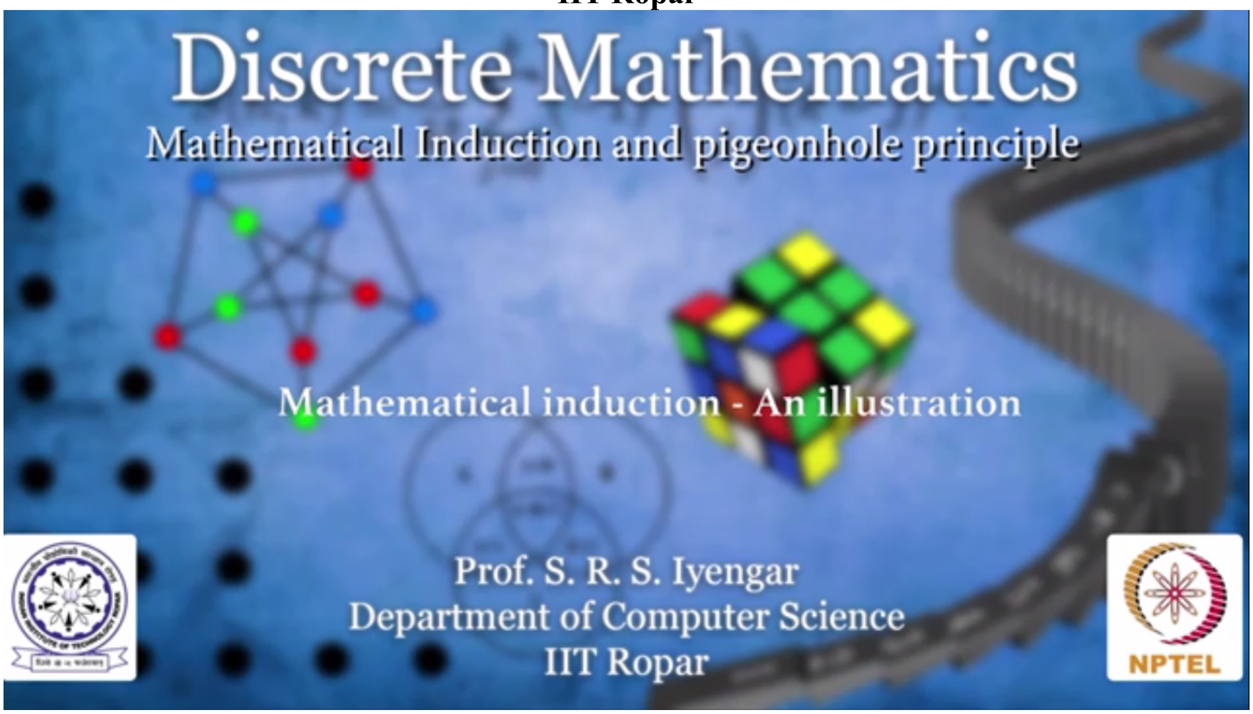
NPTEL
NPTEL ONLINE COURSE

Discrete Mathematics
Mathematical Induction and pigeonhole principle

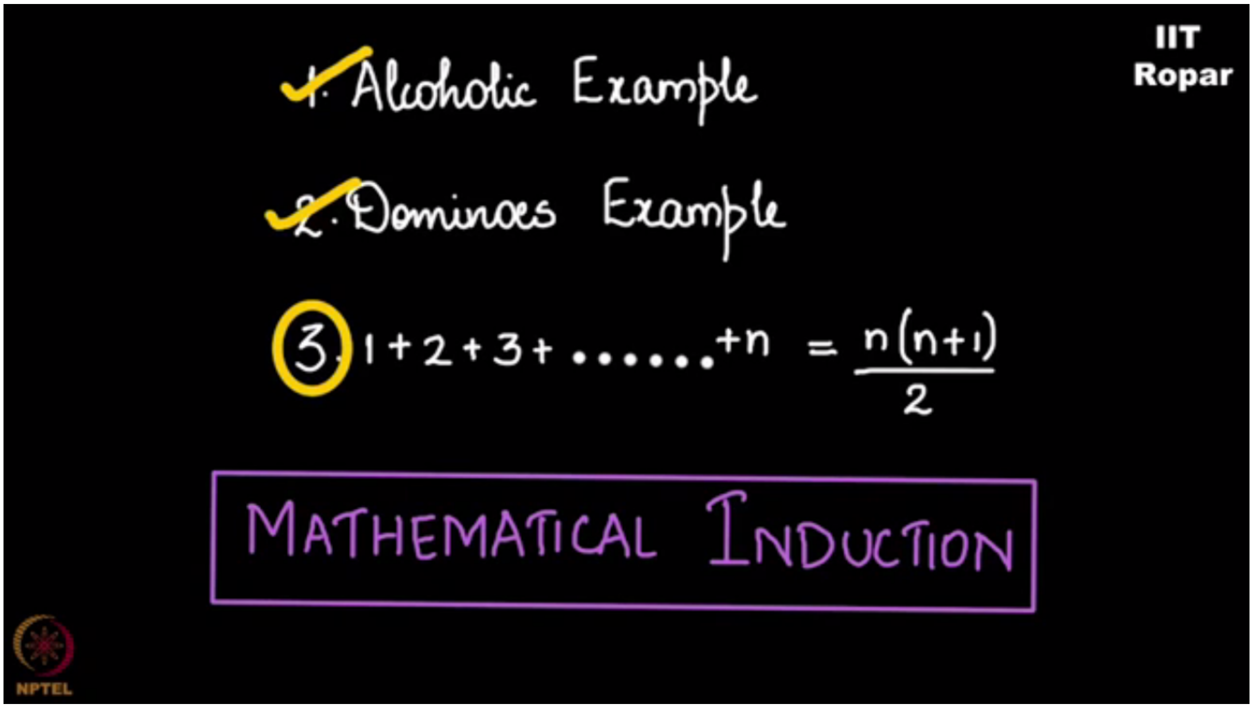
Mathematical induction - An illustration

Prof. S. R. S. Iyengar
Department of Computer Science
