Introduction to Wireless and Cellular Communications

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Lecture - 15
Multipath Fading Environment
Rayleigh Fading and Statistical Characterization

We begin lecture number 15 and todays lecture. We will cover our introduction our entry into small scale effects in a wireless multipath channel.

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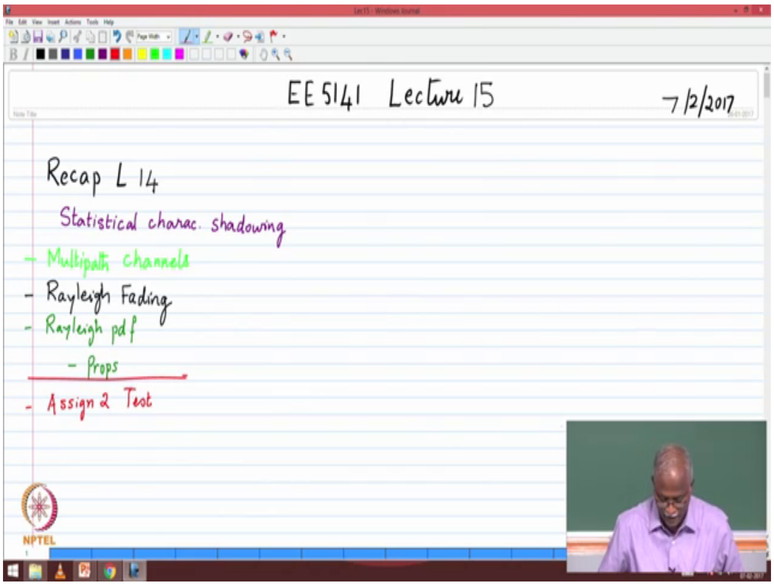

And this gives us a very interesting insight also very rich mathematical framework as we will go through the concepts in detail.

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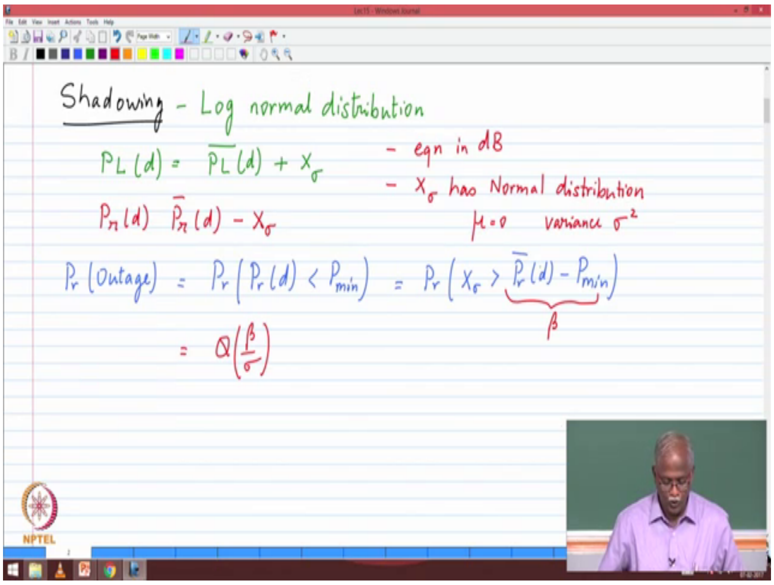

Let me begin with the quick overview of what we have mentioned in the last class, and using that as a starting point. In the last class we have talked about the large scale effects and in specifically, about shadowing we refer to it as a log normal shadowing and as I mention that is because in the logarithmic, when you write the equation in the log domain the variable that we are a we are looking at has got a Gaussian distribution and so in it is normal form it will it will we have a log normal distribution.

So, the path loss we model as a average value plus a statistical quantity which is Gaussian or a normal with 0 mean and variance sigma squared, that tells us that the receive signal will actually be the average value minus the random variable. And we related this to outage by saying outage occurs when my received signal is less than a pre determined threshold which you call as p min which you then relate to x of sigma and a we relate to the Q function.

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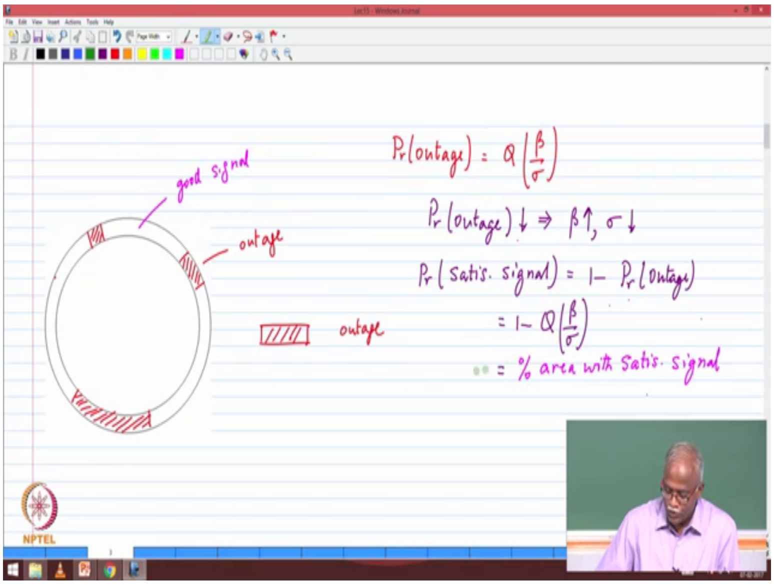

So, the probability of outage can be written as a Q function and a it is a it is 1, where if we think of it at a certain distance think of it as an annular region the Q of beta by gamma tells us what percentage will be in outage. So, basically the red shaded region give us the outage probability. If you want to reduce the outage you look at the parameter that is defining the amount of the percentage of outage it is Q of beta by gamma, if you want to reduce that we have to increase the argument of the Q function. Which basically means that I must increase beta is the margin or I must reduce the standard deviation of the shadowing again sigma is something that we may not be able to control.

So, what is in our control is the margin beta and of course, the complement of the outage region is the region that is having satisfactory signal no decided not say a good signal because all you are asking for is the signal be above the threshold and therefore, that is an acceptable signal.

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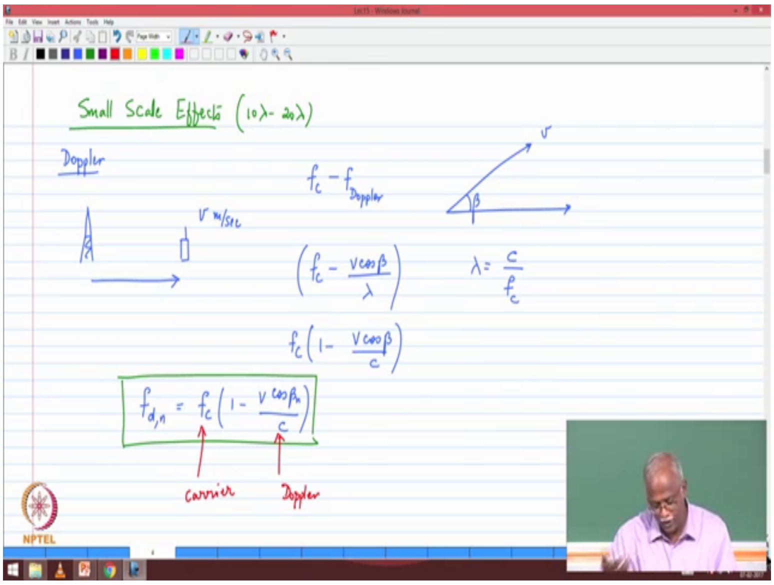

So, that was our understanding of the log normal shadowing. We now move into a after that we went into the effects of small scale. Small scale as suppose the large scale we are talking about changes in the signal which happens when you are moving of the order of 10 to 20 lambda. And one of the things that immediately come into play is that there is a prob Doppler shift of the received signal of the centre frequency of the received signal depending upon the direction of motion.

