Discrete Mathematical Structures

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Lecture - 5

Logical Inference

In the last class we saw about logical quantifier, universal quantifier and existential quantifier and how to use them we also saw the rules regarding them as to whether you can interchange the order in which they are occurring and so on. In any logical system you have some axioms and rules of inference, and making use of the rules of inference from the axiom you try to derive true statements about the system. And if you use the rules in a proper manner you arrive at a correct argument and if you use the rules in improper manner the argument will be invalid. So next we shall see how to find out whether an argument is correct or wrong starting from the axioms and rules of inference. Towards the end of the last lecture we saw some rules of inference we have also seen them as logical identities earlier so let us recall those rules.

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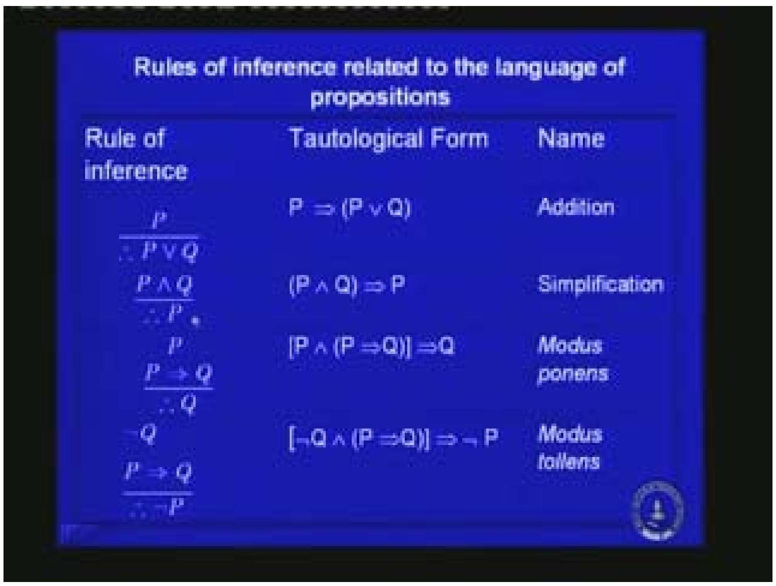

From P you can conclude P or Q this is called addition and from P and Q you can conclude P this is called simplification, from P implies Q AND P you can conclude Q this is called Modus ponens, from P implies Q AND NOT Q you can conclude NOT P that is called Modus tollens and from P OR Q and not P you can conclude Q this is called Disjunctive syllogism and from P implies Q AND Q implies R you can conclude P implies R this is called Hypothetical Syllogism.

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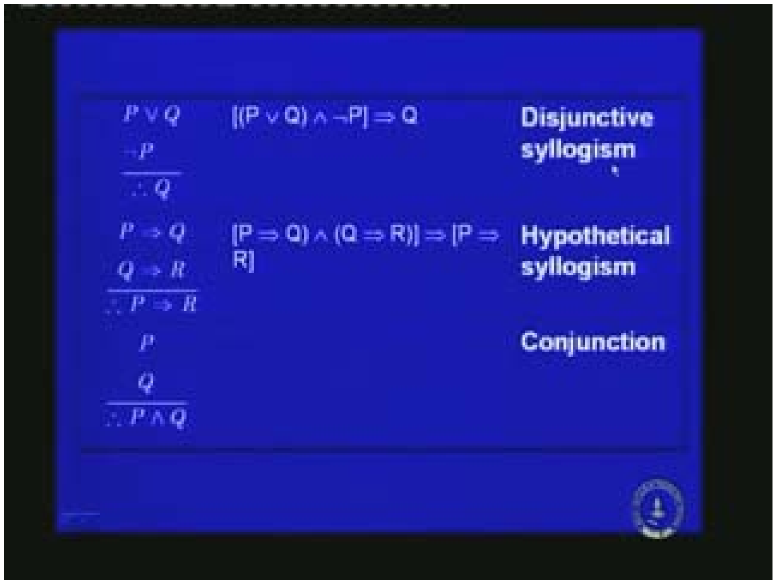

And from P AND Q you can conclude P AND Q this is called Conjunction. Then from P implies Q AND R implies S AND P OR R you can conclude Q OR S this is called Constructive Dilemma and from P implies Q AND R implies S AND NOT Q or NOT S you can conclude NOT P or NOT R this is called Destructive Dilemma.

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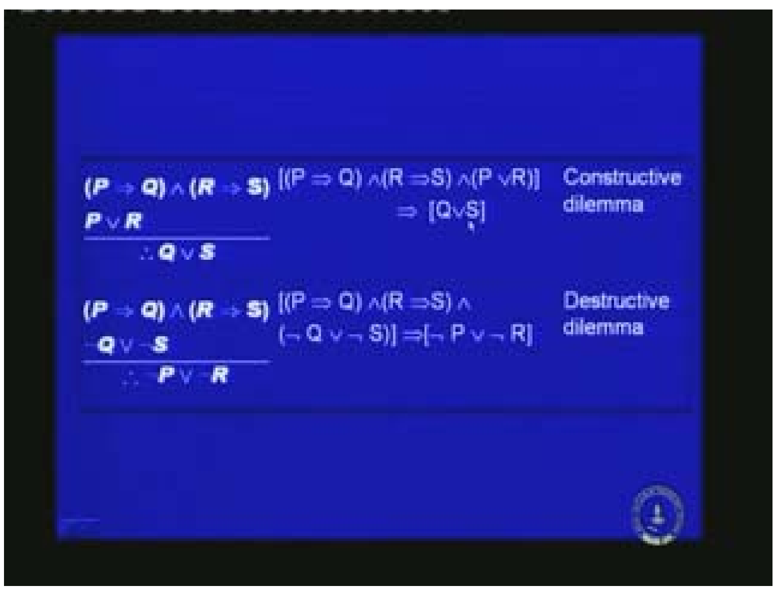

Mostly we will be using these two rules Modus ponens and Modus tollens in our arguments. Let us see how we can prove whether an argument is correct or whether an argument is wrong and so on.

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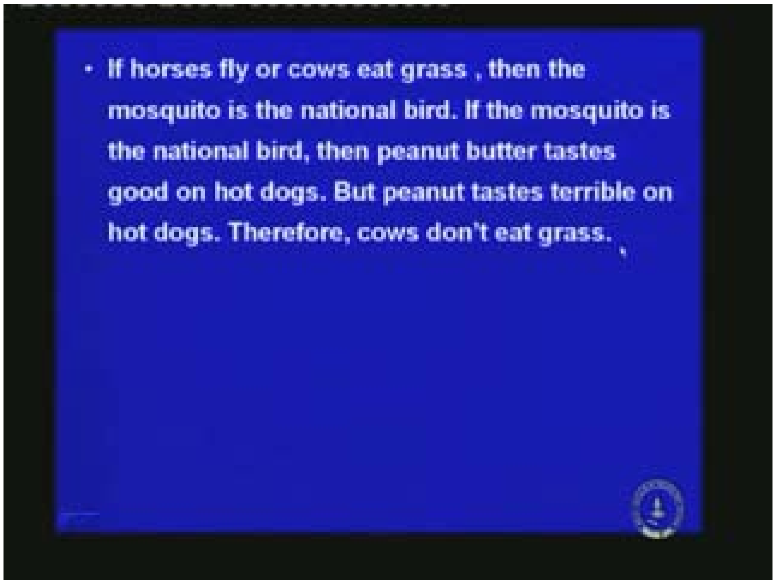

Let us take this example; it may look a very silly argument. But given the premises and the conclusion is the argument correct this is what we want to check. It is like this; if horses fly or cows eat grass then the mosquito is the national bird. If mosquito is the national bird then peanut butter tastes good on hot dogs. But peanut butter tastes terrible on hot dogs. Therefore cows do not eat grass. Is this argument correct or not? Let us see how we go about. This is a correct argument even though it looks very silly. Let us see how we go about proving this. Let us denote horses fly by H, cows eat grass by G, mosquito is the national bird by M, peanut butter tastes good on hot dogs by P.