IIT Ropar



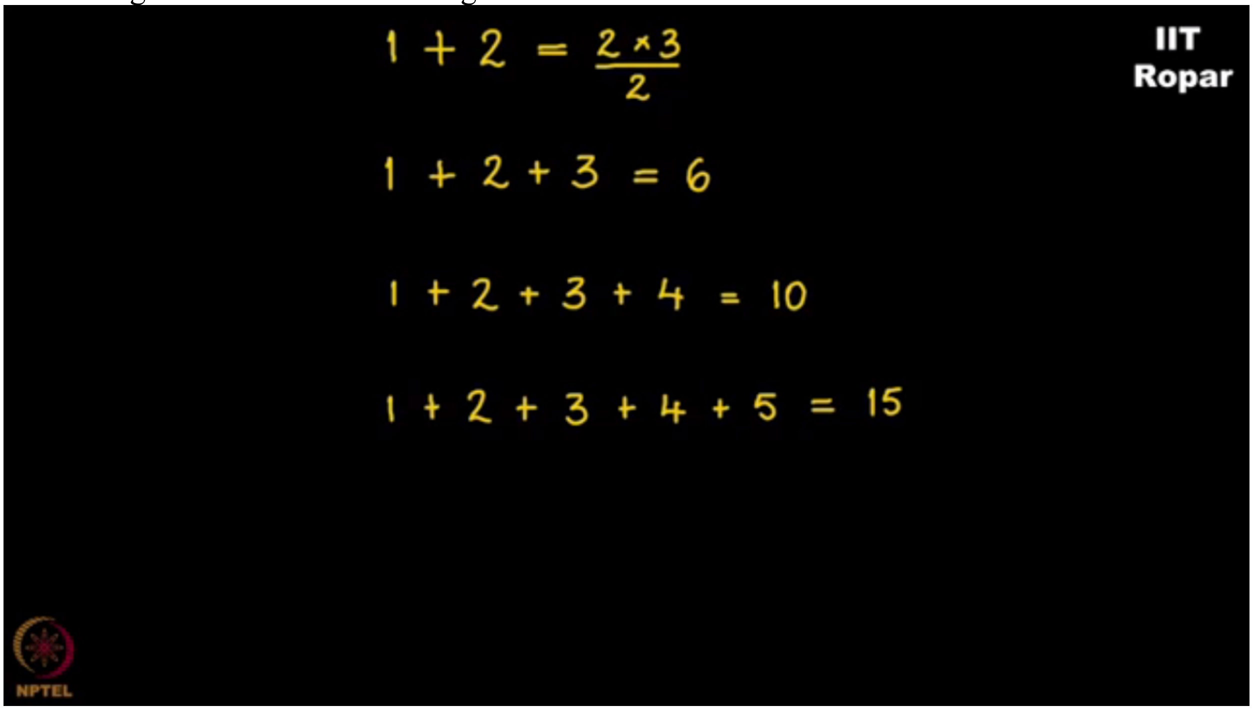
So we have been talking about three things: the alcoholic example, the dominoes falling on each other, and then 1 + 2 + 3 up to some n; this is equal to n(n+1)/2 and the connection between the three.