So, for the n th multipath component assuming the beta n is the angle subtended between that the direction of propagation and the direction of motion the carrier shift can be positive or negative depending upon the angle subtended. We denote it as a carrier the original carrier minus a shift which is caused by Doppler. So, the second component is what is the Doppler component, and again we normally in in the in digital communication we do not worried too much about the Doppler, but we will see why this is going to be an important component for this is the Doppler component the original this is the career frequency. So, in other words the modulated signal which is centered around a carrier frequency is tends to be shifted. And that shift depends upon the velocity with which we are moving and also the angle that is subtended between the 2.

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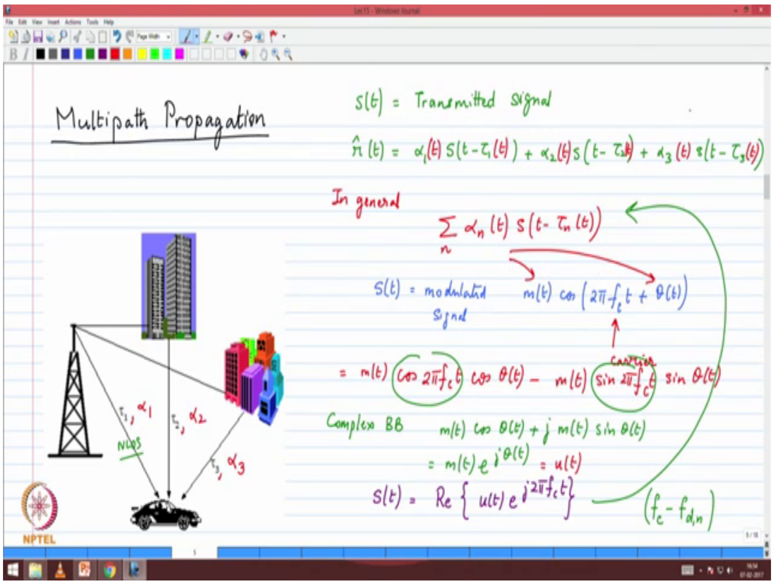

So, given these 2 parameters we now recognize that each of the received signals each of the received multiple components is not exactly at fc, but it is slightly offset to the right or it in a higher or lower depending upon the angle. We then said that we would characterize multipath propagation again I have reused the slide from the last class. So, that you will be familiar with it we took the case of a 3, 3 a component system where it was alpha 1 alpha 2 alpha 3 all of them are gains that can vary as a function of time. The delays also can vary depending upon where your position is, again this is a model that we are deriving under the assumption of non line of sight. Basically none of them are direct line of sight. So, this first path you have to take as a also as a reflected or a diffracted path.

So, in general it would be a summation over n where n goes over the number of multipath components. Each of them having a gain alpha n, and a delay tau n and again each of these can vary as a function of time and s of t is the modulated signal. The point at which we start the discussion was we write down a general expression for the modulated signal in terms of the information part. And the career part wrote it down into the complex baseband and said this is the complex baseband representation. The received signals are going to be the transmitted signals with certain gains certain delays and the Doppler shift all of which we would have to incorporate. So, that is where we stop we pick it up from there and move forward.

So, a any questions on the multipath model, basically we have moved over from the large scale effects to the small scale effects. These are effects that will become a visible or we will start having an impact when we start moving in small perturbations around a around the current position.

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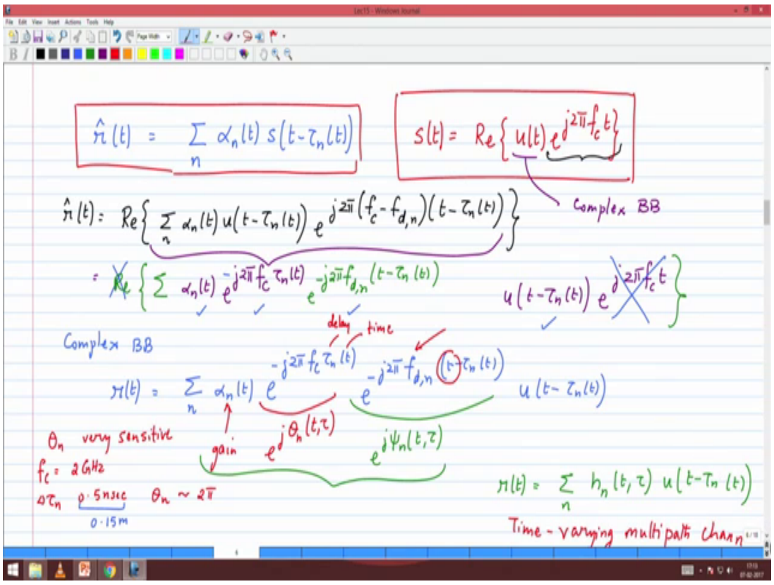

So, a basically the received signal, as we as we mentioned before will be represented in this fashion. This is the bandpass signal. The s of t can have a representation in terms of the baseband signal in terms of u of t. And now what we would like to do is write to the received signal in terms of the modulated signal and then get a base band representation of the signal.

So, basically I begin with writing down r hat of t, r hat of t I basically you have to substitute the values of s of t into this expression. This is the modulated signals. So, I would write it has real part of the summation over n alpha n of t u of t minus tau n of t right. And e power j 2 pi no longer only fc it will be fc minus f Doppler of the n th component, t minus tau n of t this is a representation of my received signal modulated signal.

So, what I need to do is, if I can isolate the career component e power j 2 pi fct, then what is left becomes my complex baseband signal. So, my goal is to take out the e power j 2 pi fct keep the remaining terms and therefore, I can get the representation that we are interested in. So, let us write down the write down the expression. So, basically I would to look at the terms inside the bracket. So, this the terms inside the bracket is what I am going to simplify. So, the terms inside the bracket are alpha n of t. I am going to take u of t minus tau n of t all the way to the right side, followed by e power j 2 pi fct that is one component, that is what I am interested in everything else has to be included inside the as the coefficient of u of t minus tau and t. So, this would be e power j 2 pi fc, into tau n of t let me use a different color. The next term will be e power minus j 2 pi fd comma n t minus tau n of t, and a basically I would have to take the summation of these terms. And the real part of these terms that would be my total expression that would be that would be involved.

So, I am going to go to the complex baseband notation. So, what remains in the complex baseband notation are the following terms. This will remain this will remain this will remain this will remain this goes out. So, I had drop the real part. So, the complex baseband representation, complex baseband, representation says r of t it is a complex valued signal ii Q signal this would be given by summation n alpha n of t e power is there minus sign here. There is a minus sign e power minus j 2 pi f c tau n of t, e power minus j 2 pi fdn t minus tau n of t times u of t minus tau n of t, if we seen a little bit you know complex, but again it is actually nothing, but a magnitude term this is like you can think of this as a gain. And this is one phase term and it is a function of t and basically there are 2 things that are coming in to play.

One is the tau that is the delay that is what we saw as the delayed coefficient and then t is the function of time. So, there are 2 variables here which need to be handled carefully one is delay the other one is time. And I am going to emphasize that a lot of very significantly in today that is going to be one of the key differences in terms of our understanding of the multipath channel that we are working with. So, this gain term or this phase term I am going to write it as e power j theta n corresponding to the n th multi pass com multipath component, it is a function of 2 variables it is both t and tau and the second one is also another phase term.

