Artificial Intelligence:

Propositional Logic: Axiomatic Systems and Hilbert Style Proofs

Prof. Deepak Khemani

Department of Computer Science and Engineering

Indian Institute of Technology, Madras

Module – 02

Lecture – 05

So we have been looking at direct proof methods in the last class we saw an example which I am beginning with today which is an example given here that given these three formulae P and Q implies R not Q or S implies T and not T and P we can derive R. we had also said that any direct proof systems needs to work with axioms to start with and we had said that in our case the axioms are the ones which are given to us from the domain. But there is an interesting theorem which says that proving certain things based on certain premises in this case we have three premises and one conclusion is equivalent to proving this formula which says that you have P and Q implies R and not Q or S implies T and not T and P the whole thing implies R. there is a theorem in logic a very well-known theorem it is called the deduction theorem which says that every time you make a deduction based on the given set of premises its equivalent to showing that a certain formula which is the one we have written in black here which is made up of the given and conclusion is a tautology.

So notice that this is a this formula is a tautology. So if you have any problem that you are trying to solve in which certain premises are given to you and you want to show certain conclusion it is equivalent we can always find the equivalent formula which is a tautology and if you can show that this formula is a tautology then the earlier proof holds and this property is given to us by a theorem called deduction. Which we can write as follows that if you want to show using KB alpha then if KB is equal to a set of sentences let's say A B C D then in our old notation you would have written as you are given A you are given B you are given C you are given D and you want to show alpha which is the same as what I have written here. You can derive alpha from a knowledge base if and only if the following sentence can be derived A and B and C and D implies alpha.

And notice that when we have written that A and B and C and D implies alpha we have not written anything on the left hand side of derivation which means we don't need any premises to derive it which means it is true under all valuations which means it's a tautology. So in that sense if you can find a proof system which is able to derive all tautologies then in effect you can solve all problems all deduction problems in propositional logic. And as I said that was one of the earliest proof systems which was given to us was by a well-known logician called Frege.

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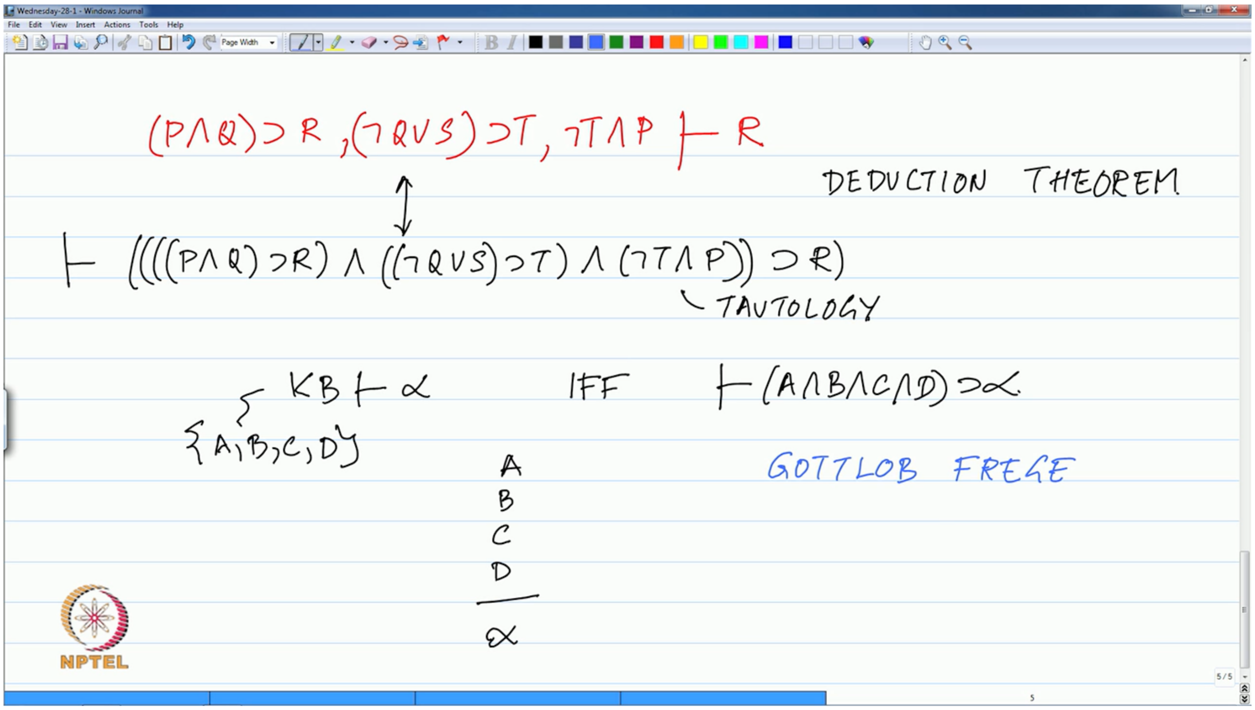

So I had said earlier that you know what logicians were trying to do is find out what is the minimal set which will allow you to prove all tautologies in propositional logic. So what do mean by that. That given an alphabet let's say P Q R S T all true sentences should be able to be derived. So such a system is called complete system. So there was a kind of a competition between logicians to try and devise a system which are tiny but which are complete at the same time. What do we mean by tiny? So when we look at this Frege's propositional logic we will see that he gave us a logic in which there were two symbols which is negation and implication. So first of all you must convince yourself we have gone through this argument we will not go through this again that anything you can say using other connectives can also be said using a complete set of connectives. So what Frege showed along the way is that this pair of connectives the negation and implication is a complete set. If you want to say that A and B or A and B or C everything can be said using implication and the negation sign. So for example a simple formula like A or B is as you can see is equivalent to saying not A implies B. I can always express it using just a negation and implication. And in fact the way to show that this set is complete is to show that every other connective can be expressed using these two connectives. So here we have shown that all can be expressed using not and implication and we can show for other connectives as well.