Kindly note, what we are going to discuss right now is a landmark result in the history of mankind, not just sciences, not just mathematics. The very idea of what is called the mathematical induction comes in very handy in proving several aspects of -- several mathematical theorems that one cannot imagine solving without this tool.



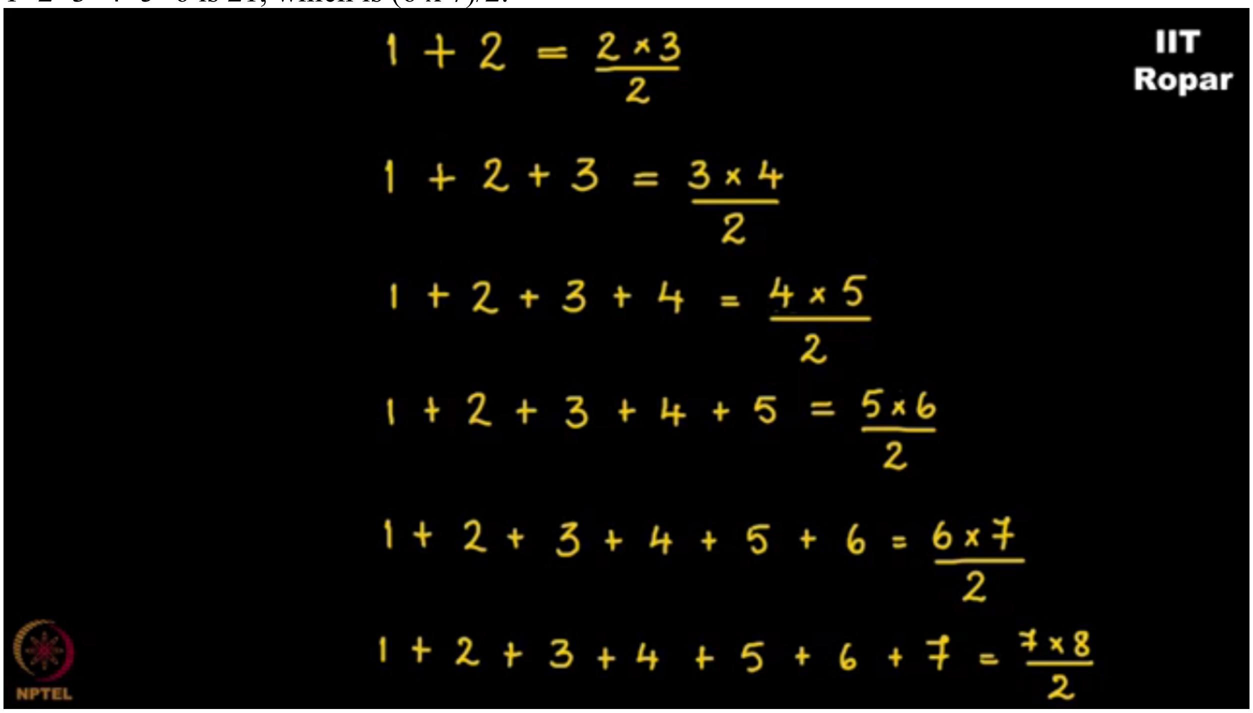
Not just that. This is also a computer engineer's toolkit. Most of the programming questions can be solved if one has this ability to think inductively. We will try giving you all an example in the later stages, but as of now let us go ahead and see what is mathematical induction in its own right.