So, basically there is a gain term and 2 phase terms. This will be represented by e power j psi n of t comma tau. So, and of course, I can combine all of these and write it in the following form, r of t is summation over n h of n t comma tau u of t minus tau n of t. So, basically h of n corresponds to each of these terms that are there inside the summation. So, basically what we have done is, they can be representation of a baseband signal use the multipath model. So, it would have been straight forward except for the fact that there was a shift in the carrier frequency, due to the motion of the of the of the receiver and therefore, cause a different Doppler shift for each of the multipath components depending upon which direction the multipath component was coming from.

So, that basically gave us the substituted equation basically gave us a representation in terms of a gain term, which was there in the original equation plus there are 2 phase terms. One important point to note is that these phase terms are a function of 2, 2 variables one is time the other one is delay, now is in delay also some form of time the answer is typically yes, but in our context we need to differentiate between the 2 and I will and explain that a little bit more.

So, each of these phase terms are function of t and tau and so therefore, the net result is this this expression. So, we refer to this as a time varying response of the channel. So, basically it is a time varying response. So, this is the received signal through a time varying multipath channel, time varying multipath channel, and that is what we have represented here, in general form any questions about the notation or what we have used. Here now I want you to pay very close attention to the following aspects.

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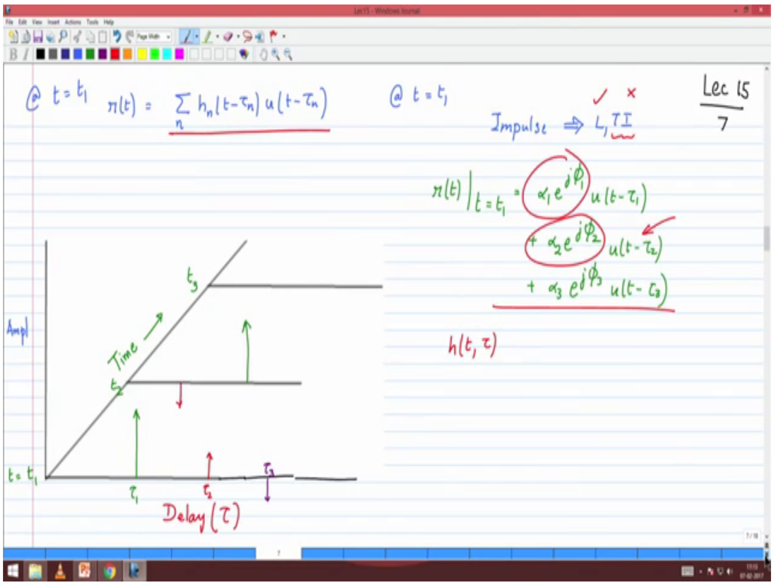

So, a basically I want to write down the expression for r of t, r of t we said was summation over the n the multipath components, h of n t I am going to represent the output of the multipath channel. So, let me take let me come back to this equation let me say that at time t equal to t 1 time t equal to t 1.

There are 3 distinct multipath components, and the delays of the multipath components are tau 1 tau 2 and tau 3, and they have certain amplitudes, that is the net resultant am amplitude. So, a basically I would capture that in, a in this in this equation in the following manner h of t minus tau n, u of t minus tau n, may be I have, but I am writing it at time t equal to t 1. Now there is a very settle temptation for us to call it the impulse response. Now impulse response if you will recall is defined for what type of systems.

Linear time invariant systems, so impulse response implicitly assumes that it is linear and time invariant. In our case linearity is guaranteed, but time invariant is not guaranteed because it is a time varying channel. So, basically there is no notion of time invariance, but this does look like a convolution. So, what is actually going on? So, what is what we the way we understand it is we have to differentiate between the time axis and the delay axis. So, if I were to represent the delay axis along the x direction amplitude along the y direction, and in the in the z direction basically going into the plane of the paper is the time axis. So, these are the being the axis that we have represented.

So, at time t equal to t 1, I can tell you what the channel behaves like it says there is a there are 3 multipath components. One at tau 1, tau 2, tau 3 and the games are alpha 1 alpha 2 alpha 3. Now at time t equal to t 2 it may be something totally different. It may be just 2 multipath components with the following a impulse response. And a tau time t equal to t 3 it could be something totally different again. This is because of the time varying nature of the channel. So, now how do we interpret this particular equation? So, what it says this the received signal r of t at time t equal to t 1, at time t equal to t, I can I am I am defining this equation only at time t equal to t 1 at time t equal to t 1 it has got 3 multipath components with the following delays.

So, basically we can write this as alpha 1 e power j phi 1, u of t minus tau 1 that will be one of the components that we would get second term would be alpha 2, e power j phi 2 u of t minus tau 2 plus alpha 3 e power j phi 3 u of t minus tau 3. So, it is alpha 1 e power j phi 1 times the signal u of t that was transmitted t tau 1 seconds before then again a second one, but this one is multiplying the signal that was transmitted tau 2 tau 2 before and tau 3. Now you may say this looks awfully like convolution, but please be careful because the minute t changes this equation changes. So, this is not a. So, all we can say is at time t equal to t 1, the output looks like a convolution in the tau domain in the tau direction it looks like a convolution.

But do not do not generalize it because across time it does not hold. So, therefore, this is something that we always visualize the multipath channel in the following way is in terms of a 3D plot. There is the delay dimension, there is the amplitude dimension typically for most channels time in variant channels, that is what you would have you would have just the time and delay or both on the same axis, but for us the time axis has to be different, because the impulse responses are different at the different instances of the time. I know that this is a probably a maybe a new way of looking at it, but this is what I would like you to start thinking about anytime you think about a multipath channel time varying multipath channel, I want you to think about it in this direction in this in this way.

Any questions is the notion that there are 2 different axis both of which are time in some sense and both of which have units of seconds, but you have to differentiate the 2 because that is what is the nature of the channel that we are dealing with yeah.

Student: Time axis the transmission time or something.

You can think of time axis as the actual physical time, basically like the clock a time you wrote t 1 t 1 plus 1, t 1 plus 2 like that. Basically it is the clock that is moving forward at each clock instant you can take a snapshot of the channel and you get some kind of an impulse response. So, that is what we are that is what we are working with. And typically yeah you may not want it is actually a continuously varying channel right, but we may interested only in discrete time instances when we observe the channel. So, again we have discretized a the channel. So, time the time access is continues the delay access can also be continuous.

So, in principle what you would have to define a multipath time varying channel would be t comma tau where both are continuous variables, but what we have done is we have discretized just for simplification of an to help us in our understanding. Now I want you to a go back in your notes, and look at the previous expressions, a very interesting thing you make an observation fc is carrier frequency. It is a one giga hertz or something is a large very large number, fd if you did a little calculation if you did the you know 60 kilometres for hour you will find that it comes out to be some 100 hertz or 100 and 50 hertz.

So, these 2 are 2 very different you know in terms of magnitude they are very different. So, one thing that we observe is this theta, n s is a very sensitive parameter, even if the multipath changes slightly. It will make a big difference because if you are talking about a carrier frequency of 2 gigahertzes, the time period of a 2 gigahertz is only 0.5 nano seconds. So, if I have a change if delta tau n changes by 0.5 nano seconds, theta n will change by 2 pi. That is the carrier frequency right basically one period, so and 0.5 nano seconds at the speed of light basically corresponds to 0.15 meters. So, 15, 15 centimeters is what you need to move at 2 gigahertz to cause a completely different looking channel, which is very similar to what we already observed that if I move slightly there is a perturbation of the channel and therefore, the channel conditions are different. So, this theta n is something that is very sensitive, even a small change in the delays of the multipath components can make a big change.