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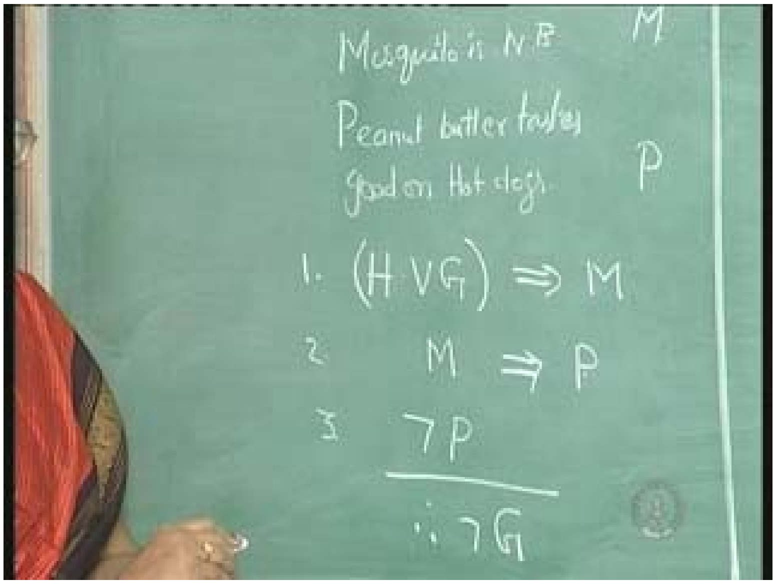

So in the logical representation how do we represent these statements? If horses fly or cows eat grass then mosquito is the national bird. That will be denoted by if horses fly or cows eat grass then mosquito is the national bird. Then second statement is, if mosquito is the national bird then peanut butter tastes good on hot dogs and that is denoted by this. But peanut butter tastes terrible on hot dogs and that is denoted by NOT P. So the conclusion is therefore cows do not eat grass therefore NOT of G. This is the argument. The argument you can represent in logical notation in this form. There are three premises and this is the conclusion. Does the conclusion logically follow from the premises? Are we using the correct rules of inference and arriving at the conclusion? This is what we want to check. Let us see whether from these three premises we will be able to prove the conclusion.

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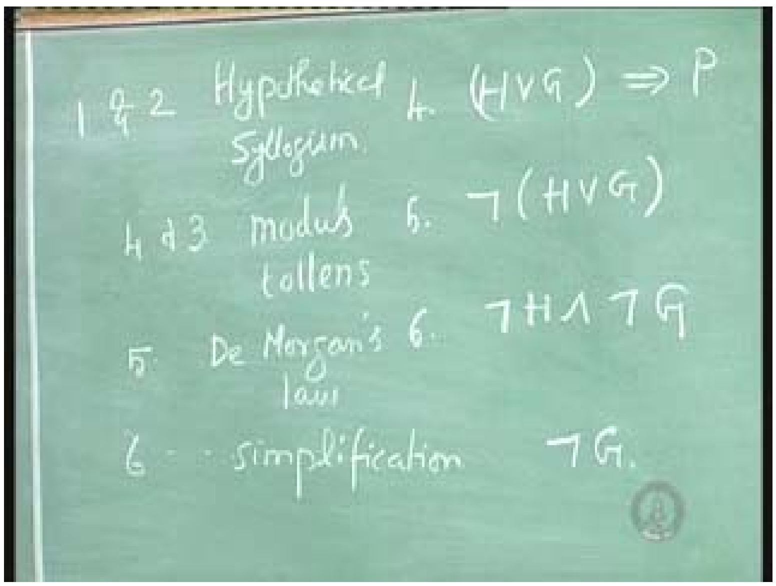

Now look at statement one and two and if you use from one and two if you use Hypothetical Syllogism what do you get? H OR G implies M and M implies P so from that you can conclude H OR G implies P. Let us call it as statement four. The fourth assertion we have arrived at. Now from 4 and 3 what is 3? So 3 is NOT P so from H OR G implies P and NOT P what can you conclude? By using Modus tollens you can conclude NOT of H OR G. Let us call it as statement 5 and from statement 5 use De Morgan’s laws, this you can expand as NOT H AND NOT G call it as statement 6. Then from 6 BY simplification you get NOT of G. So you are able to arrive at the conclusion from the premises by using proper rules of inference. So even though these sentences may not convey any meaning it may look a silly argument but still from the premises using the rules of inference you can arrive at the conclusion. So this argument is a valid argument.

Now let us see some more arguments. It is not the case that IBM or Xerox will take over the copier market. If RCA returns the computer market then IBM will take over the copier market hence RCA will not return to the computer market. Let us see whether this argument is correct. It is correct but let us see how it is. So again IBM will take over copier market denote it by I.

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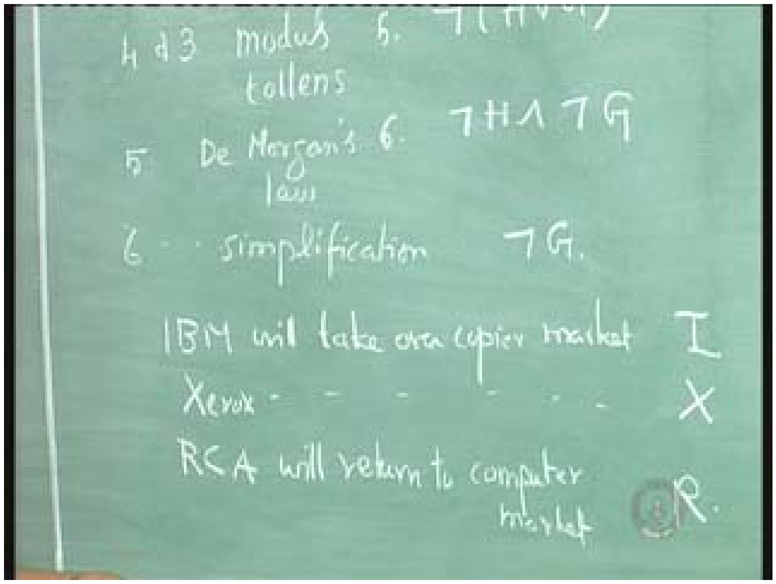

It is not the case as Xerox, Xerox will take over the copier market denote it by X. RCA will return to the computer market. Now how do you represent the statements in logical notation. It is not the case that IBM or Xerox will take over the copier market. NOT I OR X is the first statement.

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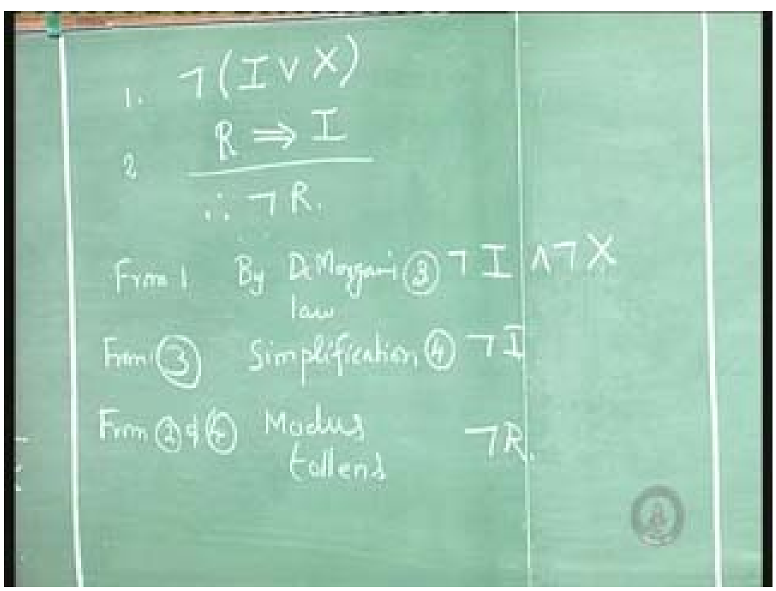

If RCA returns the copier market then IBM will return to the copier market. R implies I so the conclusion is therefore NOT of R. Is this the correct argument? There are two premises here and this is the conclusion. Can you derive the conclusion from the premises assuming the premises to be correct. Again let us see how we derive the conclusion from the premises from 1 by De Morgan's law this will be NOT I AND NOT X.

Then let us call this as statement 3. From 3 alone by using simplification you will get NOT I call this as statement 4. Then from 2 and 4 R implies I and NOT I if you use Modus tollens you will get NOT which is the conclusion. So you are able to arrive at the conclusion from the premises and this is a valid argument. Now let us see one more example; if today is Tuesday then I have a test in Computer Science or a test in Economics. If my Economics professor is sick then I will not a have a test in Economics. Today is Tuesday and my Economics professor is sick. Therefore I have a test in Computer Science. Obviously this is much simpler than the other one. This looks like a correct argument. Let us see how we can prove it starting from the premises.