So now we saw the two examples: example 1, example 2. Now let us look at example 3 in detail. Let me write the numbers 1 and 2 and then add them. I get a 3. And then I add 1+2+3, I get a 6. 1+2+3+4 gives me 10. 1+2+3+4+5 gives me 15.



Now observe 3 can be written as (2 x 3)/2. This is a place where you must be wondering why on Earth should anyone write 3 as (2 x 3)/2? You can instead write 3 as some 1,000-997 too, right? So please bear with me. There is a small trick that I am going to show you right now.

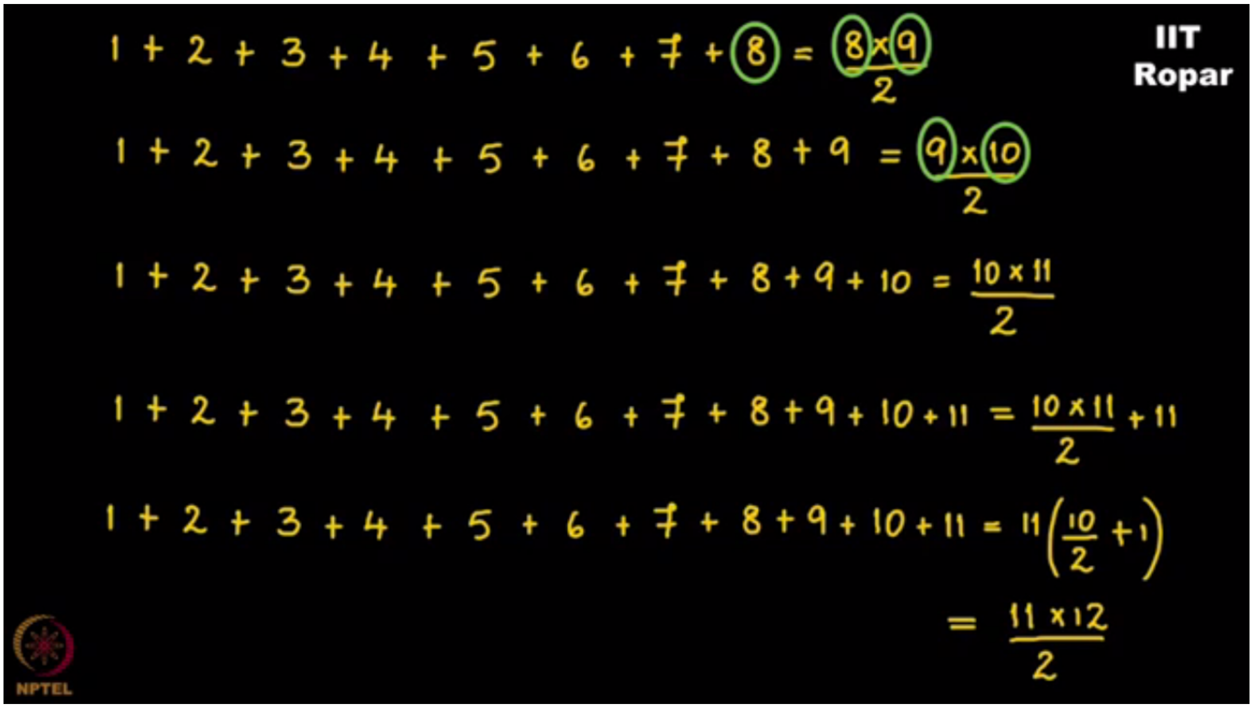
So let me write 3 as (2 x 3)/2; 6 as (3 x4)/2; 10 as (4 x 5)/2. We will get back and we will see why we are writing it like this. And next 1+2+3+4+5 is 15 and that is (5 x 6)/2. 1+2+3+4+5+6 is 21 and that is (6 x 7)/2. Let me go on. Bear with my repetitive narration of these sequence. 1+2+3+4+5+6 is 21, which is (6 x 7)/2.



And then 1 2 3 up to 7 is 28, which is (7 x 8)/2. You see that, right, seven 4s are 28 and then 1+2+3+4 up to 8 is 36, (8 x 9)/2. Do you see a pattern here? The number you end with, 8, you should write 8, add 1 to it, write the next number and then divide this product by 2. Right?

The next one is 1+2+3 up to 9 is 45 and this 45 can be written as (9 x 10)/2. Again, observe that 9 as it is, plus 1 gives you 10, by 2. Okay. This there seems to be some pattern here you see.