So Frege system used on these two connectives and includes only one rule of inference which is modus ponens and he showed we are not going to go to the proofs because our interest is more in algorithms that we will compare and in any case we are not going to be too much interested in direct proofs.

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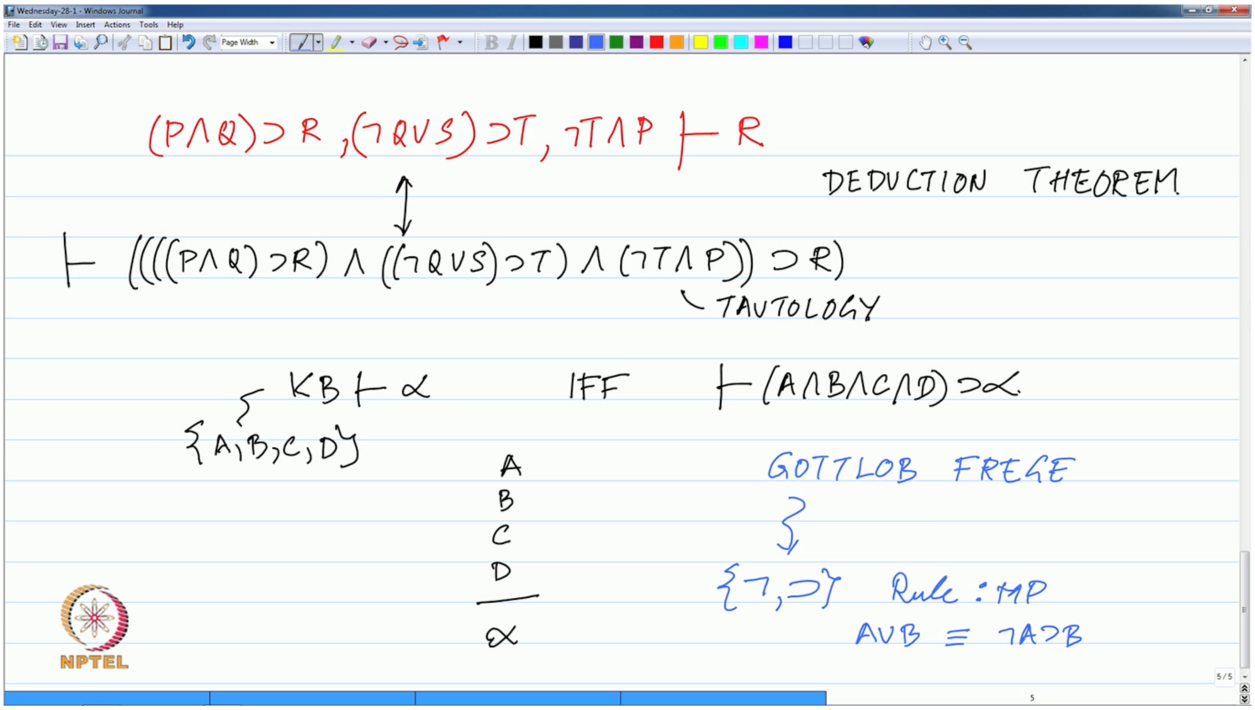

But he showed that his logic or his propositional calculus both sound and complete. Anything that you can you just choose an alphabet of primitive propositions and anything that you can say using that alphabet can be expressed using these two connectives negation and implication and all tautologies can be derived. So our emphasis is on derivation. So of course he gave us a set of axioms which is what I need to state next.

So we are looking at Freges propositional calculus. If you just search for this phrase on the web you will find on Wikipedia a very nice description of frege's calculus along with a sort of very interesting example proofs. So I do urge you to go to Wikipedia. So freges calculus is made up of certain axioms so remember we had said there are two kinds of axioms one is the logical axioms which are tautologies and other is domain specific axiom of the kind we saw in the example previously solved. Freges propositional calculus works on logical axioms and there are 6 axioms which are written as follows let me write here.

So they have been given name the first axiom is called Then one A implies B implies A. again remember that A B are propositional variables you can plug in any large formula inside this all you are saying is that this formula is an axiom whatever you can plug in for A you can in fact plugin this formula itself for A and something else for B and the whole thing will turn out to be an axiom. One way to verify that this is a tautology is to simply construct a truth table another way if to try to find a proof in some other system we will try and do that. For a moment let's just accept them as axioms. We say that if A implies B implies C then if A implies B then A implies C.