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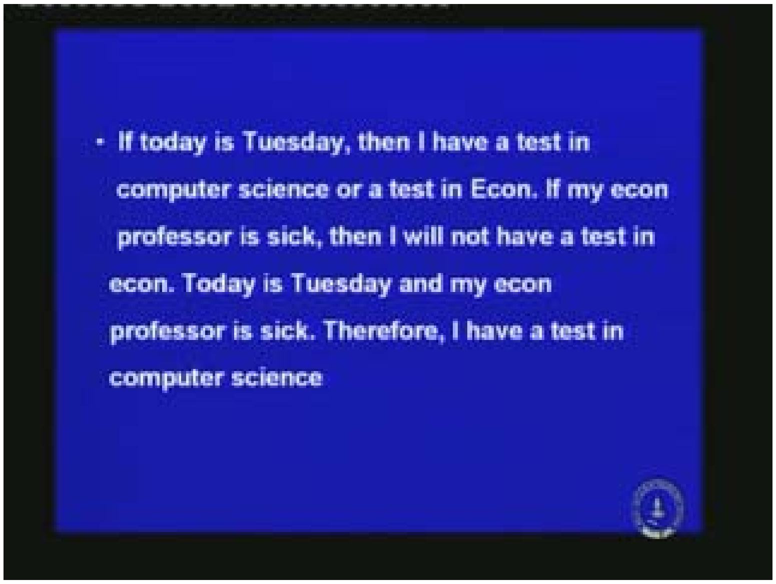

So let us see today is Tuesday denote by T. I have test in CS by CS.

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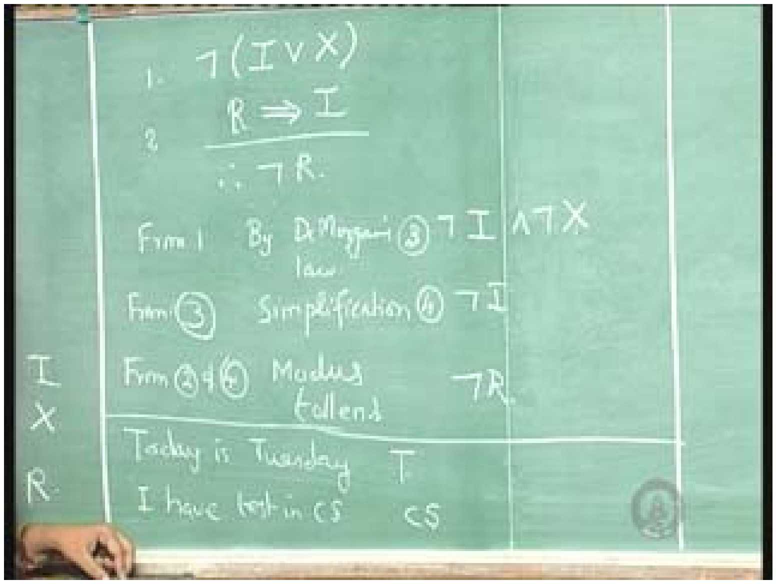

I have test in Economics by E. Economics professor is sick by S. Then how will you represent the statements in logical notation?

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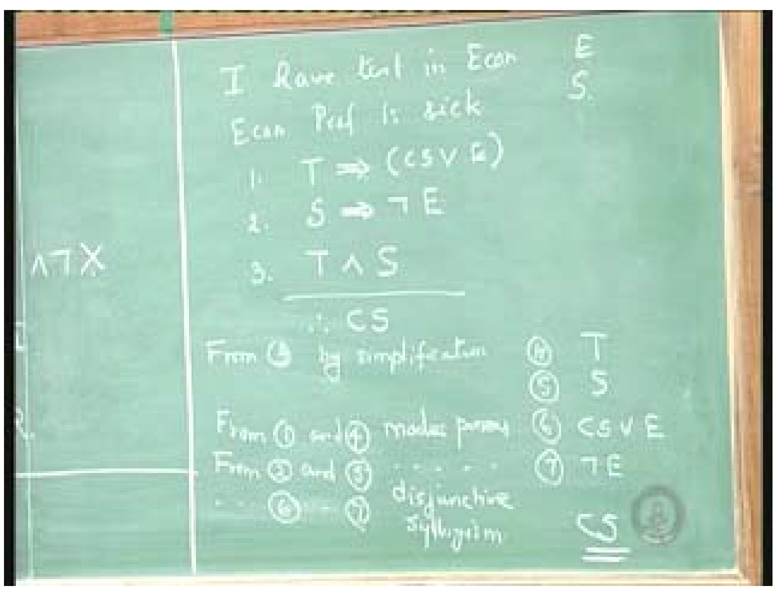

The first statement is if today is Tuesday then I have a test in Computer Science or in Economics. And second statement is, if my Economics professor is sick then I will not have a test in Economics. And third statement says today is Tuesday and my economics professor is sick. So these three are the premises from which the conclusion is therefore I have a test in Computer Science.

Now let us see whether this is a correct argument and how do we arrive at the conclusion from the premises. Now from 3 by simplification you can get T and you can get S. And from 1 and 4 T implies CS OR E AND T by using Modus ponens you will get CS OR E. Similarly, from 2 and 5 S implies NOT E and S by again using Modus ponens you will get NOT E. This is again by the application of Modus ponens to 2 and 5. Now you have CS OR E and NOT E so from 6 and 7 use Disjunctive syllogism that is from CS OR E and NOT E you can conclude CS which is the conclusion.

So starting from the premises by the application of proper rules of inference you are able to get the conclusion and the argument is valid. So this is the sort of a proof you give saying that this argument is correct or not. Now if some argument is not correct you must give a counter example. What you mean by that. There will be some premises and conclusion and if you want to show that the argument is not correct you must give the truth values to the statements involved to the assertions involved such that the premises become true and the conclusion is false.

Usually people make two types of fallacies frequently and let us see what they are. One is called the fallacy of affirming the consequence.

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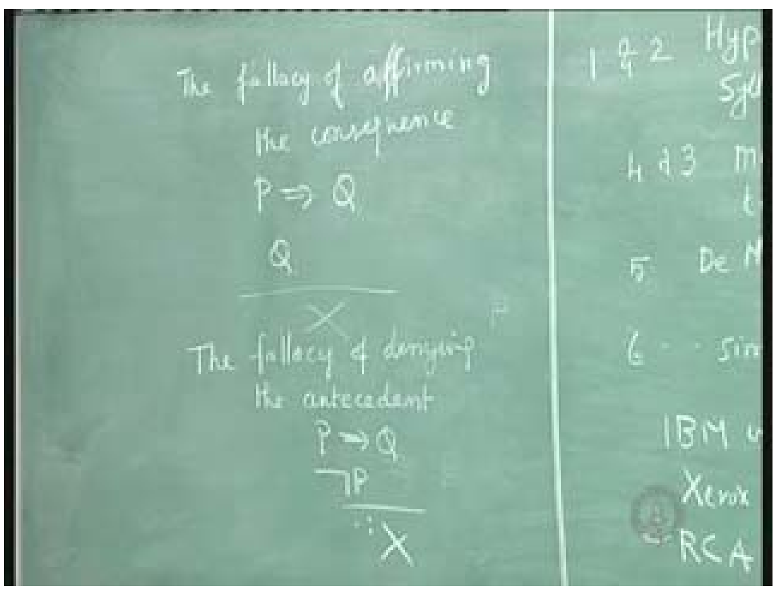

So, from P implies Q and Q you cannot conclude anything. For Modus ponens from P implies Q and P you can conclude Q. For Modus tollens from P implies Q AND NOT Q you can conclude NOT P. But from P implies Q and Q you cannot conclude anything.

Suppose if you make an argument like this; if the butler is nervous then he committed the murder. Or I will put it this way, if the butler committed the murder he will be very nervous. The butler is nervous so he committed the murder, is this argument correct?
This is not a correct argument because here the statement is, if the butler committed the murder he will be nervous and because he is nervous he committed the murder that is a

wrong statement because when the police interrogated the butler just because seeing the police itself he might have become nervous. So it is not necessary that he should have committed the murder. This is a wrong argument, this sort of an argument is called the fallacy of affirming the consequence.