And then finally, 1+2+3 up to 10 is 55, which is what? Let me just try by the sequence. 10 times, 1 more 11, by 2. This is indeed true. You can check this. Right? Okay. Very good.

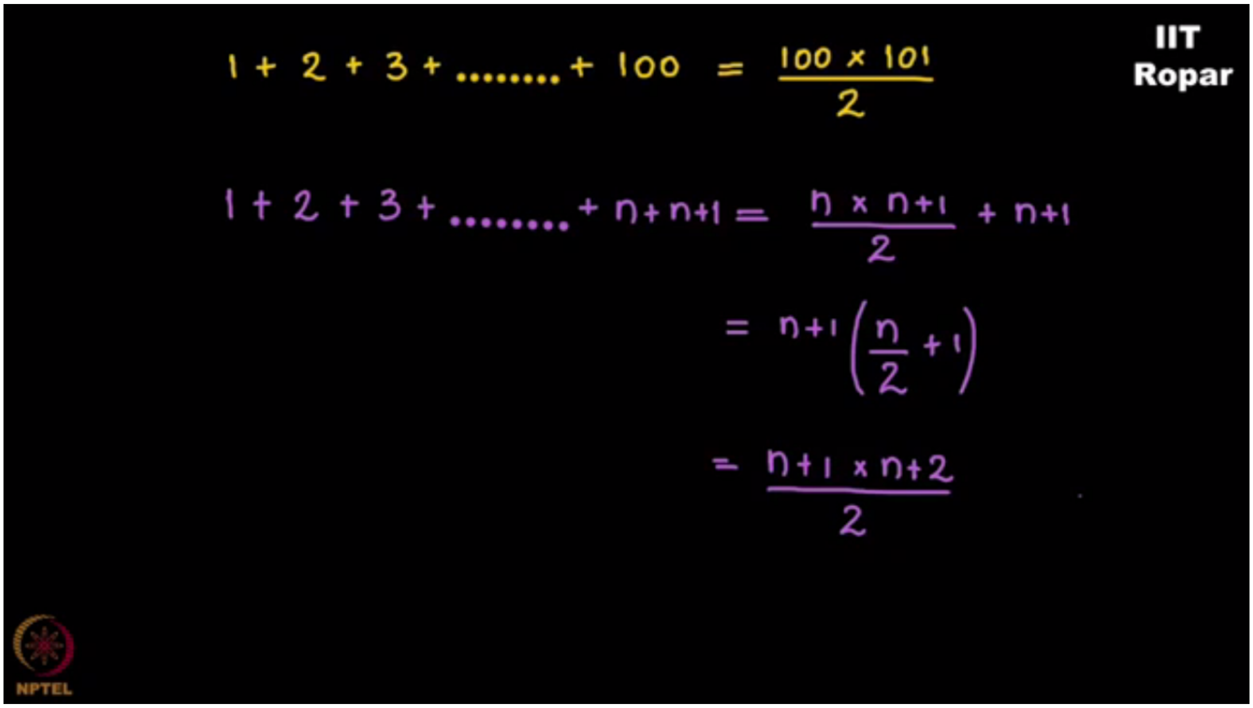


Now 1 to 10 is 55, which is (10 x 11)/2.

Let me now try to write the answer for 1 2 3 up to 11. What is the answer? Now you will say it is 11 as it is, plus one 12, by 2, which is also true. You can check that, but then I'm going to take a pause and then see what is happening here exactly. You see, 1+2 up to 10 if I write down, this is equal to (10 x 11)/2. I am just adding 11 on the left side and that 11 should get added to the right side as well.

Why? That is mathematics. When LHS is equal to RHS, if you add something to the left-hand side, you must compensate it by adding it to the right-hand side, so +11 on the left is equal to +11 on the right. Right? Perfect.

So now you will observe that when you add +11 here, this becomes 11, can be taken as common you see into (10/2)+1 and this becomes 11 into (10/2)+1 can be written as 12/2. You see this became (11 x 12)/2. Correct. That was the answer. Now your question would be we are observing the pattern. Why on Earth are we doing this? There is no need for us to do this. This seems to be true. This seems to be true is not a valid mathematical proof. This seems to be true needs to be observed carefully microscopically and then you must understand why exactly this is happening.