On the other hand fd of n is small, but notice that it has got a dependence on t; that means, it will go on changing. It is slow, but it goes on changing. So, that is why the channel in a wireless environment multi wireless multipath environment is not a constant things are constantly changing and small perturbations, suddenly there will be a big jump and so you can see there is a interplay between these 2 terms and again keep in mind the a the expressions and that will give us a lot of insight into the a into our understanding of the system.

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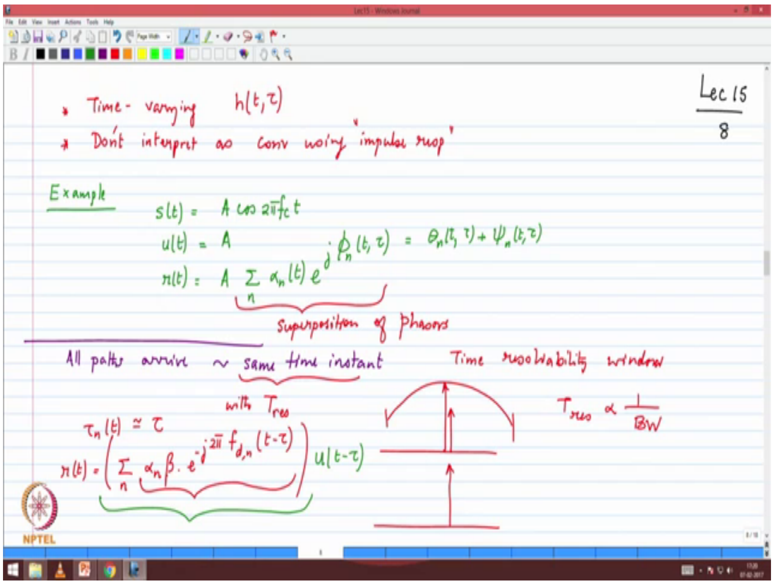

So, let us let us move on and we will come back to explaining more aspects regarding this. So, the first point is that we have a time varying channel time varying channel, which is denoted by h of t comma tau. Second thing is do not interpret as conf as do not call this impulse response, do not interpret as a convolution using impulse response it is tricky. You have to be careful how you implement it. So, therefore, I would say that be careful in how you interpret it is not it is a time varying impulse time varying channel response, but we do not call it impulse response.

So, the let us look at one I want to gain some insight. Let us look at some aspects of this particular multipath channel. I want to look at what happens if my transmitted signal is a sinusoid, a cosine 2 pi f ct. Just a simple sinusoid with an amplitude a what is the baseband signal for this he able to go back and forth it is a, a e power j 2 pi fct and if I were to basically if I to look at the remove the e power j 2 pi fct what will I left you with a. So, in this case what is the base band received signal. This will be a is the constant summation over n alpha n of t, e power j phi n t, comma tau where this is equal to theta n t comma tau t comma tau, plus psi n t comma tau I basically have combined the 2 terms into a single thing.

So, this is what we gives us a lot of insight because what is what is happening here, it is a superposition of phasors. What we get here is a superposition of phasors. So, now, there are several interesting insights that you would have already come across, when I take this kind of superposition of random phasors, which have magnitude and phase angles and I add them up, then I get some interesting behaviors in in the system and that is what we want to we wanted well upon.

So, this is a case where I took only a complex sinusoids. Now I want to extend it to our understanding of the multipath channel, and it is only a one I say single step that is away from here. So, if I were to assume that all the paths multipath components are arriving arrive at approximately the same time instant, same time instant now first question would be is well is it really a realistic assumption, because these are going through different paths is in there going to be some difference between the paths, that when they arrive. The answer is yes there will be of course, even if there is a slight difference in the distance travels there will be a difference in the time, but in our processing of the signals there is something known as time resolvability, window time resolvability I am sure you have come across this supposing there were 2 vectors that were very close in time, but you did not have the ability your time resolving window was much wider.

Basically you cannot differentiate between these 2, the resultant signal comes out to be it looks like something which is just one multipath component, because you are not going to resolve these 2. So, that is what happens for us also because the time resolvability window or the resolution the time resolution is inversely proportional to the bandwidth. The larger the bandwidth of the signal the more time resolution I will have, but never the less there is always a finite time resolution of the bandwidth. So, all we are saying is when we say that they are coming in the time instant, it is within the time resolvability window which is the reasonable assumption because I cannot I cannot differentiate it in my processing.

So, I am going to make the assumption that all of my tau n of t s are approximately tau single value now please tell me what would be the received signal r of t will be summation, over n alpha n notice that the first term which consists of the of the career component will, now become a constant because it is it is all the tau n s are the are the same career frequencies same, for fc is the same in all those terms. Then the next term will be e power minus j 2 pi fd n t minus tau t, minus tau. And then of course, you will have u of t minus tau, but notice that you can factor out u of t minus tau, you can now see that this is very similar to the superposition of phasors that you saw in the previous one previous example.

So, what is happening is for all those multipath components that are arriving approximately at the same time, it is a superposition of random phasors. That is what is the resultant signal. Now we are making the assumption that all the multipaths are arriving within the same. So, this whole thing multiplied by u of t minus tau. So, this is what we are interested in, this is the going to be the resultant of the signal that I observed. So, for this we moved into a mathematical formulation, but before that I just want to sort of anchor it in some very good intuition.

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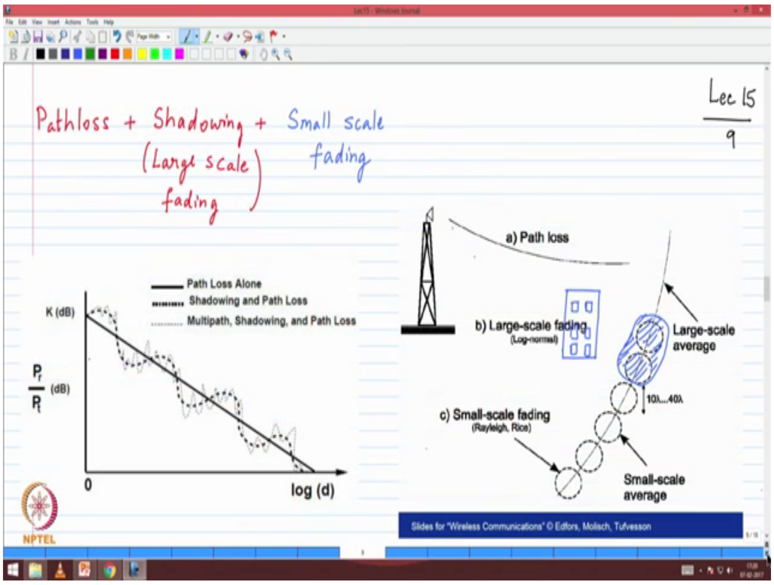

So, there are 3 things that we have talked about. So, far one is path loss as you move away from as you move away from the base station, there is a path loss component that is number one that is large scale.

Another large scale phenomenon that we saw was shadowing. So, basically if there is a building, all of a sudden the signal level will drop and therefore, you have to account for that, and we saw that that could be modeled as a log normal shadowing. So, if it was only path loss it will be some straight line, with the path loss exponent as this slope. On top of that if I have shadowing, in some cases the path loss exponent is the path losses slightly more some cases the path losses slightly less. So, there is some variation about the path loss which is caused by the shadowing. So, straight line is only the path loss the solid dash line are the path loss with the shadowing taken into account.