Similarly, you also have another fallacy which is called the fallacy of denying the antecedent. That is from P implies Q and NOT P you cannot conclude anything. So again some argument like this, if the butler's hands are covered with blood then he committed the murder. The butler's hands are clean therefore he did not commit the murder. This is not a correct argument because you are trying to deny the antecedent. The butler could have washed his hands after committing the murder. So just because his hands are clean you cannot conclude that he did not commit the murder. So this sort of fallacies may occur during your argument and we have to be careful against not allowing for fallacies. So if you use some such wrong applications of rules of inference then the argument will not be correct.

And if you want to show that some argument is not correct you have to give counter example that is you have to give truth values for the assertions which will make the premises true and the conclusion false.

Now in all these cases we have used proposition variables or single statement proposition. There may be occasions where we have to use quantifiers in arguments. So what are the rules of inference related to quantifiers? Let us see some rules of inference involving quantifiers.

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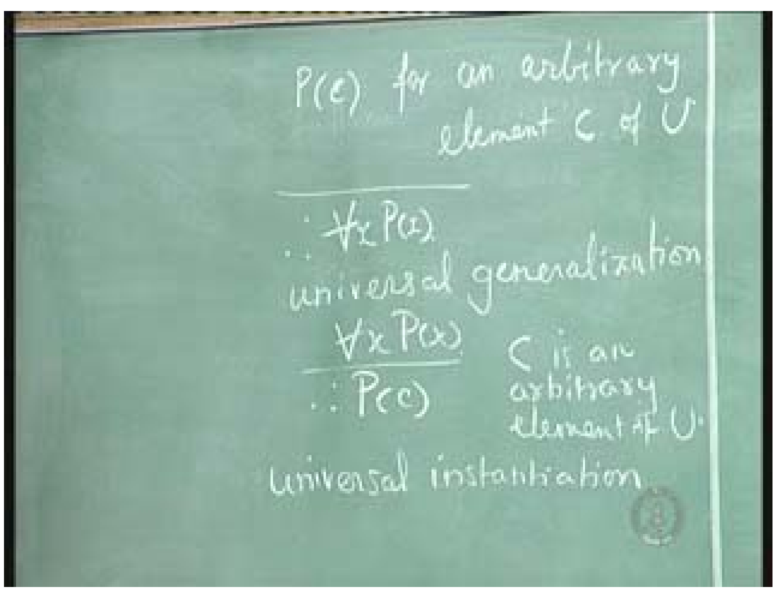

Now, if you have P(c) for an arbitrary element C an arbitrary element C of U of the underlying universe. C is an arbitrary element then from this you can conclude therefore for all of x P(x) and this rule is called Universal generalization.

And from for all of x P(x) you can conclude P(c) where C is an arbitrary element of the universe. This sort of a root is called Universal instantiation.

Now if you have P(c) for some C belonging to the universe it is not arbitrary but for some C belonging to the universe if you have P(c) then from that you can conclude there exist x P of x. This rule is called existential generalization. And from there exist x P(x) you can conclude therefore P(c) where C is some element not arbitrary belonging to the universe. This rule is called existential instantiation.

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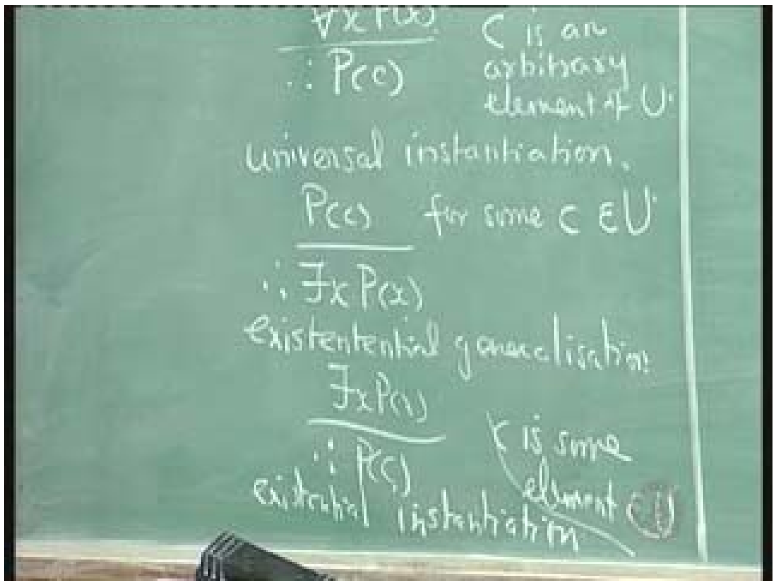

So for using quantifiers we have four types of rules; Universal generalization where if you have P(c) for some arbitrary element you can conclude for all of x P(x). Then you have Universal instantiation where from for all of x P(x) you can conclude P(c) where C is an arbitrary element of the universe. If P(c) is true for some C belonging to the universe then you can conclude therefore there exist P(x) and this rule is called existential generalization. And from there exist x P(x) you can conclude P(c) where C is some element this rule is called existential instantiation.

Now let us see how we can give some arguments using quantifiers. Now let us see some arguments involving quantifiers. The first example which is very common stated in all logic books is like this. All men are mortal. Socrates is a man therefore Socrates is mortal.

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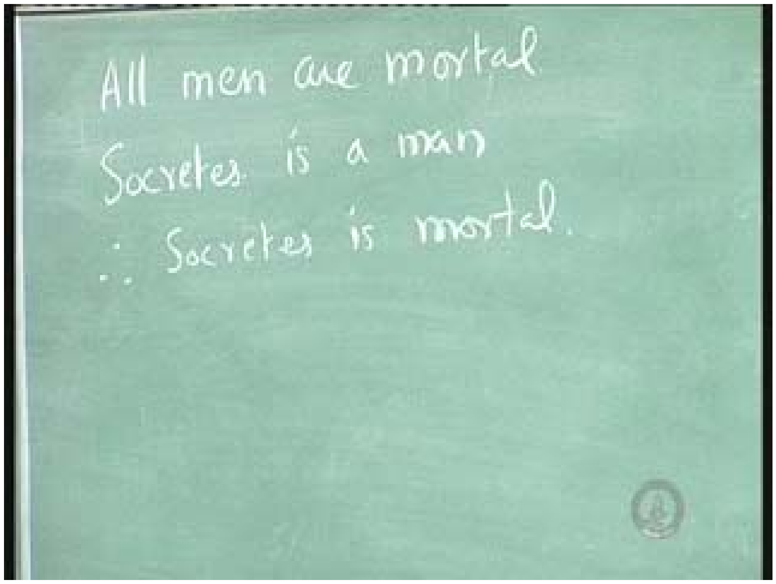

Obviously it is a correct argument and how do we prove this?
Now man of x denotes X is a man X is a man. Mortal of x denotes X is mortal. So the argument is like this, if you transfer to logical notation it will be like this for all of x man of x implies mortal of x. Then Socrates is a man, man (Socrates) the conclusion is therefore Socrates is mortal i. e. mortal (Socrates). This is the argument let us see how you prove it.

There are two premises and the conclusion. So from first using Universal instantiation from using one and from using Universal instantiation you get man (Socrates) implies mortal (Socrates). That is x is taking the value of Socrates and the rule applied is Universal instantiation. And call this as statement 3 then from 3 and 2 by using Modus ponens that is p implies q and p you can conclude q so from this you can conclude mortal (Socrates). So this is the way you have to give the argument.

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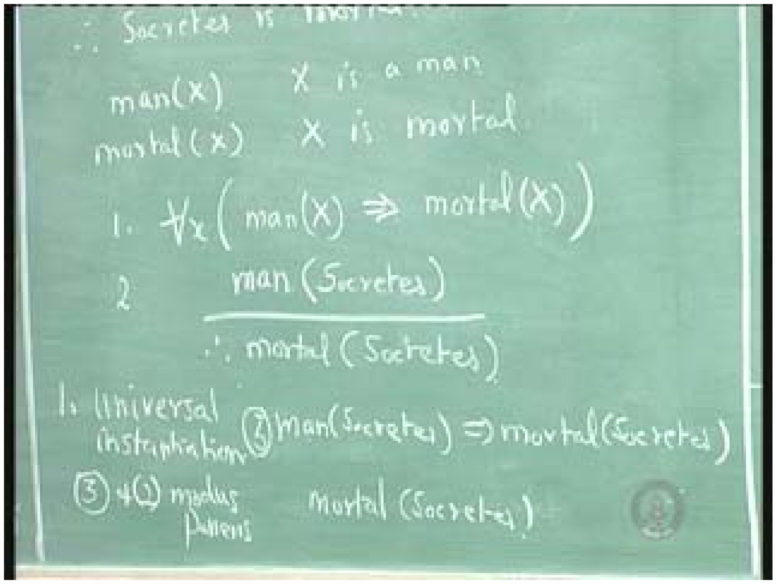

Now let us take some more examples and find how we can prove statements and that some arguments are not correct by using these universal quantifiers and existential quantifiers. This is an argument which we proved that it is correct. Let us take some argument which is not correct. Look at this one, some trigonometric functions are continuous some continuous functions are periodic therefore some trigonometric functions are periodic.