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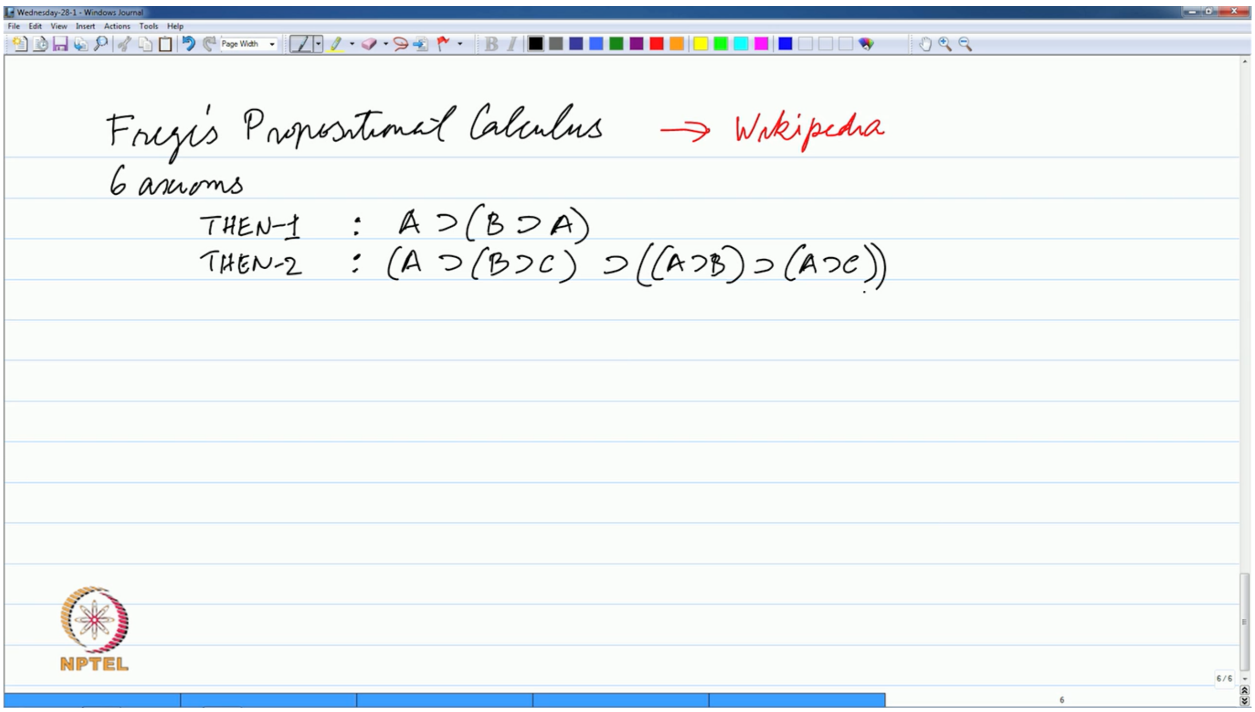

And three more axioms which deal with negation sign which are given the name based on Frege. So you see some connection with modus tollens here connection between modus ponens and modus tollens here whenever A implies B is true not B implies not A is also true. And there is only one rule of inference which is modus ponens which says that P and P implies Q gives Q. so this is Frege's propositional calculus.

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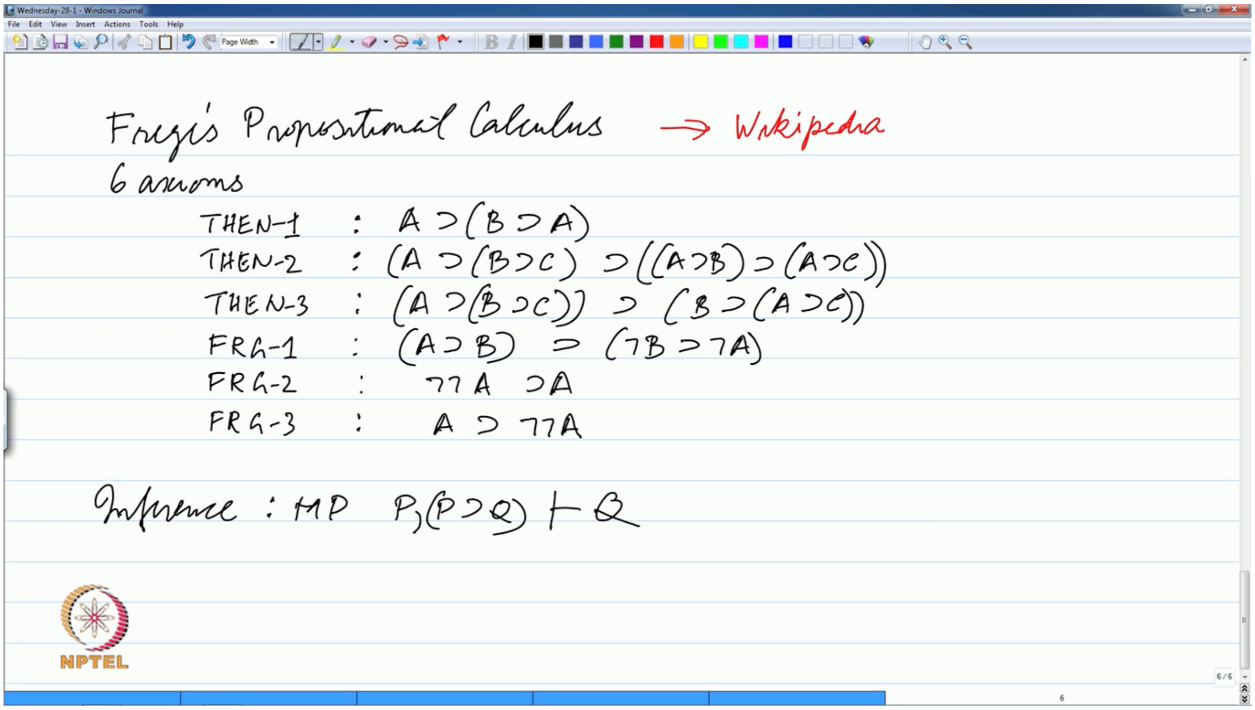

6 axioms one rule of inference and he showed that all tautologies of propositional logic can be derived from here. So what we will do is we will look at a couple of proofs just to get us a flavor of how this works. And then we will try to see that to find sort of slightly more what should I say methods which can be automated more easily. So we will see that a lot of guess work is kind of involved here.