There is a third element I am not sure if you can see it on the screen, there is a dashed line not a bold dash line, but a normal dashed line superposed on the shadowing effects. That is there is path loss there is shadowing, and if I move within a 10 lambda to 20 lambdas, within this window it is not constant, but there is some fluctuation. And that fluctuation depends on what happens with these multipath components, and how they are adding they are adding constructively or destructively that is the intend of our.

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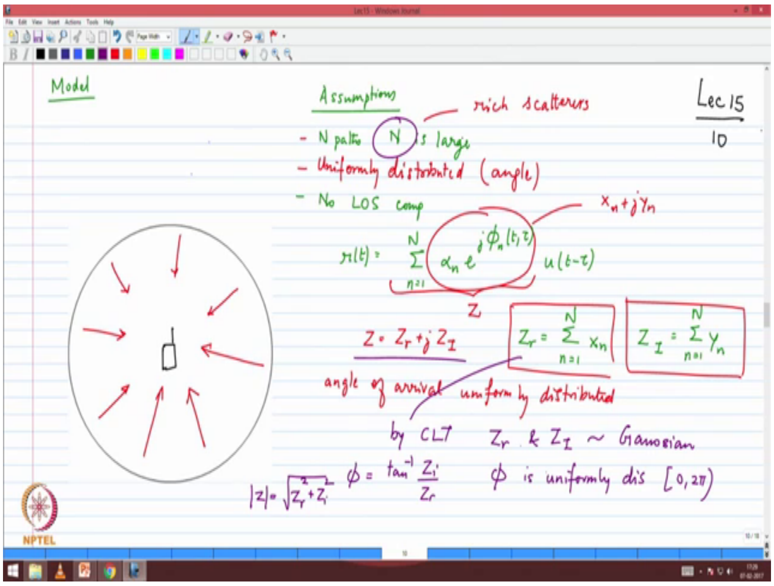

So, now, we move forward to a model that will help us a mathematical model that helps us understanding what is going on this phenomenon. So, basically we will now derive a model, for multipath comp multipath propagation channel where all the multipath components are arriving within the time resolution window.

So, we make certain assumptions, simple assumptions which help us to analyze the so we will say that there are n paths, where n is large what; that means, is there are different objects in the vicinity that either have reflected or have scattered. So, such an environment is called a rich scattering environment; that means, there are large number of scatterers. Scatterers present large number of scatterers either reflection or diffraction. So, basically there are several objects that are caused contributing to the resultant signal. Then we also assume that these objects are uniformly distributed in angle; that means, these objects are not all in one direction they are all around the mobile. So, you may ask you know how can it be if the base station is in this direction, how can there be an object that is in in in that direction.

But basically what we are saying is they these scatterers are uniformly distributed in around the around the device. So, we make the assumption that they are uniformly distributed. All around they all around in terms of the angle in terms of the angle; that means, the angle of the arrival is not in any is not biased in one direction the multipath components can arrive pretty much in any of the direction. And the third one which is very important there is no line of sight component, because if there is a line of sight component that will be much stronger than the others. So, basically there is no line of sight component, now these are the broad assumptions that we have and under this assumptions the received signal r of t complex baseband notation.

If summation n equal to 1 through n alpha n e power j phi n t comma tau u of t minus tau notice. I have dropped the tau n s because all the taus are arriving at the same instant of time. So, if I take u of t outside I get my total gain in terms of the summation of these phasors, let me call that as some complex variable as z. I am do doing some very simple manipulations for to get the form that we are familiar with. So, basically each of these individual components.

I am going to write it in terms of the Cartesian call chord components. So, there is a xn plus j times y n, basically the real part and imaginary part. So, the result z can be written as a real part plus j times an imaginary part, where z is the summation over n is equal to 1 through n of all the xn s the real components the real parts. And the imaginary part will be the summation of n equal to 1 through n x of x y of n. So, what I have done my net channel gain z is a summation of so a phasors I have done a decomposition of each of the phasors into real part and imaginary part. Then over all z can be split up into real part and imaginary part that gives me 2 expressions that that we have here.

Now, what are some of the assumptions that that we have made. We have made that these angles are uniformly distributed. Angle of arrivals are uniformly distributed. Angle of arrival is uniformly distributed uniformly distributed. So, the statistical model that we are what we have derived is at zr and zi are both super positions of a large number of identical parameters because there is no line of sight component. All of these are some path multipath component coming in different directions. So, by central limit theorem, by central limit theorem z r and zi, will tend to Gaussian and because we have already made the assumption that n is large these are Gaussian, Gaussian in terms of their distribution zr and zi are Gaussian.

Now, if I were to think of a angle a basically phi as tan inverse of zi by zr think of it as the resultant zr and zi are the real and imaginary parts, phi is the angle we can think we can by the assumption that we have made phi can be from any direction I do not know which direction z does not. So, phi is uniformly distributed anywhere in 0 to 2 pi uniformly distributed in the range 0 to 2 pi. So, this is the statistical characterization that we have made let me quickly summarize, a whenever all the multipath components are arriving at the same time the equation that we work with is a superposition of phasors. The phasors can be decomposed into real part and imaginary part if I have a large number of these phasor components, then by central limit theorem the real part of the resultant channel and the imaginary part both Gaussian the resultant angle between them is uniformly distributed in terms of the angle of arrival.

Now, why is all of this important because of the following the following reason. So, basically if I ask you what is mod z, mod z will be square root of z r squared plus zi square.

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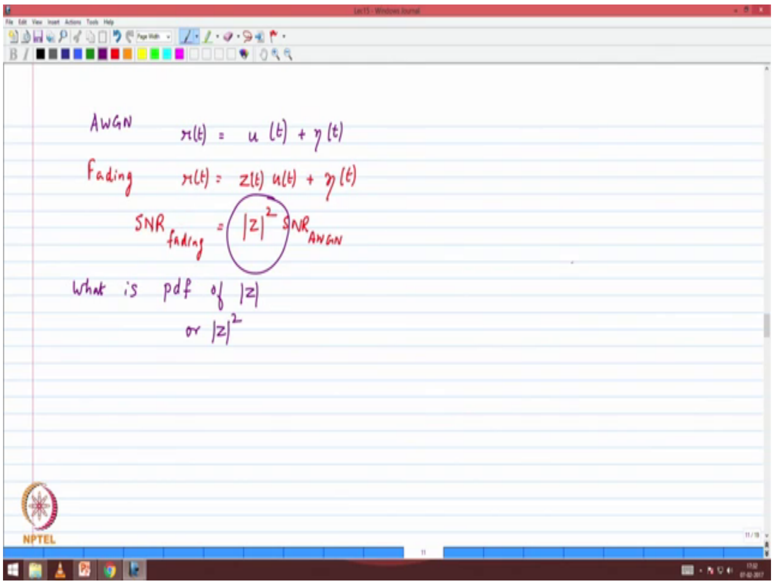

Now let me explain why this is important, because of the following reason if I were dealing with an awgn channel my received signal r of t complex baseband would be u of t plus eta of t. And you have already done some computer experiments where you know how to measure SNR. Now what will it be if the same signal same awgn impairment, but was passing through a fading channel, what would be the received signal.