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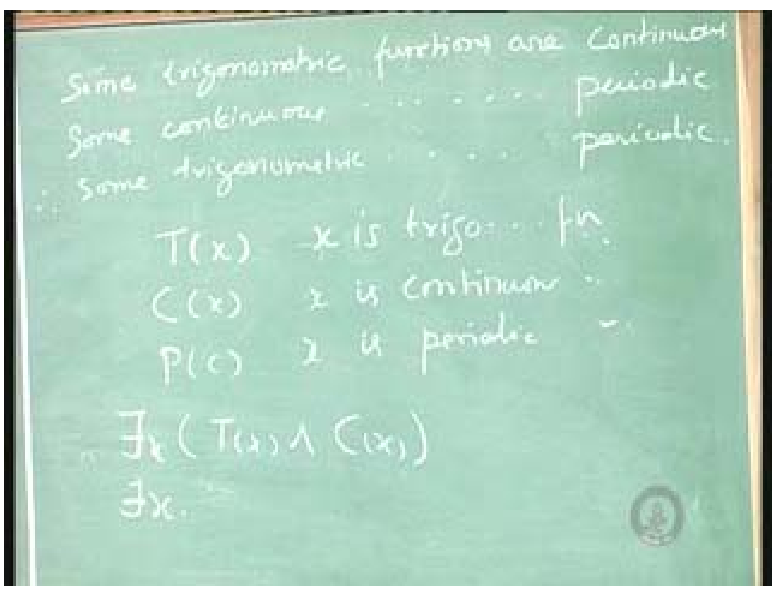

Please remember that we are only looking at an argument. The premises we take to be true and we want to check whether we can derive the conclusion from the premises using rules of inference that is what we want to check. Actually if you look at this in this case all the three statements are really true but whether the conclusion which is some trigonometric functions are periodic will follow from the premises. The premises here are some trigonometric functions are continuous and some continuous functions are periodic.

So if you denote by trigonometric function x is a trigonometric function T(x), x is trigonometric function and C(x) continuous function x is a continuous function P(x) x is a periodic function. The statements will be; there exist x T(x) AND C(x) this is the premise. And the second statement is there exist x C(x) AND P(x). The conclusion will be; there exist x T(x) AND P(x) this is the conclusion.

But you can see that this will not follow from this by the rules of inference in this case you can give a counter example like this; some trigonometric functions are continuous using the Venn diagram this may represent something like this, some trigonometric functions are continuous the intersection is nonempty, the intersection of this is nonempty.

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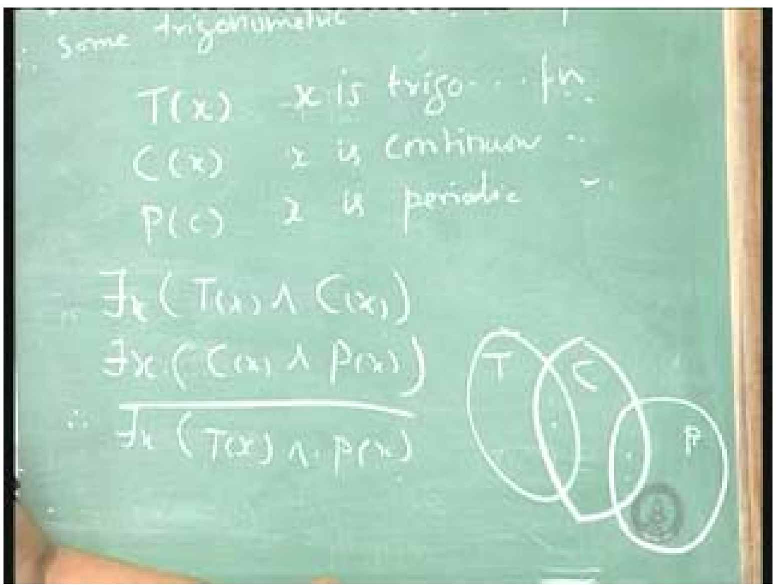

Then the second statement is, some continuous functions are periodic. So if you denote the periodic functions like this the intersection of C and P is nonempty. There are some functions which are continuous and periodic. So the first two statements may represent a situation like this; but you can see that this does not mean that T and P intersect. The intersection of T and P may be empty also by this argument. By this argument we cannot conclude that some trigonometric functions are periodic. So if you want to show that some argument is wrong then in that case you can draw a proper Venn diagram and show that it is correct. But for proving it is not enough if you just draw the Venn diagram you have to also use proper logical argument.

Let us see how you do this in this case. First you can check whether the argument is correct by drawing a Venn diagram but then you have to give the proper argument for this. Babies are illogical, so if you denote B of x to be x is a baby and Il of x x is illogical, D(x) x is despised and M(x) x can manage a crocodile.

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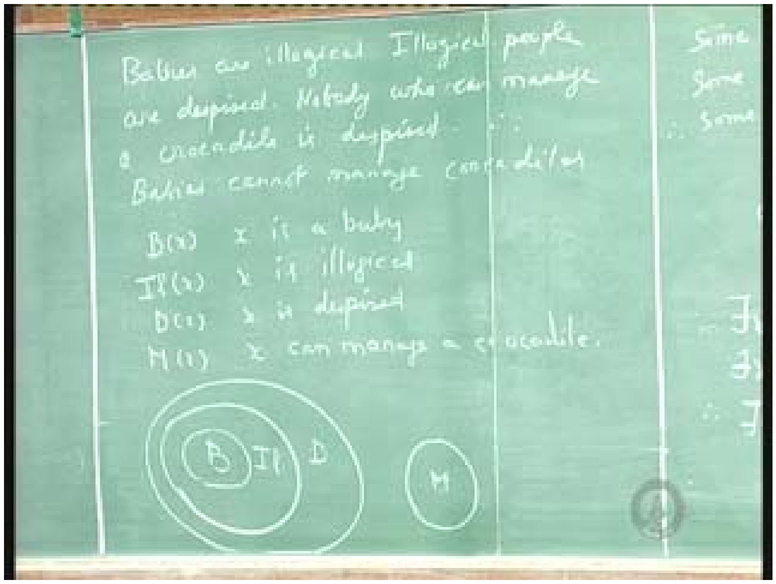

Using Venn diagram how will you represent this?
Babies are illogical so babies will be inside illogical. And all illogical people are despised so that will be inside D and nobody you can manage a crocodile is despised that is the intersection of M and D is empty that means babies cannot manage crocodile that is B and M are disjoint, that is true from this diagram. So this argument is the correct argument. You can very it by drawing the Venn diagram but that is not the proof you have to prove step by step. But if you want to show that some argument is not correct it is enough if you draw the Venn diagram and show that there is a situation where the premises may be true but the conclusion is not true. Let me prove this now. Babies are illogical for all of x B(x) implies Il of x.

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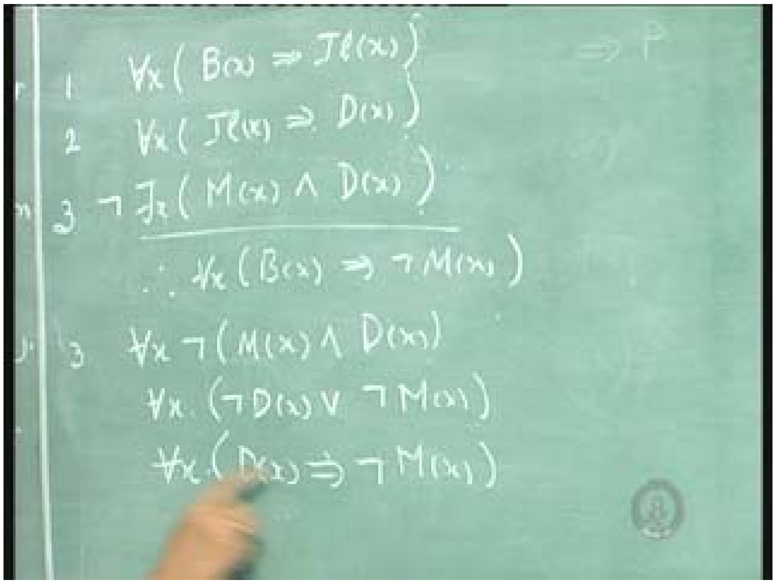

Illogical people are despised for all of x Il of x implies D(x). And nobody is despised who can manage a crocodile NOT there exist x M(x) AND D(x). Therefore for all of x B of x implies NOT M(x). If you write in logical notation this is the way it should be written. So denote by 1, 2, 3 like this and let us use Universal instantiation. We can use for first and second Universal instantiation but before using that we can convert the third one in the usual notation bringing the NOT inside. So before this let us see.