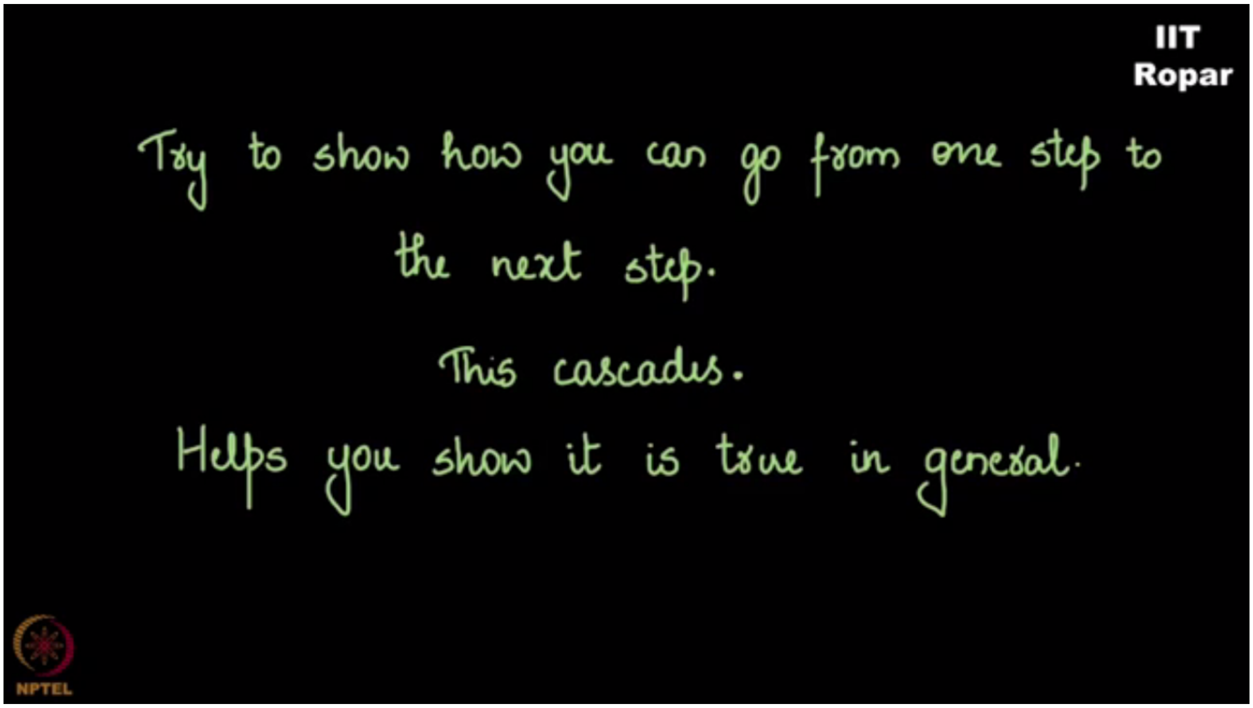


So let me try unraveling it by giving you, in general, an example. 1+2+3 up to 100 will be (100 x 101)/2. How do you know this up to this level it is true?