So, in fading let me use the different color if this was a fading environment multipath fading environment where all of them were arriving at the same time r of t would be equal to z of t times u of t plus eta of t. What is the impact the SNR in fading? SNR in a fading environment let me call it as SNR subscript fading, if I were to ask you it will be mod z squared I am just a omitting the t bracket it will be mod z square times the SNR in awgn that correct. Because I it got scaled by a complex factor alpha which got with the phase term, but what is important for me is the power of the signal. So, it will be actually mod z.

Because I am taking in amplitude it will be mod z power will be mod z squared. So, this is why it is very important because my SNR is no longer a constant, it is now a random variable and what it what type of random variable is I am going to ask we are going to we are going to ask the question what is. So, the important thing that we need to get from this discussion is what is the distribution. What is the distribution of mod z because that is going to tell me how my signal is going to vary it is also going to tell me, how my SNR is going to be what is the pdf what is the pdf of mod z or mod z squared.

If I am talking about SNR, I would be interested in mod z squared that will also be a random variable with the distribution mod z will also be. So, as you can see our basic model has resulted in a particular resultant received signal where the SNR is no longer constant, but is actually going to be a something that is going to be fluctuating any questions on what we have said. So, far because what we what we are trying to work towards is a good understanding of what the multi path environment is going do to my signal, at a first level we have found out that it is going to multiply the received signal by a complex value z, which is a statistical quantity right statistical quantity, but I am interested in taking it further looking at the distribution of mod z.

Because we know that the individual components the real part and the imaginary part are Gaussian, and therefore, what is the magnitude that is the pdf of the magnitude any questions.

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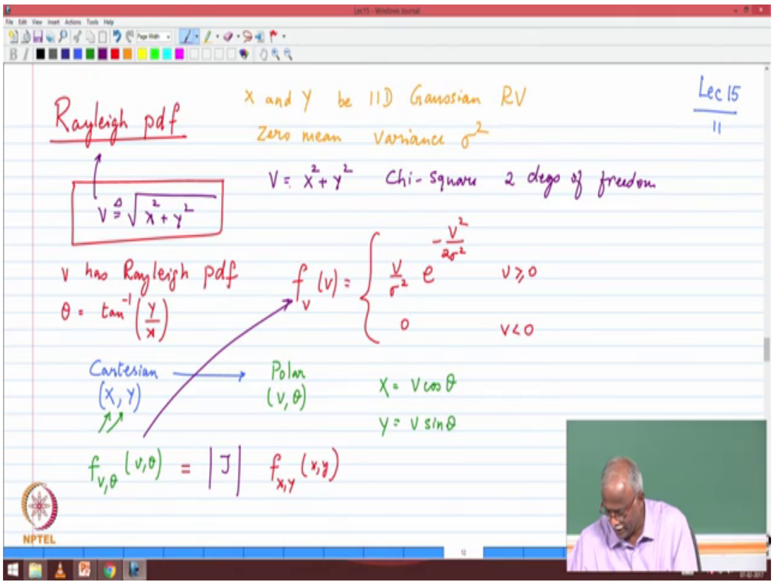

Now, this is a problem that you may have already have a come across in a in your probability and statistics, but let me just repeat it just in case someone is not very familiar a with this particular result. So, we are going to make the following assumptions, x and y, x and y are id independent identically distributed Gaussian random variables, Gaussian random variables 0 mean and both of them have got a variance sigma squared, variance sigma square, x and y are identical independent identical distributed Gaussian random variables, with the 0 mean and variance sigma squared.

Now, if I define a random variable which is v equal to x squared, plus y squared will that be a statistical quantity yes what is distribution of v it is called a.

Student: Chi squared.

Chi squared it is called a chi square distribution with 2 degrees of freedom. 2 degrees of freedom, but my interest now is v is equal to square root of x squared plus y squared. That is what we are interested in because I am interested in knowing what is the amplitude resultant amplitude of the complex signal. Now this is said to have just like v is equal to x squared plus y squared as a chi square distribution, square root of x squared plus y squared is said to is said to have a Rayleigh distribution. So, a basically the definition is x and y are id independent identically distributed Gaussian random variables. 0 mean variance sigma squared if I define the variable v in this following fashion, then v has Rayleigh pdf, Rayleigh pdf which is given by f v of v is equal to v by sigma squared e power minus v squared by 2 sigma squared.

If v is greater than 0, than equal to 0 v less than 0. Because basically we are looking at the magnitude of the resultant vector. So, does not make sense to talk about negative values of v, we are only interested in the in the positive sign. So, basically the Rayleigh pdf is given by this by this expression. Now you may be familiar with it or may not be familiar, but I need you to be comfortable with this result because this is the one of the foundational results. So, let me quickly run through the steps, I just need to indicate you how this pdf expression is obtained, and then request you to work on it if there are any doubts and clarifications we can do that in subsequently.

So, the relationship that we have is v is equal to square root of x squared plus y squared we can think of an angle theta which is tan inverse of y by x. So, you can think of the transformation of variables that is happening is from the Cartesian, Cartesian where we have x comma y, it is going to a polar form polar form, where we have amplitude and angle v comma theta. And we know that x and y have got Gaussian I, id Gaussian random variables. So, basically the relationship would be x is equal to v cos theta, y is equal to v sin theta. I want to know what is the joint pdf of v of theta, I want to obtain I want to compute this right. Because once I, get the joint pdf from there I can get the from there we can get the marginal pdf, the pdf that we are interested in is a marginal pdf of the joint distribution of v and theta basically x and y is a Cartesian form v and theta is the polar form or the polar form and I just want to get this representation.

So, when I do this sort of a transformation, again I am assuming that some of these results are familiar to you, what we do is take the joint distribution of x and y make the appropriate substitutions and very important we have to multiply it by what is called the Jacobean matrix right. I am assuming that you are familiar with this, but let me just give you the expressions. So, that even if you are not very familiar we can you can comfortable with the result.

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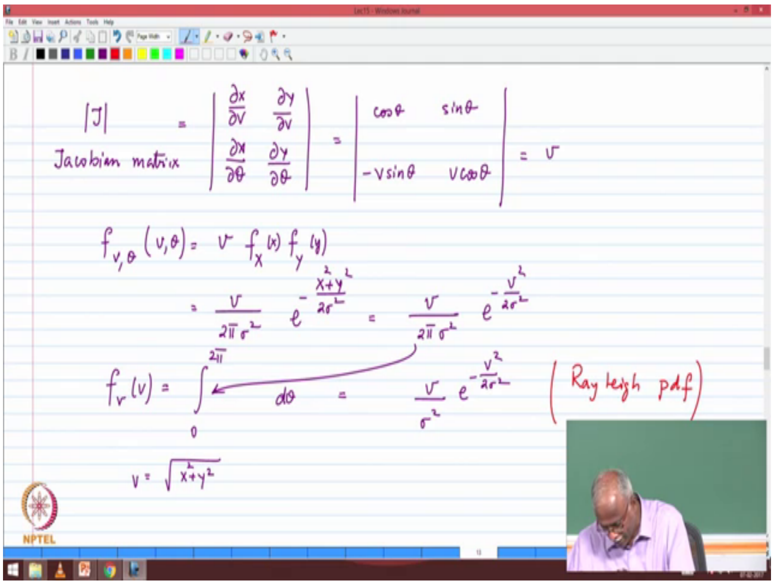

So, the determinant of j the Jacobean matrix this is called the Jacobean matrix, always comes when we have to do transformation of variables, again this is something that you would have worked with. This is given by the it is a determinant basically has the partial derivatives of the old variables with respect to the new variables, dou x by dou v, dou y by dou v, dou x by dou theta, dou y by dou theta. And modulus of that, and again we know that x is equal to v cos theta y is equal to v sin theta.