From 3 you can write it as for all of x NOT of M(x) and D of x and this you can write as for all of x NOT(D(x)) OR NOT of M(x) bringing the NOT inside AND will become OR using De Morgan's law and here when you bring the NOT inside there exist will become for all.

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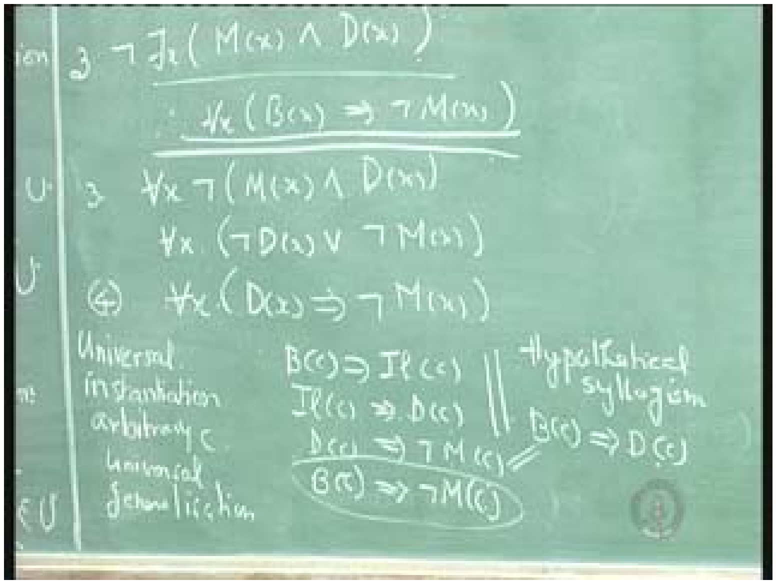

So you can write the third state statement like this and you can also write it as for all of x D(x) implies NOT of M(x) because NOT P OR Q is equivalent to saying P implies Q. So the third one can be written in this form, I will write it as 4. So using Universal instantiation taking an arbitrary C you can write these statements 1, 2 and 4 in the following manner. B(c) implies Il of c Il of c implies D(c) and D(c) implies NOT M(c).

And using Hypothetical syllogism for these two using hypothetical syllogism you will get B(c) implies D(c) and from these two again using hypothetical syllogism you will get B of c implies NOT M(c). Using this and this and using hypothetical syllogism you get B(c) implies NOT M(c) because you have B(c) implies D(c) and D(c) implies NOT of M(c) you will get B(c) implies NOT of M(c). And here from this use Universal generalization you will get the conclusion for all of x B of x implies NOT(M(x)). Let us consider some more examples in logical inference.

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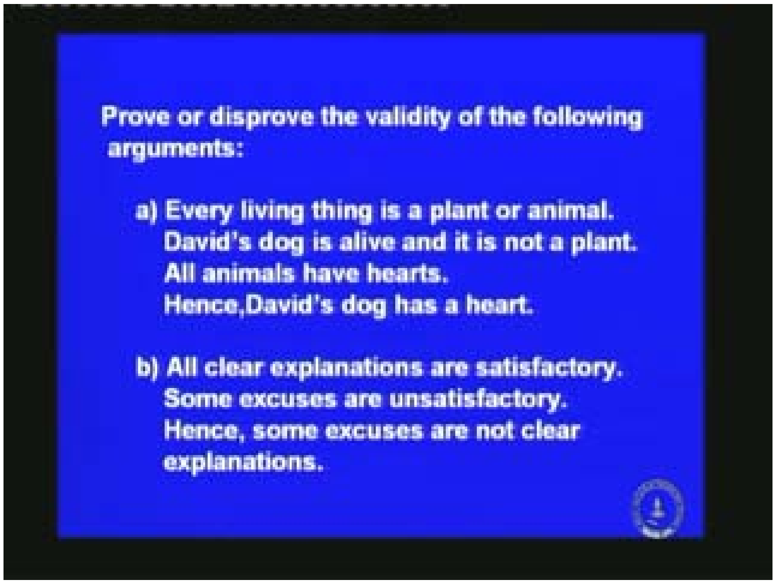

Take this problem, prove or disprove the validity of the following arguments:
Take the first one a; Every living thing is a plant or animal. David's dog is alive and it is not a plant. All animals have hearts. Hence David's dog has a heart. Let us see whether this argument is correct by writing in logical notation. Let L(x) denote x is a living thing and P(x) denote x is a plant, A(x) denote x is an animal and H(x) denote x has a heart.

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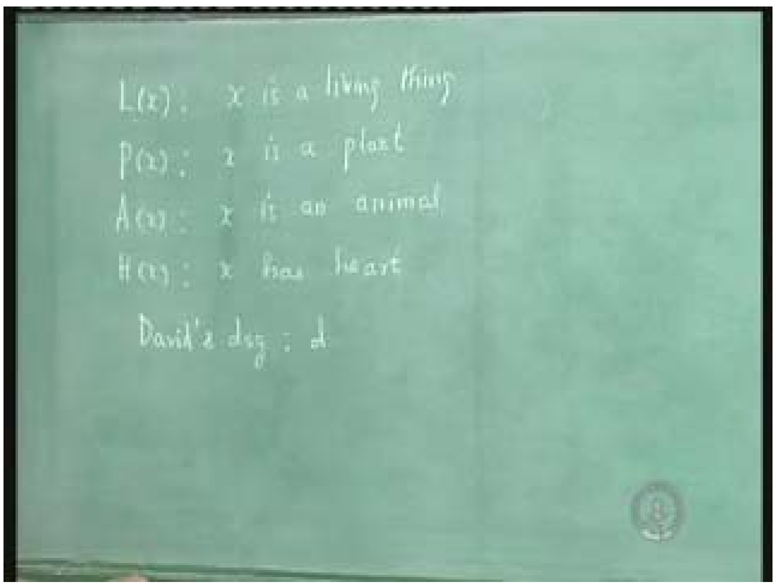

And let David's dog be denoted by d. Now what is the first sentence? The first sentence is every living thing is a plant or animal. You can write it in this way; for all of x L(x) implies P(x) OR A(x).

The second statement; what is the second statement? David's dog is alive and it is not a plant, L(d) and NOT P(d) because d denotes David's dog.

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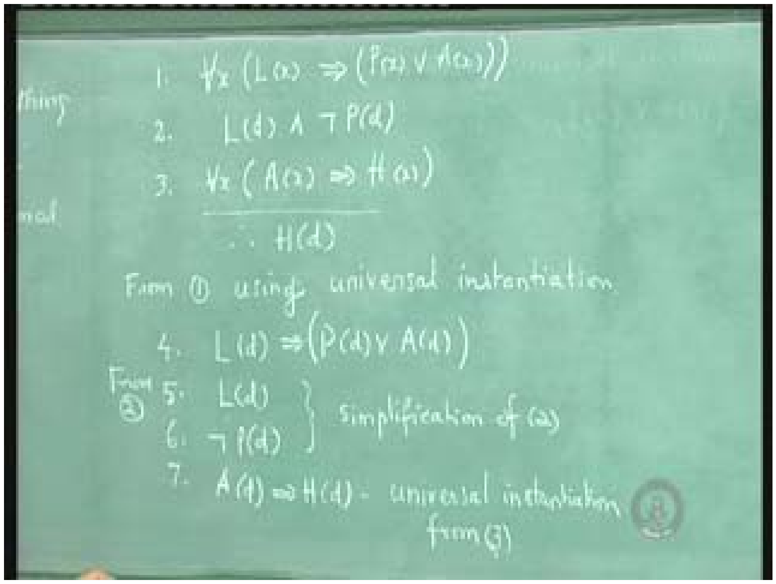

Third sentence is, all animals have hearts. For all of x A(x) implies H(x). From this you have to conclude that therefore what is the conclusion David's dog has a heart Hd is the conclusion you have to get.