So let's look at a couple of examples. So we want to show this and so this is called Derived rule. So if you try a hand a frege's calculus you will see that proofs so in fact in a moment we will look at a proof of a very simple statement called which is P implies P. we will see that proofs can be terribly long proof. And you are not really going to be interested in long proofs so what logicians often do is that they have derived rules of inference which says that once you show that another rule is sound you are allowed to do that as a shortcut. So we will look at one derived rule which is called as Th1 star and this says that if you have A implies B and if you have B implies C then you can conclude that A implies C.

So if you look at the Frege's axioms ok so if you look at the Frege's axioms then this is all that's given to us. And we now want to do is to derive this one. So let's see how we can do this. We start with B implies C where did this comes from this is a premise that's given to us we are given true statements and based on those statements, the true statements are A implies B and B implies C and now we want to show that A implies C. know that this notation that we have left hand side of this symbol here is the antecedents or what's given to us and the right hand side of this symbol are what we want to show to be true.

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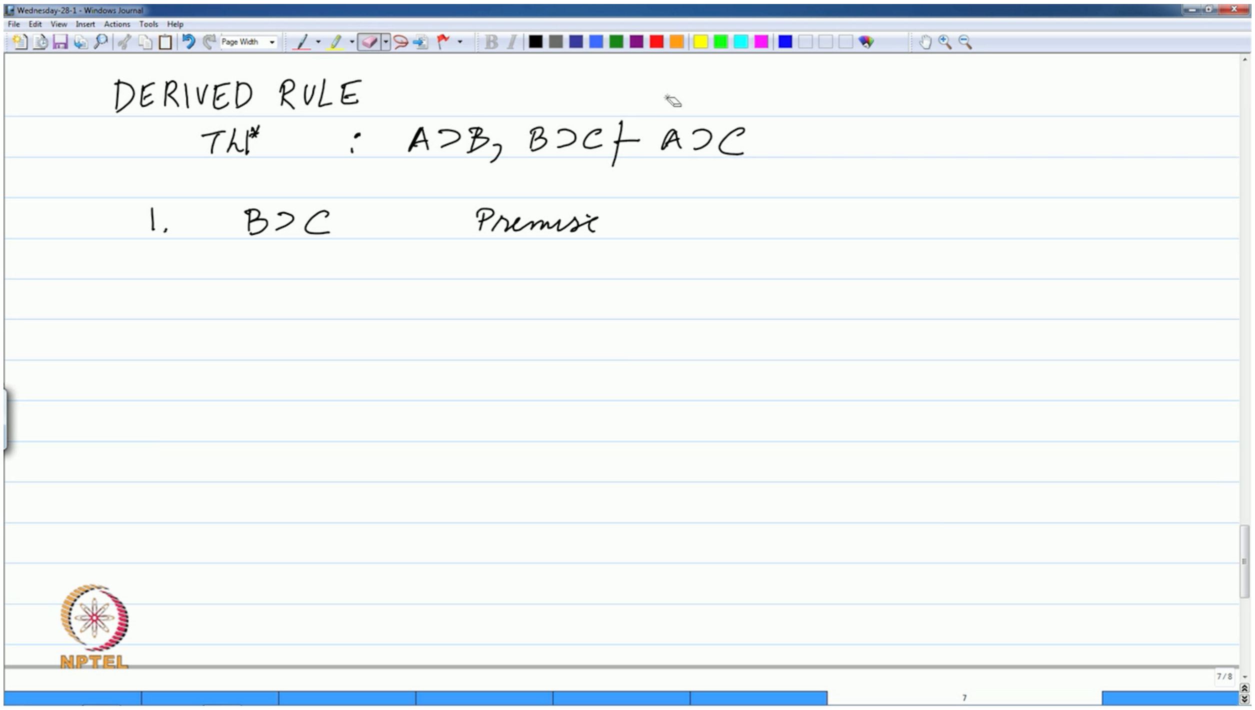

So this B implies C is given to us then we look at an example of B implies C implies A implies B implies C. where did I get this from the first axiom so this is from Then1 infact it is Then 1 except that instead of A implies B implies A I have written B implies C implies A implies B implies C. now I can see that I can write A implies B implies C how because of 2 1 and modus ponens. This as you will recognize is Then 2 the second axiom by Frege and then we apply modus ponens again so I get A implies B implies A implies C. so by 3 4 Modus ponens. Then A implies B is given to us and therefore we can conclude A implies C by 6 5 and modus ponens.