Please do substitute here is a simple result you will get cos theta, sin theta minus v sin theta v cos theta, basically gives us resultant of v. So, f of v comma theta is actually a very simple and a compact expression, v of theta is the Jacobean determinant of the Jacobean is v. Then we have the 2 pdf the joint pdf s, but because they are independent we can write them as a product of their marginal pdf fx of x, fy of y each of them is a Gaussian please substitute the expressions for the Gaussians, what we will get is v divided by 2 pi sigma squared e power minus x squared by y squared by 2 sigma square. This is nothing, but in terms of the a new variables it will be v divided by 2 pi sigma squared e power minus v squared by 2 sigma squared. I think you can already see the result more or less in front of us to get this is the joint pdf of v comma theta.

So, what we need to do is to get the marginal pdf five of v would be I have to multiply this joint p.d.f, with the and integrate it basically over the range of the over the range of the other variable theta. So, basically f I do that integral 0 to 2 pi of this quantity times d theta. Basically that will come out to be 2 pi. So, this will give me v by 2 sigma v by sigma square v by sigma squared e power minus v squared by 2 sigma square. So, again the key steps, visualizing this as a transformation of variables x and y treating it as the as a Cartesian form, v which is the ampl magnitude and theta being the angle as the polar form or the transformation of variables using the Jacobean, and using that to from there to get the marginal pdf.

So, the expression for the pdf of v where v is equal to square root of x squared plus y squared has been obtained. And this is what is called as the Rayleigh pdf. Now the reason we obtain the distribution is because we can work with it. So, the first thing whenever you get because this variable v is going to affect your amplitude, it is also going to affect your SNR and it is going to affect outage it is going to affect many things in terms of your the working of your wireless system. So, our ability to work with this is work with this with this statistical framework is extremely important.

So, let me just do the following, give you a series of a tasks for you to work with here are here are some simple things that I would like you to do. First thing how do you verify that something is a valid pdf?

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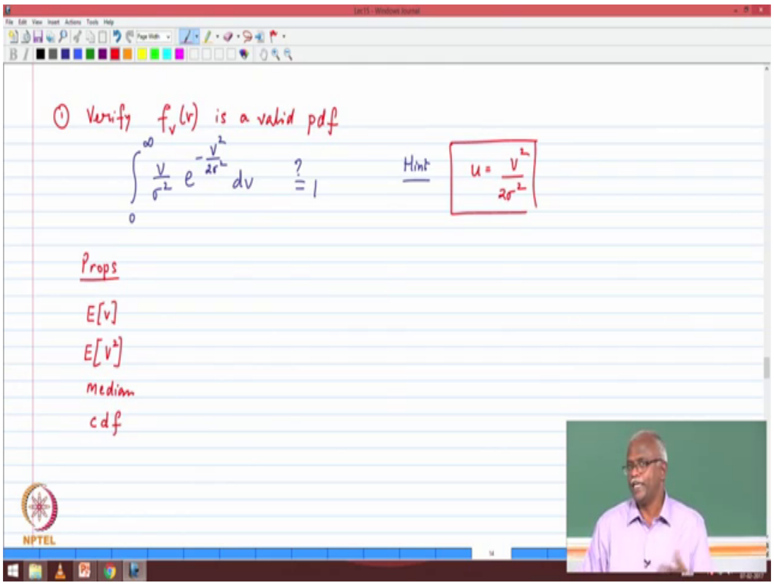

Student: Integrated.

Integrated over it is entire range and show that it is equal to 1. So, first thing I would like you to do is verify that f v of v is a valid pdf, valid pdf. So, basically what that will entail is you take it, from 0 to infinity, v by sigma squared e power minus v squared by 2 sigma squared d v, is this equal to 1, we need to verify now one of the things that we would very much need as a tool in in this course or this portion of the course is familiarity with these known integrals again we do not expect you to some of these are fell easy some of them are known standard integrals wherever you can use a standard integral feel free to use it because the idea is to what is the final result.

Now, in this in this case you can do a simple transformation, let me give you a hint should may be if you have not already have seen this. If you do the substitution u is equal to v squared by 2 sigma squared. Then you will get a very simple form which comes out to be gives you the expression that it is equal to 1. So, this simple substitution sometimes gives us a good result for us to work with. Now I want to do a couple of things one is I need us to understand the properties of the Rayleigh pdf one. Of course, I have verified this the properties of the Rayleigh pdf is one is what is the mean, what is the mean squared value we expect a value of v squared of course, that will give me standard the standard db, the variance and standard deviation. I would also be interested in median value I will be interested in understanding the cumulative distribution function.

Again I will explain why all of this. So, there are series of properties that we want to understand because Rayleigh distribution is an important one, but before we go in to that let me give you a little bit of insight in to what is happening.

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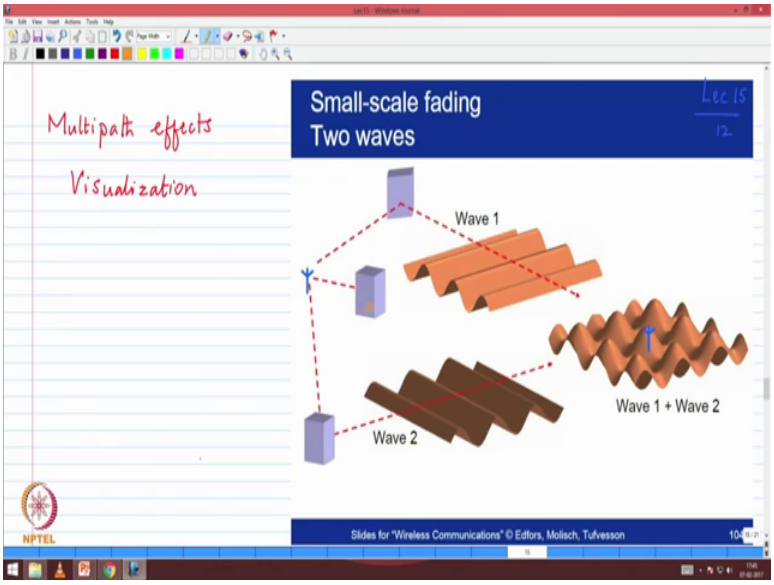

Always very important that we get good intuition, in to what we are working with. So, think of these scenario where the electric field is propagating the signal is propagating in 3, 3 directions from the base station. One is being completely abstracted by one of the buildings. So, nothing propagates behind that another path gets reflected. So, therefore, you get a certain direction of the electromagnetic wave, and then a third direction that is happening.

Now, this particular scenario where you just have 2 of these wave coming in different directions with different phases. When they add together, you will find that there is a peaks and troughs are happening that is basically interfere these 2 signals adding constructively or destructively. Now this phenomenon extended to arbitrary number of multiple path components. So, then what you have is a very complex set of peaks and troughs the amplitudes of which the real part and imaginary part, if we have to represent this in complex each of them looks Gaussian, if you have to look at the magnitude of the peaks and troughs the distribution that is happening here that would be Rayleigh distribution.