Now, statements 1 and 3 have Universal quantifies but statement 2 does not have a quantifier and the conclusion does not have a quantifier. Now you have to see whether you can infer this from this. Now you can use Universal instantiation for these two and try to use the other rules of inference. So from 1 using Universal instantiation what do you get? I can replace x here by any arbitrary value from the universe. Here you have to take the Universe as a set of living things. So Ld implies Pd or Ad. I am just taking it to be d. It should hold any arbitrary value because it is the Universal quantifier taking it to be d because rest of the statements involves d. Then from 2 what is the rule you use? You can use simplification and get two statements you can get Ld and you can also get NOT Pd using simplification from two that is simplification you get them separately. Then again this you can use Universal instantiation so from 3 by using Universal instantiation you will get Ad implies Hd. I can choose any arbitrary value for x, I am choosing it as d. This is again Universal instantiation from 3. Now, from 4 and 5 use Modus ponens. From 5 and 4 by Modus ponens what do you get? Modus ponens rule is P AND P implies Q gives you Q. So P AND P implies Q gives you this so you will get Pd OR Ad.

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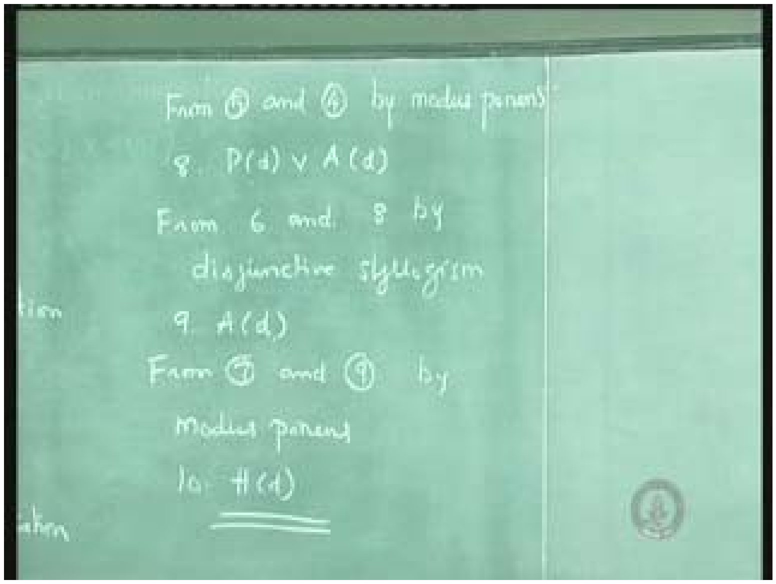

Now the sixth statement is NOT Pd from NOT Pd and the eighth statement Pd OR Ad. From this, that is from 6 and 8 by Disjunctive syllogism what do you get? What is Disjunctive syllogism? That is if you have P OR Q and NOT P by Disjunctive syllogism you will get Q. So this is what we are going to use now. So 6 is NOT P of d and 8 is P of d or Ad so from 6 and 8 by using Disjunctive syllogism you will get Ad. And what is 7? So 7 is Ad implies Hd, 7th statement is Ad implies Hd. So from Ad and Ad implies Hd that is from 7 and 9 by Modus ponens you get Hd so this is the conclusion. So you are arriving at a conclusion from the premises so this argument is correct.

In a similar manner you can prove this part b also. All clear explanations are satisfactory some excuses are unsatisfactory. Hence some excuses are not clear explanations. This is a correct argument. Look at these two things; the following propositions involve predicates that define sets. Use the properties to conclude relationships between these sets. Use Venn diagram to check the validity of the arguments.

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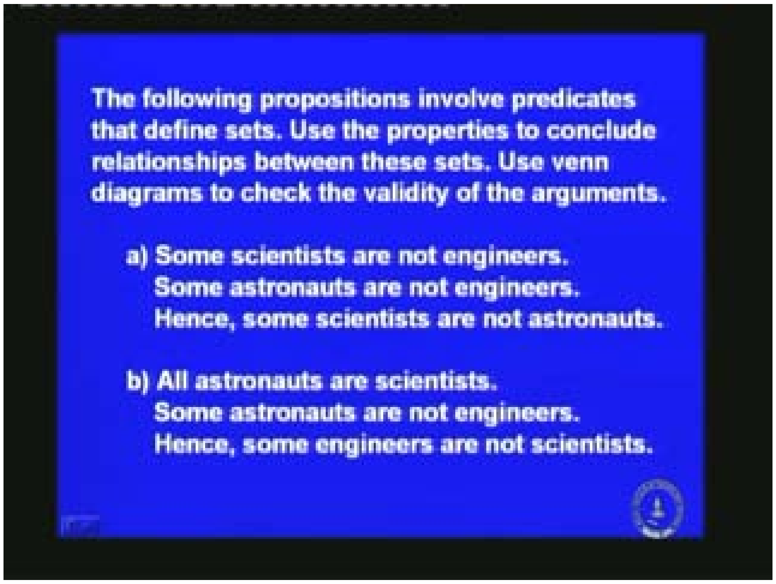

So you can use the Venn diagram to show that some thing is not correct some argument is not correct. Consider the first one; some Scientists are not Engineers, some Astronauts are not Engineers hence some Scientists are not Astronauts. Let me draw a Venn diagram, if you draw a Venn diagram which will satisfy the premises but not the conclusion then the argument is not correct then the conclusion cannot be inferred from the premises.

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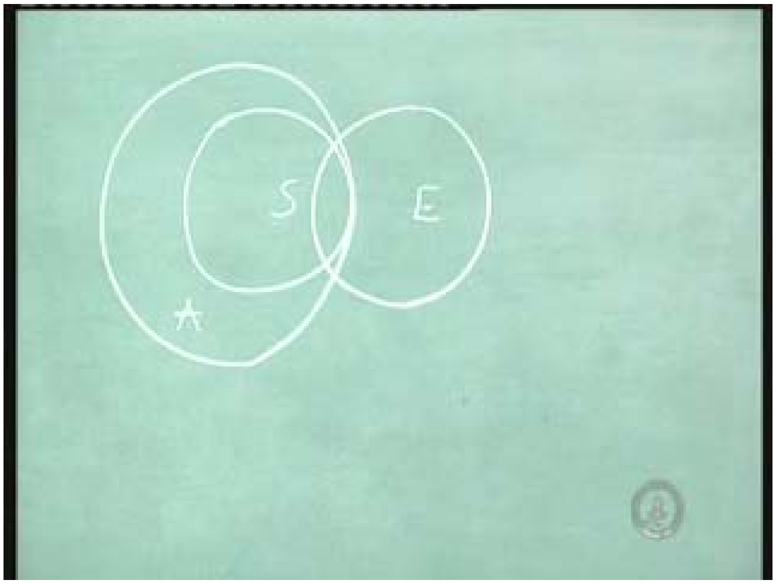

Now the first statement is some Scientists are not Engineers. You can have something like this:

Some Scientists are not Engineers. The second statement is some Astronauts are not Engineers. You can have something like this. Some Astronauts are not Engineers. This diagram satisfies both the premises. Some Scientists are not Engineers. Some Astronauts are not Engineers both of them are satisfied. But from this you cannot conclude some Scientists are not Astronauts because here all Scientists are Astronauts. So from the diagram you cannot infer the conclusion or from the given premises you cannot infer the conclusion the argument is not correct. For such things you can use then Venn diagram and show that the argument is not correct. This is known as giving a counter example.

Whenever an argument is correct you have to write the proof step by step to show that the argument is valid. But when you want to show something is not correct you have to use a counter example. In ordinary propositional Calculus you have to give values to the variables such that the premises are true but the conclusion is false. In the case of arguments like this we can draw the Venn diagram and from the Venn diagram we can show that the premise can be satisfied but the conclusion does not follow from that.

So we have seen how to give an argument how to show that whether an argument is correct or not and so on. Such rules of inference are very useful in the logic programming language prolog. Prolog uses what is known as a resolution principle. We shall see in a moment or may be in the next lecture what a resolution principle is. For that we have to write the premises in a proper form which is called clause form and in order to know what a clause form is we have to see some normal forms of Boolean expressions or propositional expressions or well formed formula of the propositional logic. We have to write them in a proper logical form which is called normal forms.