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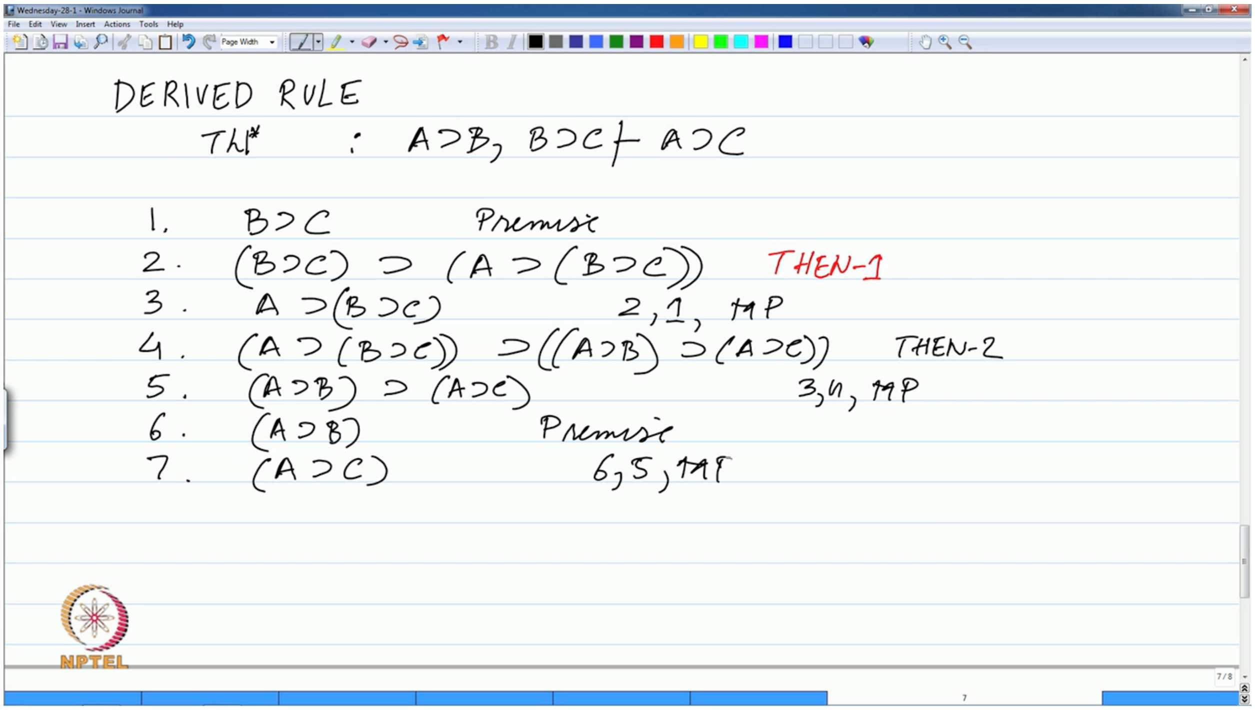

So this is a proof in Frege's propositional calculus we have shown that so you can see that basically it's the rule of transitivity or there is a corresponding rule which I had mentioned sometime called hypothetical syllogism. It says that if A implies B and B implies C then you can infer A implies C. we will look at other ways of arriving at these proofs a little bit later.

Let's look at a proof of a simple sentence which is to show that A implies A. show that A implies A is a tautology. Now obviously this looks like a very trivial statement and we accept it to be true we can simply construct a truth table and see that if will be true and so on. But we have thrown away the notion of using truth tables and semantics and things like that. We are saying that we are going to rely on Frege's machinery to produce sentences for us so can we produce this sentence. And on the webpage that I mentioned to you the Wikipedia page there is a nice proof which I am going to reproduce here.

Audience Question

Answer: well that's true and we will see that some of the indirect methods we will look at the tableau method in fact is trying to do semantic analysis which much later actually was recognized by people as a proof system. So it's true that we can try to show that the formula is unsatisfiable and when we fail to show that it is unsatisfiable we conclude that it is satisfiable and things like that. But a more compelling reason for resorting to proof methods is the fact that truth table don't carry forward the first order logic you just cannot construct truth table for first order logic. So there we need a formal system.

So this is the proof for A implies A.

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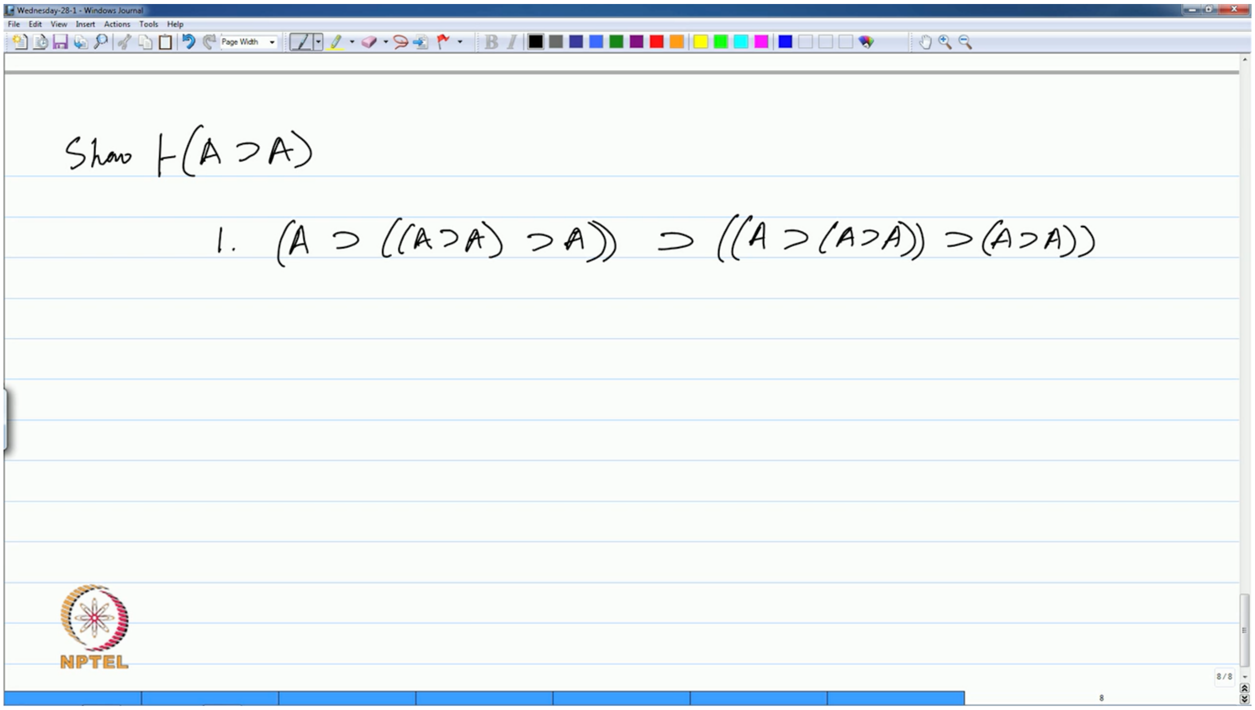