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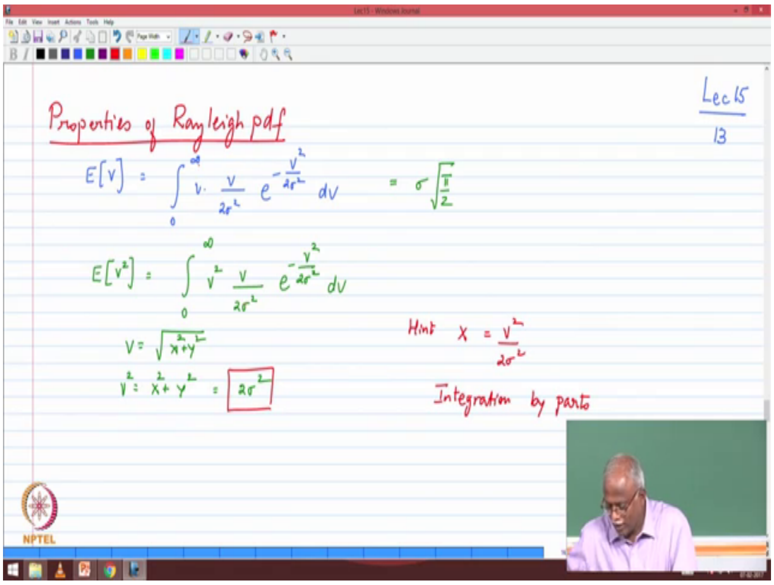

So, again what we are doing is mathematical in nature, but keep the underlying physical phenomenon happening what is happening very much in your picture. So, properties of Rayleigh pdf, the first one first property expected value of v, again all of these are integrals that you need to verify 0 to infinity v times the pdf of v which is v by 2 sigma squared e power minus v squared by 2 sigma squared dv now many of these integrals are not easy to do because of the of their form.

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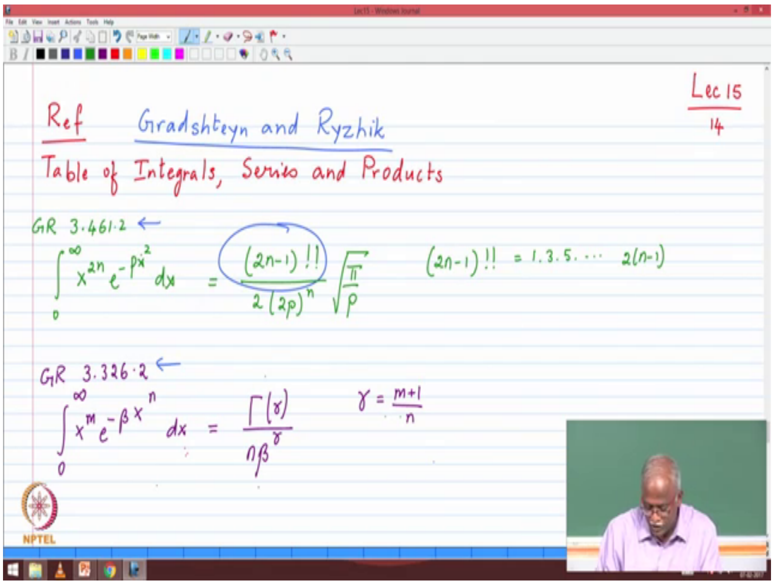

So, what we very commonly use in an engineering practice is a book called gradshteyn and ryzhik, table of integrals series and products you may have used it in communication course. And usually their formula that is formulae that are given are given using some numbers right the basically which chapter which section and all of that.

So, if you know the indexing of the formula you can actually come. So, basically I had given you a 2 examples of formula from gradshteyn and ryzhik, it turns out that these 2 are used in in in defining the properties. So, feel free to use any of the standard integrals make sure you interpret them correctly, because gradshteyn and ryzhik has got some notations that we do not come across. For example, in this particular 0 to infinity x of 2 n e power minus p x squared d x, has got a something with a double what is this factorial double factorial 2 n minus double factorial is only the odd terms 1 3 5 2 n minus 1. It is notation again whatever is the notation used please make sure you are understanding that because it is important that we use this results correctly.

So, the result that we would need for our discussion turns out to be one of the results in gradshteyn and ryzhik. Please do substitute the results and verify that this comes out to be sigma root of pi by 2, again doing it the long way would be quiet cumbersome. Just use the standard results and verify that this is indeed the case. Now expected value of v square expected value of v squared would be expect you expected to be a slightly more difficult integral, v squared in to v divided by 2 sigma squared e power minus v squared by 2 sigma square dv. Again we would have to use our in gradshteyn and ryzhik and look for suitable formulae or you may know that you know expected value of v squared is v is equal to x plus iy right. This would be v squared would be equal to x squared plus y squared or let me not confuse you we defined v as v as square root of x squared plus y squared.

So, v squared will be x squared plus y squared. So, expected value of v squared will be expected value of x squared plus y squared each of these. So, you can write down without even doing any calculations it should be 2 sigma squared. Because each of those has got variance sigma squared real and the real the x and y both of them individually have got, but of course, what I would like you to do is also be able to do it using the integration method, let me give you a hint for this also x is equal to v squared by 2 sigma squared, please do the substitution it will come out to be a very simple integral you will have to do integration by parts, but it is still a easy enough integral that you can you can work with integration by parts. And please verify that the answer must come out to be 2 sigma squared that is the correct answer.

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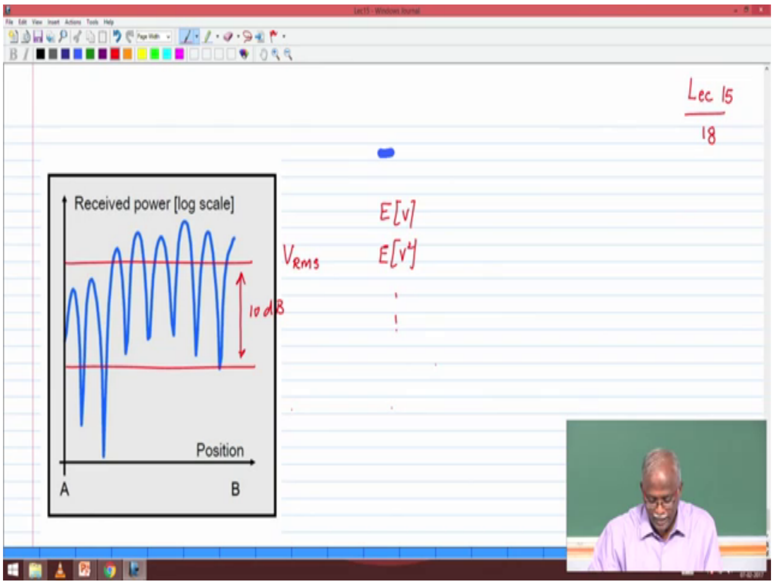

So, let me give you one more insight, which I will repeat again in the next lecture, but what I want you to think about is the following. When we have the small scale fading what the path loss and the shadowing tells me is what is the rms what is the level, that I am average level that I will receive I will call it v rms. So, that is the signal level now about that level I am going to see fluctuations to fast fluctuations, based on my position. I move a little bit suddenly there is a dip if I move again there is a. So, this is a very tricky phenomenon that I need to worry about.

So, one of the things that we are going to ask is how much margin you know that how much margin you should keep for shadowing because we know that that is going to result in outage. Now in addition to the margin that you have kept for shadowing you must keep some margin for this fast fading, otherwise the fast fading will cause you to go in to outage. So, one of the things that we are why one of the reasons why we are so interested in things what is the expected value what is the expected value of v, what is the expected value of v squared all of these properties is because I want to know how much margin I should have for fast fading.

It is an important element in our understanding of the small scale effects it is also important in our design of our systems. So, that we get a good and efficient design of the system that will work in the presence of path loss in the presence of a shadowing and in the presence of multi path fading. This is where I will end, but this is a figure that we are going to come back to do it please make sure that you are able to just verify the pdf the properties of the pdf again; we will build on that in the next lecture.