So let us see what normal forms are. There is one called Conjunctive Normal Form or the CNF. There is another normal form called Disjunctive Normal Form or DNF. You can write any well formed formula either in DNF or in CNF and that is what we are going to see now. Now let us see what CNF is and what DNF is.

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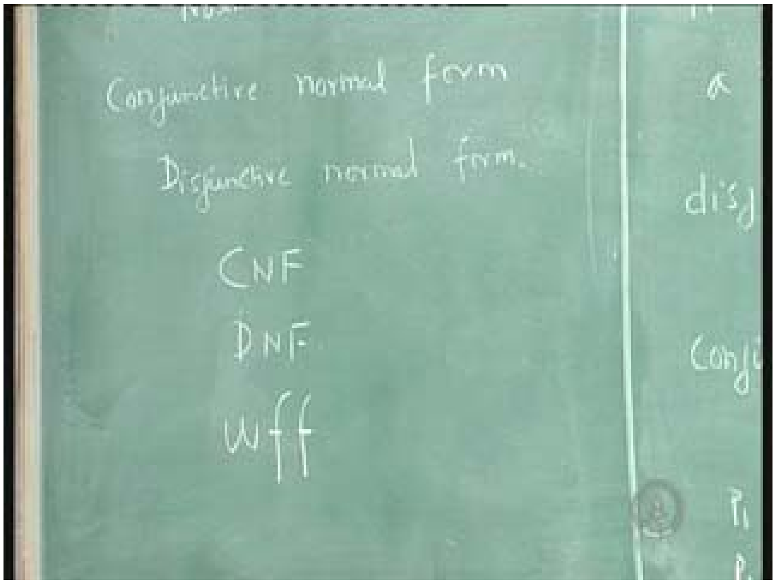

Let us look at these definitions. A variable or the negation of a variable is called a literal. Something like P or NOT P this is called a literal.

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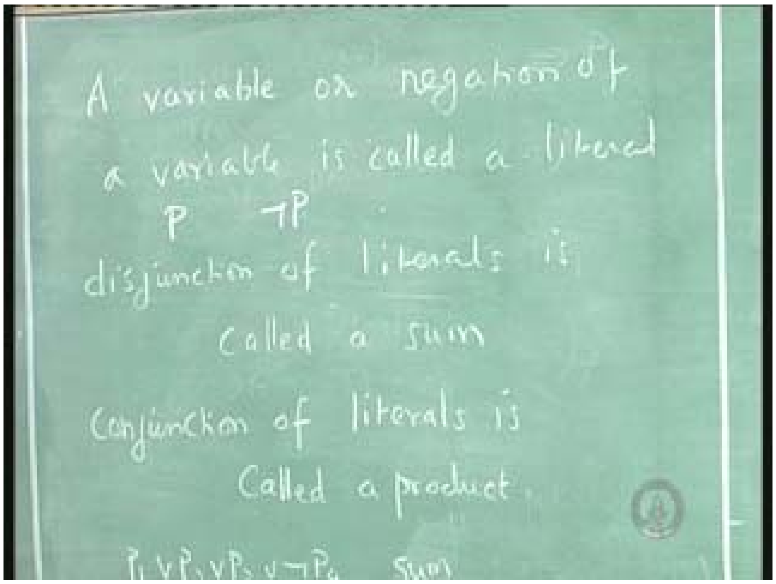

A Disjunction of literals is called a sum. Something like P_1 OR P_2 OR P_3 OR NOT P_4 each one of these is a literal a Disjunction of this is called a sum. A Conjunction of literals is called a product. Look at this P_1 AND P_2 AND NOT P_3 each one is a literal and the Conjunction of this is called a product. A sum of products is called a Disjunctive Normal Form. Something like this is called a Disjunctive Normal Form P_1 AND P_2 AND NOT P_3 OR NOT P_4 AND P_5.

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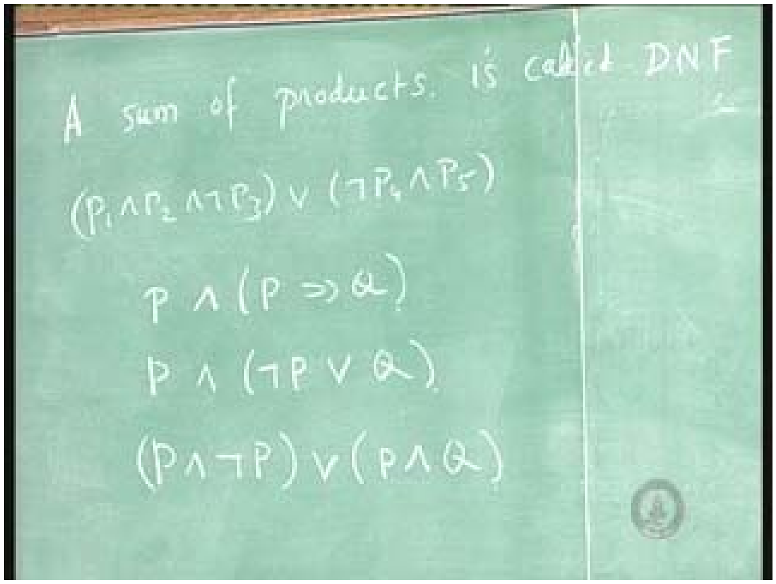

This is a Disjunction; this is the sum of products. Each one is a products and you have a sum of products this is called Disjunctive Normal Form. You can bring any expression well formed formula of the propositional logic into Disjunctive Normal Form.

Let us take an example; P and P implies Q how will you bring it to Disjunctive Normal Form? You can write it as P AND NOT P OR Q. And use distributive laws this will become P AND NOT P OR P AND Q. This is the Disjunctive Normal Form for this expression. You see that it is a sum of product. Each one is a product and you have a sum of product. This is called Disjunctive Normal Form.

Similarly, you have Conjunctive Normal Form which is called product of sums. For example, something like this is a Conjunctive Normal Form P_1 OR P_2 AND P_3 OR NOT P_4 OR P_5 look at this each one is a sum, this is a sum, this is a sum and you have a product of sums. Such an expression is called Conjunctive Normal Form. And we can bring any well formed formula into Conjunctive Normal Form. For example P AND P implies Q how will you bring into Conjunctive Normal Form?

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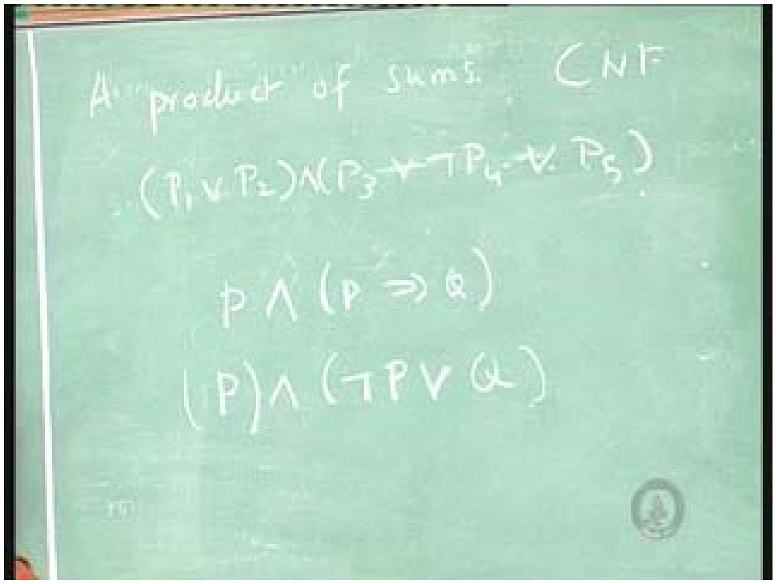

P AND this one you can write as NOT P OR Q so this is a single sum this is another sum you have product of sums. So this is the Conjunctive Normal Form for this. So you can bring any well formed formula into Conjunctive Normal Form.

So we have seen what is meant by a Conjunctive Normal Form and a Disjunctive Normal Form. We will see how to make use of this and write the argument in a different way. And that is called a resolution principle and this is very much used in the logic programming language Prolog. There we make use of this Conjunctive Normal Form and try to write the argument by using what is known as resolution. And this is very useful in logic programming languages like Prolog and we shall see how to go about it in the next lecture.