Where does this come from can you see. It's from Then 2 if you look at A implies A here and A implies A here this is where we substitute it for B. and this A the third A so this was B and this one is C so we have this. When you look at this you can see that you know we are trying to use modus ponens all the time. There is only one rule of inference available to us. So once we have a large implication we are trying to generate a formula to match the antecedent so for the first implication you can see that second sentence is something which matches the first one and allows us to use modus ponens. So where did this come from this is an example of Then 1 right where this corresponds to D. now you can infer what is there in right hand side A implies A implies A implies A implies A which is from 1 2 and modus ponens. Already you can see that the conclusion of this statement is what we are looking for which is A implies A. so if you can generate the left hand side then we can get to the right hand side. So the left hand side is easy enough to generate. A implies A implies A this is just an example of Then 1 where we have substituted A instead of B and then finally from modus ponens.

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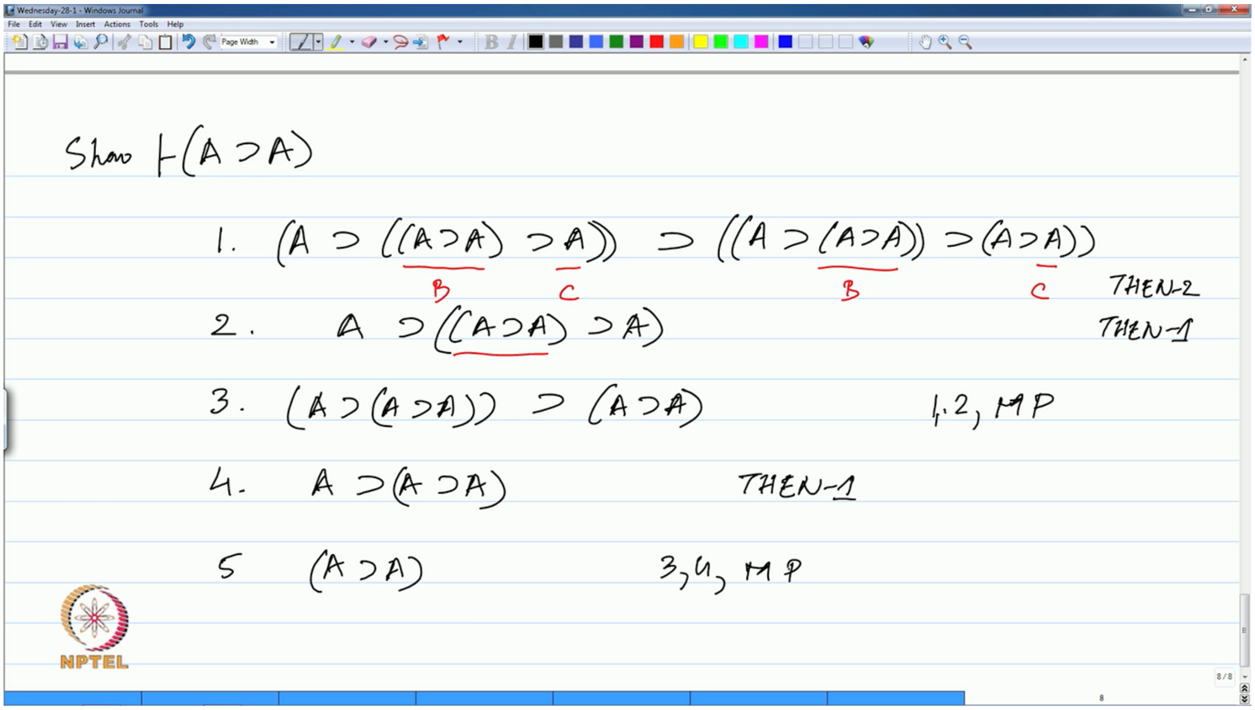

So whats the difficulty with this proof method. We can show that it is complete but you can see that it involves a lot of guess work. Or you might say involves lots of instantiations you have to produce those particular you can see that the first formula is an instance of Then 2 which means that we plugged in certain values for B and C and we got this. So you have to somehow produce magically that right instance if you can do that then ofcourse you can generate a proof. But even for such a small thing we might end up producing such a long proof and you can imagine that for more complex formulae the proofs tend to be even longer.

So let's look at another variation in direct proof in which you are allowed to use assumptions. So very often if you make certain assumptions so if you remember your geometry proofs you would say okay draw a line from here to here so again you have to be creative here or you have to do an informed guess work to make the right assumptions but if you can do that then you can get the proof fairly quickly. Some people have called it as Hilbert style proofs Hilbert as you know was a greek mathematician who at the beginning of the last century had posed a set of 30 40 problems and said that if you can solve these problems then you have solved the world of mathematics. And unfortunately he received a great setback because somewhere along the line Dodal …came along and he showed his proved his incompleteness theorem he said that you can never build powerful enough system which are complete.

Okay so basically Hilbert style proofs take make certain assumptions and they allow you to come to conclusions based on assumptions. So the basic idea is this that if you make an assumption alpha you are still given the knowledge base but we are adding certain assumptions to this. Then you do some steps and derivation and somehow produce beta. Now you can see that this beta is dependent upon alpha and it may turn out the case that because you assume alpha to be true you could prove beta. What Hilbert style proofs do is you can so some people like to draw a box around the assumption say that you want to come out of the assumption you must replace this with alpha implies beta.