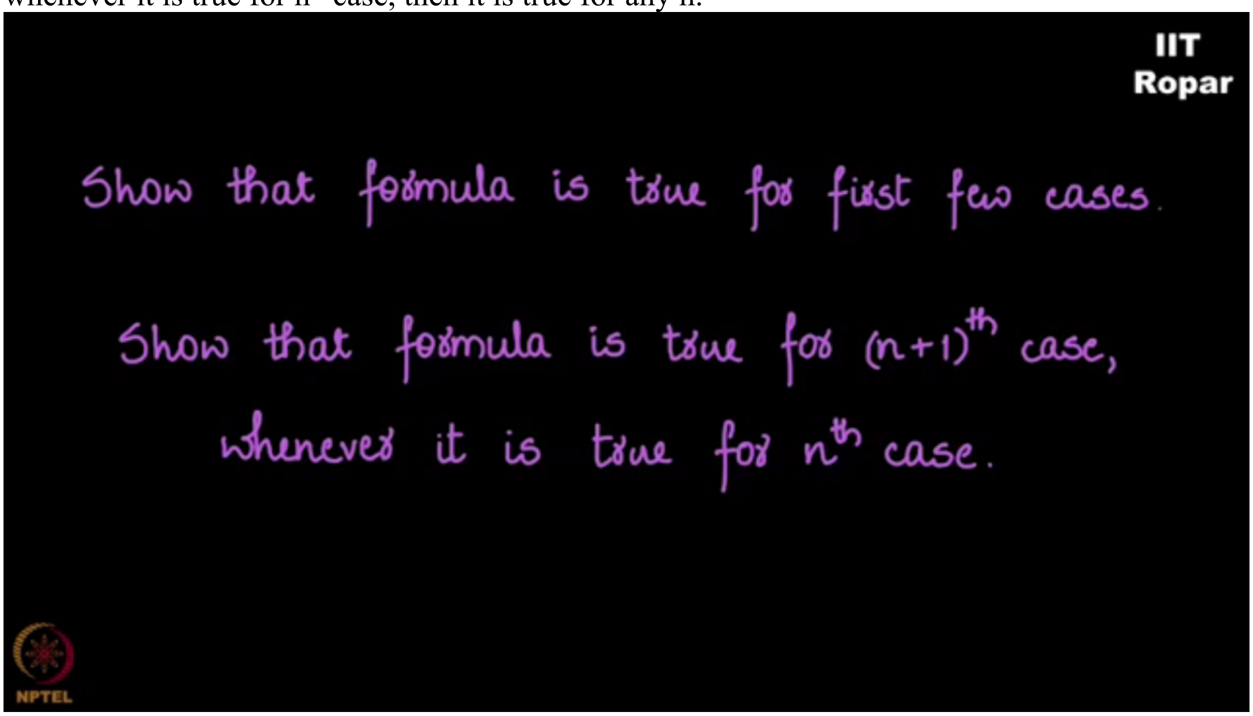
Now for up to 10 it was (10 x 11)/2. If you add 11 on both sides, it becomes (11 x 12)/2. In general, when you add up to n, just in case it is (n x n+1)/2, then when you add n + 1 on both sides, what am I doing right now? The case when n equals 10, we added 11, right?

I am trying to see if that case 10, the 10 case can be generalized. Whenever 1+2+3 up to n adds up to (n x n+1)/2 and you add the next number on both sides, what will you see? 1+2 up to n+n+1 will be n+1 common, you see (n/2) plus 1, which is equal to n+1. (n/2) plus 1 becomes (n+2)/2. Ta-da, you have n+2, (n+1) times (n+2) by 2. This is true for any n, not just 10 that we saw, which means from 10 to 11 you can go. What do you mean by you can go from 10 to 11? By adding 11 on both sides, you will get (11 x 12)/2. Right? And then by adding 12 on both sides, you will get (12 x 13)/2. By adding 13 on both sides, you will get (13 x 14)/2 so on and so forth and finally you will get up to 100 is (100 x 101)/2.

Take a pause and watch this video once again. You may want to watch it several times to understand what just happened. So let's recollect. Any formula that is true in general for n can be proved using this fascinating method, which goes like this. Try to show how you can go from one step to the next step. If something is true for 10, you show that it's true for 11. If something is true for 11, you show that it's true for 12. You don't just directly go and say that something is true for 12. You first assume that it is true for 11 and then you say it's true for 12. You say that it is true for 12 and then you show how it's true for 13 and now this cascades. This argument helps you to cascade and then show that it is true in general.



If it is still unclear to you, it will get clear with time, but note the following. We show that the former is true for the first few cases. We show that the formula is true for $(n + 1)^{th}$ case, whenever it is true for n^{th} case, then it is true for any n.



Things in life are very simple, but when you write it in an abstract way, it gets very complicated. Here is a spectacular example of how simply pushing a few -- pushing the first domino and ensuring that the i^{th} domino falls on $(i+1)^{th}$ domino is so easy for a three-year-old kid, but the very same concept translated to mathematics called Mathematical Induction looks so sophisticated and goes over the head, but bear with us. Trust me on this. It is as easy as pushing a domino and ensuring that one falls on the other and hence everything falls.

IIT Madras Production

Founded by
Department of Higher Education
Ministry of Human Resource Development
Government of India

www.nptel.iitm.ac.in