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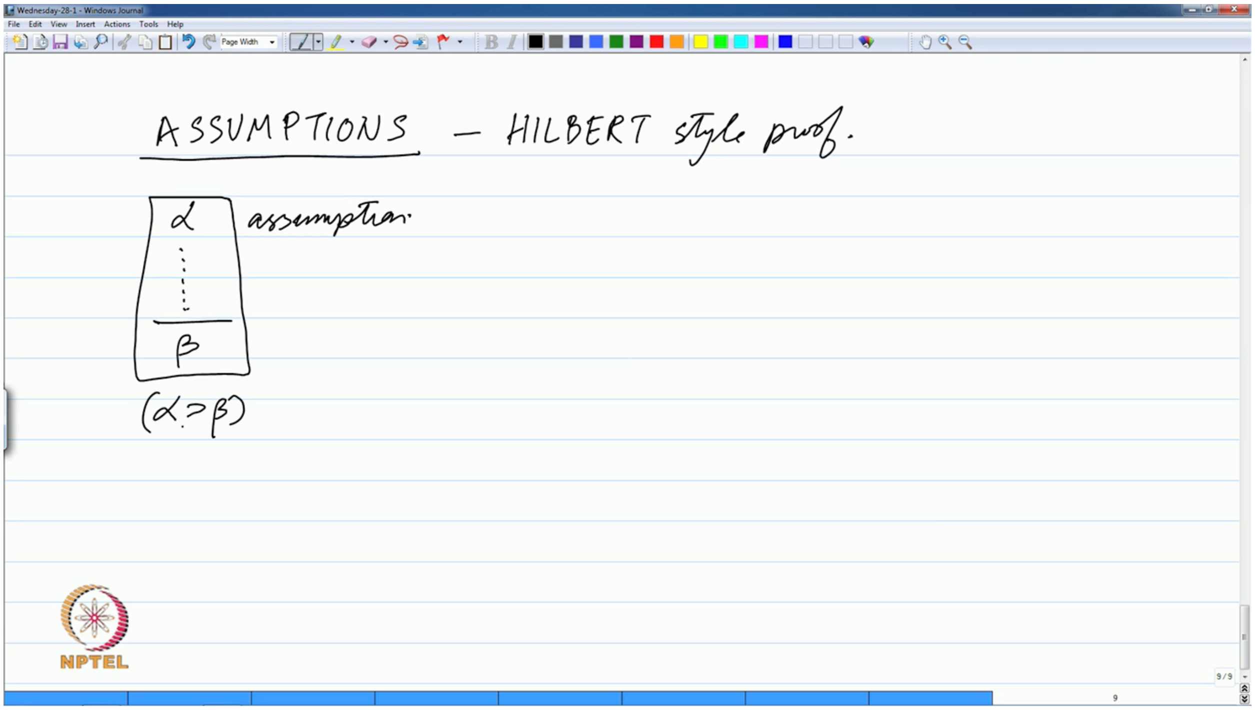

So as it turns out this gives some short proofs you can generate some really short proofs. You take an example. We make an assumption P and therefore we conclude P Identity. And therefore you conclude P implies P. now this is the derivation this is not the we are not looking at truth table or something like that. Let's look at Then 1. Ofcourse we can construct truth tables for all these things but what does Then 1 say A implies B implies A right. You can see that so Hilbert style proof they work only with the implication kind of a thing. In fact, hilbert's calculus uses only two of the axioms that we use. And he tries to show that you can so a lot of things with that. Ofcourse it's not a complete system.

So let's see if we can prove Then 1 here. It's not so hard we wil make assumption A. so let me write this in a linear fashion then you make assumption B. first statement is an assumption second statement is an assumption. Third statement is from 1 so I will just write from 1 and identity you are allowed to reproduce it again. Then the fourth statement is B implies A in the sense that here you have made an assumption B and then you showed A so therefore when you write B implies A you can throw away the assumption B and then the fifth statement is A implies B implies A which is you are going back to the first one.

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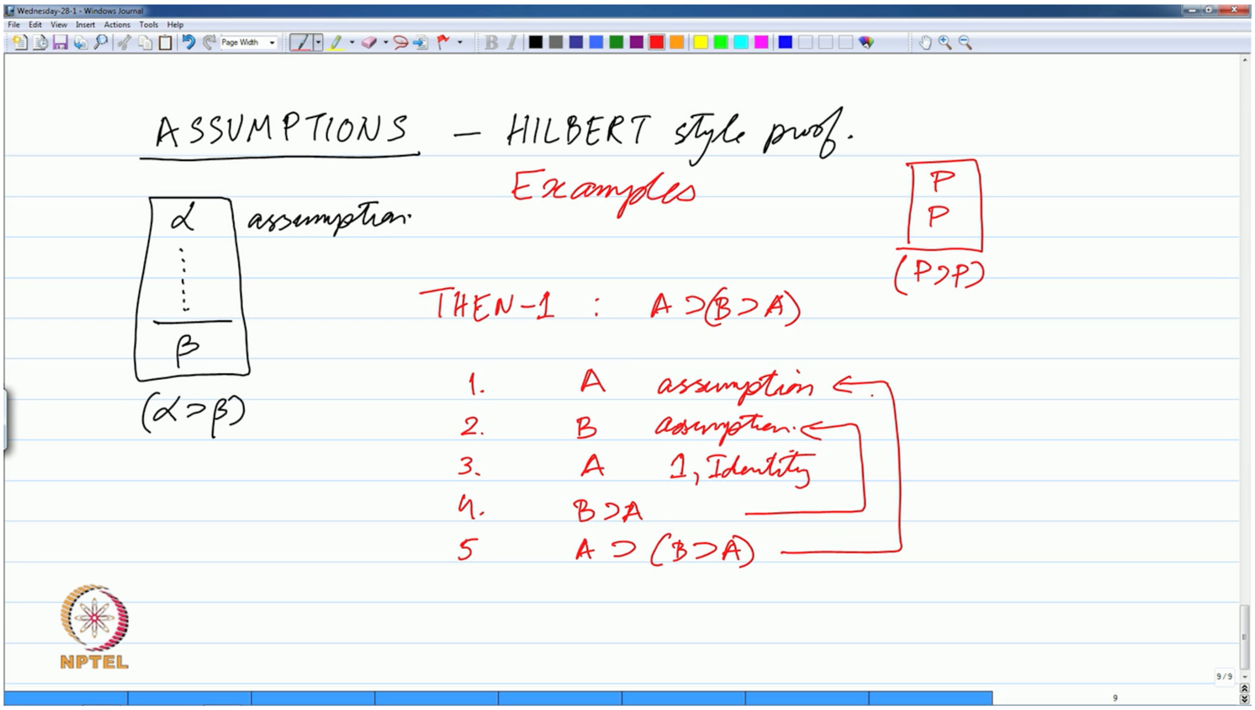

So you have done away with both the assumptions and you ended up with the formula and therefore you accept the formula to be true. Just look at a couple of more examples.

Audience Question

Faculty: Ya so let's look at this example and then it will make things clear or maybe we will show Then two as a proof for Then two. So let's start with the highest level assumption which is the left hand side of Then two so this is an assumption let's put it in a box. Then we address the right hand side so we make another assumption which is P implies Q put it in a box which is inside this outer box. So whatever is in the outer box we can use in inner box as long as we are inside the outer box. Then we make a third assumption which is P so the third box is nested inside these two boxes. So all these three assumptions P implies Q implies R P implies Q and P. now we can get Q so let's give numbers to these one two three four so from three four modus ponens then we have Q implies R from three one modus ponens. And then we have R from four five modus ponens now if we close this innermost box that means we have thrown away the assumption P. so we can write P implies R. then we close the second box so we have thrown the assumption P implies Q so we can write P implies Q implies P implies R and finally we close the outermost box and we have Then two.

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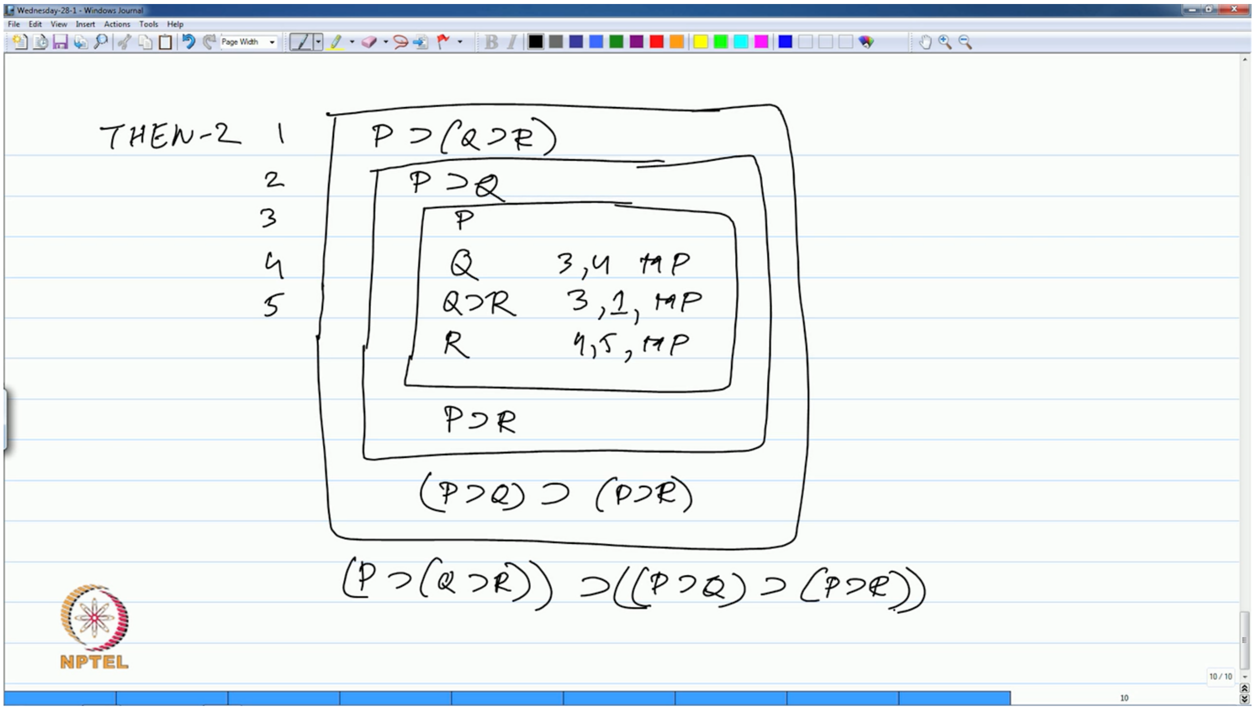

We make three assumptions and then we close all assumptions and we ended up with this formula which is actually Then two. So Hilbert style proofs tend to give us shorter proofs. But again there is guess work involved which is not so easy to implement in programs. I mean you cannot having a program make an inspired guess. So for this reason we find that direct proofs are hard to implement though when we move to first order logic we will still look at some examples of a direct proof which are used in real life so forward chaining and backward chaining for example but we also want to look at other proofs which are easier to implement in algorithms. So in the next class we will look at our first indirect proof method which is called as tableau method. So we will stop here.